Consider

$$\begin{cases} x_1 &= b_1 \\ 4x_1 - 2x_2 &= b_2 \\ 2x_1 + 3x_2 &= b_3 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b} = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

coef. of the linear combination

Therefore, b is a linear combination of columns of A.

Definition. Let $A \in M_{m\times n}(\mathbb{R})$, and $a_1, \ldots, a_n$ be columns of A. The column space of A, denoted by $C(A)$ consists of all linear combinations of columns of A, i. e.,

$$C(A) = \text{span}\{a_i \mid i = 1, 2, \ldots, n\}.$$

Remark 1. Let $A \in M_{m\times n}(\mathbb{R})$.

1. To solve for $Ax = b$ means to be able to express b as a linear combination of columns of A. That is, $Ax = b$ has a solution if and only if $b \in C(A)$.
2. $\vec{\mathbf{0}} \in C(A)$
3. $C(A) \leq \mathbb{R}^m$
4. If E is an elementary matrix, then $C(A) = C(AE)$.

2. $Ax = \vec{0}$ will have solution at least solution $x = \vec{0}$

this will always be a linear combination of columns of A w/ coeficients all zero

3. A linear combination of vectors in $\mathbb{R}^m$ will be a subspace of $\mathbb{R}^m$

4. In AE, elementary operations applied on the columns of A will not affect the subspace of linear combination of columns of A.

Example 1. Find a basis for the column space of the following matrices:

1. $\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 2 & 3 \end{bmatrix} = A$
2. $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \end{bmatrix} = B$

1. Note that C(A) is the linear combination of columns so we only need linearly independent columns of A.

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = x_1\begin{bmatrix}1\\4\\2\end{bmatrix} + x_2\begin{bmatrix}0\\-2\\3\end{bmatrix} \Rightarrow \quad A \sim \begin{array}{cc} x_1 & x_2 \\ \left[\begin{matrix} 1^* & 0 \\ 0 & 1^* \\ 0 & 0 \end{matrix}\right] \end{array}$$

columns of A

* leading 1s ⇒ vectors w/ coef. x_1 and x_2 are linearly indep.

$$\Rightarrow \text{basis for } C(A) = \left\{ \begin{bmatrix}1\\4\\2\end{bmatrix}, \begin{bmatrix}0\\-2\\3\end{bmatrix} \right\}$$

2. $B \sim \begin{bmatrix} 1^* & -2 & 3 \\ 0 & 1^* & 1 \end{bmatrix} \Rightarrow$ basis for $C(A) = \left\{ \begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}-2\\7\end{bmatrix} \right\}$

* leading 1s on the 1st and 2nd column of REF of B

Definition. Let $A \in M_{m\times n}(\mathbb{R})$. The nullspace of A, denoted by $N(A)$ consists of all solutions of the homogeneous system $Ax = \mathbf{0}$.

↳ solution set of a homo. system w/ coef. matrix A

Remark 2. Let $A \in M_{m\times n}(\mathbb{R})$. Then

1. $N(A) \leq \mathbb{R}^n$.
2. The solutions of $Ax = b$ do not form a subspace if $b \neq \mathbf{0}$, since $x = \mathbf{0}$ is not among the solutions. Note that $x = \mathbf{0}$ is a solution of $Ax = b$ if $b = \mathbf{0}$.

Example 2. Find a basis for the nullspace of the following matrices:

1. $\begin{bmatrix} 1 & 0 \\ 4 & -2 \\ 2 & 3 \end{bmatrix} = A$
2. $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -6 \end{bmatrix} = B$

1. We only have to compute for solution set of $Ax = 0$ just like in Lesson 2.

$$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow N(A) = \left\{ \begin{bmatrix}0\\0\end{bmatrix} \right\} \text{ (only the trivial solution)}$$

$$\Rightarrow \text{Basis for } N(A) = \emptyset$$

2. $B \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 3 \end{bmatrix} \Rightarrow N(B) = \left\{ \begin{bmatrix} -9r \\ -3r \\ r \end{bmatrix} \mid r \in \mathbb{R} \right\}$

free column

$\Rightarrow$ Basis for $N(B) = \left\{ \begin{bmatrix} -9 \\ -3 \\ 1 \end{bmatrix} \right\}$

Example 3. Find a basis for the column space and nullspace of the following matrices:

1. $\begin{bmatrix} 0 & -2 & 6 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 & 0 & -2 \end{bmatrix}$ $= A$

2. $\begin{bmatrix} 1 & 3 & 3 & 2 & -1 \\ 2 & 6 & 9 & 5 & 1 \\ -1 & -3 & 3 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & -4 \\ 0 & 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $= B$

Verify the ff:

1. Basis for $C(A) = \{[-2]\}$

Basis for $N(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$

Basis for $N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Remark 3. The **dimension of the column space** is the number of pivot columns (columns with leading 1s when the matrix is in row echelon form). The **dimension of the nullspace** is the number of free columns when the matrix is in row echelon form.

Definition. Let $A \in M_{m\times n}(\mathbb{R})$. The **row space** of A, denoted by $R(A)$, is the space spanned by the rows of A.

Remark 4. Let $A \in M_{m\times n}(\mathbb{R})$. Then

1. $R(A) \leq \mathbb{R}^n$, and

2. If E is an elementary matrix, then $R(A) = R(EA)$.

2. In EA, elementary operations applied on the rows of A will not affect the subspace of linear combination of rows of A.

Remark 5. Let $A \in M_{m\times n}(\mathbb{R})$ and $U = \text{RREF}(A)$.

1. $R(A) = R(U)$ (by Remark 4.2, take $EA = \text{RREF}(A)$)
2. $C(A) \neq C(U)$, but they have the same dimension.
3. $N(A) = N(U)$, since $Ax = \mathbf{0}$ and $Ux = \mathbf{0}$ have the same solutions.

Example 4. Find a basis for the row space.

1. $\underset{A}{\underset{\parallel}{\begin{bmatrix} 0 & -2 & 6 & 0 & 4 \end{bmatrix}}} \sim \begin{bmatrix} 0 & 1 & -3 & 0 & -2 \end{bmatrix}$

2. $\underset{B}{\underset{\parallel}{\begin{bmatrix} 1 & 3 & 3 & 2 & -1 \\ 2 & 6 & 9 & 5 & 1 \\ -1 & -3 & 3 & 0 & 7 \end{bmatrix}}} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & -4 \\ 0 & 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

1. Using Remark 5.1.

Basis for $R(A) = \{(0,1,-3,0,-2)\}$

2. Basis for $R(B) = \{(1,3,0,1,-4), (0,0,1,\frac{1}{3},1)\}$

Definition. Let $A \in M_{m\times n}(\mathbb{R})$.

1. The **row rank** of A is the dimension of the row space of A.
2. The **column rank** of A is the dimension of the column space of A.

Remark 6. Let $A \in M_{m\times n}(\mathbb{R})$, P, Q are invertible matrices of appropriate sizes.

1. row rank of PA = row rank of A (by Rem 4.2)
2. column rank of AQ = column rank of A (by Rem 1.4)

Theorem. *Let $A \in M_{m\times n}(\mathbb{R})$. Then row rank of A = column rank of A.*

row rank of A = row rank of RREF(A)
= number of nonzero rows of RREF(A)
= number of leading 1s of RREF(A)
= number of linearly indep. columns of A
= column rank of A ∎

Therefore, there is no distinction between column rank and row rank, so we will just say **rank.**

Definition.

1. The **rank** of a marix A, denoted by $\rho(A)$, is the common value of the row and column rank.
2. The **nullity** of a marix A, denoted by $\eta(A)$ is the dimension of $N(A)$

Corollary. *Let $A \in M_{m\times n}(\mathbb{R})$, P, Q are invertible matrices of appropriate sizes. Then rank(PAQ) =rank (A).*

By Remark 6 and Theorem.

Example 5. Find $\rho(A)$, $\eta(A)$, and give bases for $C(A)$, $R(A)$ and $N(A)$ if

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\} \Rightarrow \rho(A) = 2$

Basis for $R(A) = \left\{ \begin{matrix} (0,1,2,0,-2), \\ (0,0,0,1,2) \end{matrix} \right\}$ (Basis from nonzero rows of RREF(A))

Note that

1st row of A

3rd row of A

$(0,1,2,4,6) = 1\,(0,1,2,3,4) + 1\,(0,0,0,1,2)$

2nd row of A

linearly independent

Basis for $R(A) = \left\{ \begin{matrix} (0,1,2,3,4), \\ (0,0,0,1,2) \end{matrix} \right\}$ (Basis is not unique, but it is easier to just use nonzero rows of RREF(A) or REF(A) to get basis of R(A).)

$$N(A) = \left\{ \begin{bmatrix} r \\ -2s+2t \\ s \\ -2t \\ t \end{bmatrix} \;\middle|\; r,s,t \in \mathbb{R} \right\}$$

$$\text{Basis for } N(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \Rightarrow \eta(A) = 3$$

Observe that: Number of columns of $A = \rho(A) + \eta(A)$.

In preceding example: $5 = 2 + 3$

Remark 7. Let $A \in M_n(\mathbb{R})$. The following are equivalent:

1. A is invertible;
2. A is row equivalent to I_n ($\text{RREF}(A) = I_n$);
3. for every b, there is a unique solution to $Ax = b$;
4. $Ax = 0$ only has the trivial solution;
5. for every $b \in \mathbb{R}^n$ is a linear combination of the columns of A;
6. $\det A \neq 0$;
7. $\rho(A) = n$
8. the columns of A form a basis for $\mathbb{R}^n$;
9. $\eta(A) = 0$

Remark 8. Let $A \in M_n(\mathbb{R})$. The following are equivalent:

1. A is not invertible;
2. A is row equivalent to a matrix with a row of 0s;
3. there is a b such that $Ax = b$ has either no solution or infinitely many solutions;
4. $Ax = 0$ only has infinitely many solutions;
5. there is a $b \in \mathbb{R}^n$ that is not a linear combination of the columns of A;
6. $\det A = 0$;
7. $\rho(A) < n$
8. the columns of A do not form a basis for $\mathbb{R}^n$
9. $\eta(A) > 0$

Remark 9. Every solution to a linear system $Ax = b$ can be written in the form

$$x_p + x_n,$$

where x_p is a particular solution of $Ax = b$ and x_n is an element of the nullspace of A (solution of $Ax = 0$).

$A(x_p + x_n) = Ax_p + Ax_n = b + 0 = b \Rightarrow x_p + x_n$ is a solution of $Ax = b$.

Example 6. Find the solution set to the following systems of equations using the nullspace of the coefficient matrix of the given system:

1.

$$\underbrace{\begin{bmatrix}1&1&2\\1&2&-1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}\Rightarrow\left[\begin{array}{c|c}\underbrace{\begin{matrix}1&0&5\\0&1&-3\end{matrix}}_{RREF(A)}&\begin{matrix}5\\-1\end{matrix}\end{array}\right]$$

$$N(A)=\left\{\begin{bmatrix}-5r\\3r\\r\end{bmatrix}\middle|\, r\in\mathbb{R}\right\}$$

$$\text{Take } x_p=\begin{bmatrix}5\\-1\\0\end{bmatrix}.\quad \therefore\ \text{Solution set}=\begin{bmatrix}5\\-1\\0\end{bmatrix}+\underbrace{\overset{x_n\,\in}{\text{span}\left\{\begin{bmatrix}-5\\3\\1\end{bmatrix}\right\}}}_{N(A)}$$

2.

$$\underbrace{\begin{bmatrix}1&1\\1&2\\-2&-3\end{bmatrix}}^{A}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2\\-1\\3\end{bmatrix}\Rightarrow\left[\begin{array}{cc|c}1&0&-3\\0&1&1\\0&0&0\end{array}\right]$$

$$N(A)=\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}\ ;\quad x_p=\begin{bmatrix}-3\\1\end{bmatrix}\quad \therefore\ \text{Solution Set}=\begin{bmatrix}-3\\1\end{bmatrix}+\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

$$=\left\{\begin{bmatrix}-3\\1\end{bmatrix}\right\}$$

3.

$$\underbrace{\begin{bmatrix}1&3&3&2&-1\\2&6&9&5&1\\-1&-3&3&0&7\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2\\0\\-10\end{bmatrix}\Rightarrow\left[\begin{array}{ccccc|c}1&3&0&1&-4&6\\0&0&1&\frac{1}{3}&1&-\frac{4}{3}\\0&0&0&0&0&0\end{array}\right]$$

$$N(A)=\left\{\begin{bmatrix}-3r-s+4t\\r\\-1/3s-t\\s\\t\end{bmatrix}\middle|\, r,s,t\in\mathbb{R}\right\}$$

$$x_p=\begin{bmatrix}6\\0\\-4/3\\0\\0\end{bmatrix}\Rightarrow\ \text{solution set}=\begin{bmatrix}6\\0\\-4/3\\0\\0\end{bmatrix}+\underbrace{\text{span}\left\{\begin{bmatrix}-3\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-1/3\\1\\0\end{bmatrix},\begin{bmatrix}4\\0\\-1\\0\\1\end{bmatrix}\right\}}^{N(A)}$$