Intraday Value-at-Risk Estimation for Directional Change Events and Investment Strategies

Rui Jorge Almeida*†, Nalan Baştürk* and Robert Golan‡
*Department of Quantitative Economics, Maastricht University
P.O. Box 616, 6200 MD Maastricht, The Netherlands.
E-mail: rj.almeida@maastrichtuniversity.nl
†BISS Institute, Brightlands Smart Services Campus
Smedestraat 2, 6411 CR Heerlen, The Netherlands
E-mail: n.basturk@maastrichtuniversity.nl
‡DBmind Technologies Inc.
2829 Indian Creek Drive, Suite 1211, Miami, USA
E-mail: golan@dbmind.com

***Abstract*—Accurate risk measurement is important for making and assessing investment decisions. Recently, directional change representations of returns are proposed as a new method for analyzing and forecasting intra-day price movements and for creating investment strategies. This paper presents an FGARCH model for intraday Value-at-Risk (IVaR) estimation, for assessing risk properties in time series represented as directional change events, and for predicting the market risk for investment strategies based on directional changes. We apply the proposed method to 5-minute intraday data and report the time-varying risk based on IVaR estimates. We study the accuracy of these estimates and report the robustness of the risk estimates to the choice of the threshold parameter for directional change representations. Furthermore, we apply the proposed methodology to compare the risk properties of two investment strategies based on directional changes with a baseline moving window investment strategy. For these data, we find that the directional change strategies lead to higher returns but no substantial increase in risk compared to the baseline strategy. The proposed methodology is applicable to other intra-day data frequencies and it is generalizable to other risk metrics.**

I. Introduction

Assessing measures of risk for financial returns has an important role in investment decisions. Investors are not only interested in the expected return from an asset but also in the risk involved in the asset. As a result of risk management, activities are undertaken to reduce the possibility of failure to an acceptable range. These activities may include portfolio adjustment, hedging or insurance [1], [2]. In line with this, particularly after the recent financial crisis, accurate risk analysis has drawn considerable attention [3], [4]. One difficulty in financial risk analysis is that risk, typically market risk, varies over time together with the potential changes in the return distribution. Such time-variation in the volatility structure is important both in daily returns [5] as well as intra-day returns [6], where shorter than daily time intervals for risk assessments are required to match the horizon of investments, which is also generally less than a day.

Recently, directional changes (DC) are proposed as a new method of summarizing, analyzing and forecasting intra-day market price movements [7], [8]. Based on these summaries, it is possible to determine investment strategies based on directional changes in the data [9], [10]. This approach builds on the literature on estimation and prediction of turning points, often applied to business cycles and economic indicators [11].

In the DC approach, a time series price curve is transformed into an intrinsic time curve which records price changes that exceed a threshold level and identify upwards and downwards price changes [8], [7]. Risk properties of DC-based data summaries have been analyzed using several metrics [8], but more conventional risk evaluation methods, such as the Value-at-Risk (VaR) have not yet been implemented in assessing the risk of directional changes.

In this paper, we propose to use a fuzzy Generalized Autoregressive Conditional Volatility (FGARCH) model [5] for intraday Value-at-Risk (IVaR) estimation. The FGARCH model has been previously used for conditional density estimation of daily returns [5], where the distribution of the returns are allowed to vary in mean and variance smoothly over time, related to fuzzy sets. In this paper we also propose to use the FGARCH model to assess risk properties in directional changes events, and exemplify how risk can be assess for trading strategies based on DC. We show that the flexible distributional properties of the FGARCH model enables the prediction of market risk for returns which are represented by their directional changes as well as the prediction of market risk for investment strategies based on directional changes.

We exemplify the proposed method using 5-minute intra-day data from HSBA data (HSBC Holdings PLC returns), report the IVaR estimates, accuracy of these estimates and the robustness of the risk estimates to the choice of the threshold parameter for directional changes. In addition, we consider two investment strategies based on DC which focus on event points of the intrinsic time curve and compare the IVaR estimates with those from a conventional moving window IVaR estimates. The method proposed in this paper has the potential to be generalized to estimate and forecast other market risk metrics such as the expected shortfall.

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II. Fuzzy GARCH Model

In this paper we use the fuzzy GARCH model, FGARCH(l,p,q) to estimate a conditional density function, which combines two different types of uncertainty, namely fuzziness or linguistic vagueness, and probabilistic model. In this model the output y_t and conditional variance h_t are defined by each of l-th fuzzy rule [5]

$$R_l : \text{If } \mathbf{x} \text{ is } F_l \text{ then } y_t^l \mid x_t, h_t^l \sim N(\mu^l, h_t^l)\,, \tag{1}$$

$$\text{with } h_t^l = \alpha_0^l + \sum_{i=1}^{q} \alpha_i^l y_{t-i}^2 + \sum_{j=1}^{p} \beta_j^l h_{t-j}\,, \tag{2}$$

where h_t is given by $h_t = \sum_{l=1}^{L} g_{t,l} h_{t,l}$ and the output of the system is defined using the rule outputs and the membership function $g_{l,t} = f_l(x_t)$ which is a function of the antecedent variable x_t.

For the l-th fuzzy rule, the output is defined by a GARCH(q,p) model and it has a normal distribution with mean μ^l and variance h_t^l defined by:

$$y_t^l = \sqrt{h_t^l}\epsilon_t\,, \tag{3}$$

$$h_t^l = \alpha_0^l + \sum_{i=1}^{q} \alpha_i^l y_{t-i}^2 + \sum_{j=1}^{p} \beta_j^l h_{t-j}\,, \tag{4}$$

$$h_{t-j} = \sum_{l=1}^{L} g_{l,t-j} N(\mu^l, h_{t-j}^l)\,, \text{ for } j = 1\ldots, p\,. \tag{5}$$

The output of the fuzzy GARCH model FGARCH(l,p,q) is given by:

$$y_t \mid h_t, x_t \sim \sum_{l=1}^{L} g_{l,t} N(\mu^l, h_t^l)\,, \tag{6}$$

which can be interpreted as a fuzzy combination of normal densities. Depending on the membership functions $g_{l,t}$, the output has several distributional forms, such as a normal density, a skewed density or a bimodal density [5].

The output of the FGARCH(l,q,p) model has a proper conditional distribution, similar to a finite mixture of normal densities, under the condition that membership values satisfy $g_{l,t} \geq 0, \forall l, t$ and $\sum_{l=1}^{L} g_{l,t} = 1, \forall t$. This condition ensures that the probability density, hence the likelihood of observation t can be written conditional on past observations and past variance. The conventional definitions of the membership values in fuzzy models satisfy these conditions since $g_{l,t} = u_{l,t}/\sum_{l=1}^{L} u_{l,t}$, $u_{l,t} \geq 0$ for $l = 1, \ldots, L$, $\sum_{l=1}^{L} u_{l,t} > 0$.

The FGARCH(l,q,p) model ensures positivity and stationarity conditions h_t^l for every rule and at every time period. A sufficient condition for this is to incorporate standard GARCH model conditions for each rule $l = 1, \ldots, L$ in the model:

$$\alpha_0^l > 0,\ \alpha_i^l \geq 0,\ \beta_j^l \geq 0, \tag{7}$$

$$\sum_{i=1}^{q} \alpha_i^l + \sum_{j=1}^{p} \beta_j^l < 1, i = 1, \ldots, q,\ j = 1, \ldots, p. \tag{8}$$

The fuzzy GARCH model combines fuzziness or linguistic vagueness, and probabilistic model. We assume that the probabilistic uncertainty is model with the well understood GARCH model. In a GARCH model the conditional density of the data is a normal distribution with time varying variance depending on past variance and past observations. The fuzziness or linguistic vagueness is present in the antecedent of each rule and their combination. By using fuzzy sets to represent linguistic vagueness, the distribution of the returns is allowed to vary in mean and variance smoothly over time, where the smooth changes are related to linguistic descriptors, belonging to one or several fuzzy sets at the same time [5].

As described in [5], it is possible to estimate the model in (1) and (2) by maximum likelihood method, given that input x_t is included in the information set at time $t-1$, i.e. x is *predetermined* with respect to y. Specifically, x_t can use past y values or can be an exogenous variable. In this paper we use this property to estimate Value-at-Risk with horizons matching the investment positions taken.

Given that the type and number of membership functions $g_{l,t}$ are known, the log-likelihood of $y = \{y_{t^\star}, \ldots, y_T\}$ is:

$$\ln \ell(y \mid I_{t-1}) = \ln \prod_{t=t^\star}^{T} \ell(y_t \mid x_t, h_t) \tag{9}$$

$$= \sum_{t=t^\star}^{T} \ln \left(\sum_{l=1}^{L} g_{l,t} \phi(y_t; \mu^l, h_t^l) \right), \tag{10}$$

where $t^\star = \max(p+q) + 1$ and initial variances $\{h_1, \ldots, h_{t^\star-1}\}$ are assumed to be known. In practice, $\{h_1, \ldots, h_{t^\star-1}\}$ can be set as the unconditional data variance.

In order to obtain the parameter estimates, the likelihood in (9) is maximized with respect to the GARCH parameters and the membership parameters at the same time using a gradient search method. We define Gaussian membership functions since these membership functions constrain the search space to solutions satisfying the positive variance condition and membership functions that cover the universe of the input variables in the antecedent space [5]. In this paper, we choose an FGARCH(2,1,1) model where the model consists of $L = 2$ fuzzy rules and GARCH(1,1) models for each rule. Our choice of $L = 2$ is based on the robustness results presented in [5], which shows that parameter estimates are more stable compared to FGARCH with a higher number of fuzzy rules. The optimization problem is [5]:

$$\underset{\mu_l, \theta_{g,l}, \theta_{u,l}}{\text{minimize}} - \ln \ell(y|I_{t-1}) \tag{11}$$

subject to

$$\alpha_{0,l} > 0,\ \alpha_{i,l} \geq 0,\ \beta_{j,l} \geq 0,\ \forall i, j \tag{12}$$

$$\sum_{i=1}^{q} \alpha_{i,l} + \sum_{j=1}^{p} \beta_{j,l} < 1,\ \forall i, j \tag{13}$$

$$c_l \leq c_{l+1},\ l = 1, \ldots, L-1, \tag{14}$$

where c_l for $l = 1, \ldots, L$ are the centers of the Gaussian memberships. The first two restrictions in the optimization

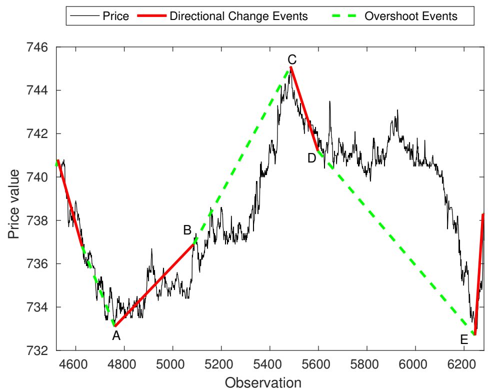


Fig. 1. Stock price data for HSBA and the corresponding intrinsic time curve for a $\theta = 0.05\%$, for a selected time period.

problem ensure a positive variance at each time period, and the last restriction ensures that the membership functions cannot permute labels.

III. Directional Changes

The directional change (DC) approach is a way to summarize market price movements. A DC event is identified by a change in the price of a given financial instrument, given a pre-defined threshold value θ. Any change below this value θ is not considered an event. DC events can be either a downturn or an upturn event.

A downturn DC event is defined as an event where the absolute price change between the current price P_t and the last high price P_h is lower than a fixed threshold θ [8]:

$$P_t \leq P_h(1-\theta) \tag{15}$$

Similarly, an upturn DC event is defined as an event where the absolute price change between the current market price P_t and the last low price P_l is higher than a fixed threshold θ:

$$P_t \geq P_l(1+\theta) \tag{16}$$

After a DC event, an overshoot (OS) event follows until an opposite DC event occurs. Under this approach, a downward trend is characterized by a downturn event followed by a downward overshoot, and an upward trend is characterized by an upturn event followed by an upward overshoot. In the DC approach, a time series price curve is transformed into an intrinsic time curve composed of DC and OS events, given a threshold θ [7], [10]. We note that different thresholds θ will create different intrinsic time curves.

Figure 1 depicts a part of the HSBA time series and the corresponding intrinsic time series composed of DC and OS events for a $\theta = 0.01\%$ [7]. In this figure, an upturn DC event starts at point A and lasts until point B, when the upturn OS events starts. The upturn OS lasts until point C, when there is a reverse in the trend, and an downtrend starts, which lasts until point D. From point D to E we are in an downtrend OS event. In the remainder of this paper, we denote points B and D, the directional change confirmation points in Fig. 1, as the Directional Change (DC) points. Similarly, we denote points A and C, the directional change extreme points in Fig. 1 is Overshoot (OS) points. We will particularly report the risk properties at these DC and OS points.

We further note that a confirmation of a change of a trend can only be confirmed after the price has changed by the pre-specified DC threshold value θ. In Fig. 1 we can only confirm that we are in a upward trend from point B onwards. Point B is called a confirmation point. Before point B, the market price had not changed by the pre-specified θ and thus the directional change is not confirmed. According to the intrinsic time series, the price is in a downward trend, which started before this point B. Trading strategies based on DC try to anticipate the change of the trend as early as possible, i.e. before Points C and E have been reached. For currency exchange rates, it was observed that a DC with a given θ and duration t is on average followed by an OS event of the same θ and duration of $2t$ [7].

A. DC Summarizing

To analyse price dynamics using the DC approach, it is necessary to extract useful indicators to describe profiles of price movements in markets [8]. In this paper we selected summary indicators that can potentially aid in the understanding of the risk an investor may take in certain DC and OS events in the intrinsic time series.

Let $t = 1, \dots, T$ denote the time intervals during which prices are observed, p_t denote the price level at time t, and $r_t = 100 \times \ln(p_t - p_{t-1})$ denote the percentage returns at time t. Starting from an initial directional change point DC_{k-1}, directional changes are defined with respect to the earlier directional change point as the reference point. For $k = 1, \dots, K$ directional change points with $K < T$, each directional change and the sign of the directional change can be calculated iteratively. Specifically, if DC_{k-1} is a downward DC, the next directional change is an upward directional change (UDC) with the following timing:

$$\mathrm{DC}_k = \underset{t \geq \mathrm{DC}_{k-1}}{\arg\min} \left(p_t \geq p_{\mathrm{DC}_{k-1}}(1+\theta)\right), \tag{17}$$

$$p_h = p_{\mathrm{DC}_k}, \tag{18}$$

$$\mathrm{EDC}_k = \mathrm{UDC} \tag{19}$$

where p_h denotes the latest 'high price' in the market, and 'event' indicates the sign of the directional change at every time period.

Similarly, if DC_{k-1} is an upward DC, the next directional change is an downward directional change (DDC) with the following timing:

$$\mathrm{DC}_k = \underset{t \geq \mathrm{DC}_{k-1}}{\arg\min} \left(p_t \leq p_{\mathrm{DC}_{k-1}}(1-\theta)\right), \tag{20}$$

$$p_l = p_{\mathrm{DC}_k}, \tag{21}$$

$$\mathrm{EDC}_k = \mathrm{DDC} \tag{22}$$

where p_l denotes the latest 'high price' in the market and DC_0 can be initialized as an upward DC point at the beginning of the sample.

In line with these definitions, the following summary metrics are useful in defining directional changes. These metrics are average time of a DC (TDC), average time of a OS (TOS) after a confirmed directional change, and average ratio of OS event length over average ratio of DC event length (TOS/TDC), number of directional changes (#DC), number of upward directional changes (#UDC), number of downward directional changes (#DDC):

$$\mathrm{TDC} = \frac{1}{K} \sum_{k=1}^{K} \left(\mathrm{DC}_k - \mathrm{DC}_{k-1} \right), \tag{23}$$

$$\mathrm{TOS} = \frac{1}{K} \sum_{k=1}^{K} \left(\mathrm{OS}_k - \mathrm{DC}_k \right), \tag{24}$$

$$\mathrm{TOS}/\mathrm{TDC} = \frac{1}{K} \sum_{k=1}^{K} \frac{\mathrm{OS}_k - \mathrm{DC}_k}{\mathrm{DC}_k - \mathrm{DC}_{k-1}}, \tag{25}$$

$$\#\mathrm{DC} = K, \tag{26}$$

$$\#\mathrm{UDC} = \frac{1}{K} \sum_{k=1}^{K} I[\mathrm{EDC}_k = \mathrm{UDC}], \tag{27}$$

$$\#\mathrm{DDC} = \#\mathrm{DC} - \#\mathrm{UDC}, \tag{28}$$

where $I[.]$ is an indicator function which takes the value of 1 if its argument is true, and the value of 0 otherwise.

The total number of DC events for a given profiled period, for a given θ, (#DC) measures the variation of DC events. Based on the same θ, a time period with lower #DC value will have less volatility than another time period of the same length.

B. Trading Strategies

One potential advantage of identifying price changes is to predict future changes and invest in the stocks in accordance with the upward or downward price movements. For directional changes, it is very important for an investor to record or anticipate DC events and particularly the OS events to increase returns from an investment on the stock [10]. Such an investment, however, also includes a financial risk which is time-varying. In this section, we consider two DC based investment strategies and a standard baseline strategy in line with [10], which will be compared in terms of their IVaR values.

As the baseline strategy, we use the Moving Window (MW) strategy which is not based on directional changes and is defined as:

$$\mathrm{MW}(L,t) = \frac{p(t) - 1/L \sum_{i=1}^{L} p(t-i)}{1/L \sum_{i=1}^{L} p(t-i)}, \tag{29}$$

where the short-term $\mathrm{MW}(L,t)$ goes above the long-term $\mathrm{MW}(L,t)$ an upward momentum is expected and the investor invests one unit (goes long) in the stock. On the other hand, when the short-term MW goes below the long-term one, this indicates downward momentum and the investor sells one unit (goes short) of the stock.

In addition, we define a Directional Change (DC1) strategy where the investor follows the directional change confirmation points, points B and D in Fig. 1, and buys the stock at an upward DC event, and sells the stock at a downward DC event:

$$\begin{aligned} &\text{If } \mathrm{EDC}_k = \mathrm{UDC}, \text{ go long while } \mathrm{DC}_k < t \leq \mathrm{DC}_{k+1}, \\ &\text{If } \mathrm{EDC}_k = \mathrm{DDC}, \text{ go short while } \mathrm{DC}_k < t \leq \mathrm{DC}_{k+1}. \end{aligned}$$

Following the DC summary, we define a Directional Change (DC2) strategy, where we consider an investor who follows the directional change confirmation points, but anticipates the OS points, points A and C in Fig. 1. In this case, the investment strategy 'predicts' an OS point, hence a change in direction at time $DC_k + \mathrm{TOS}$ following a confirmation point at time DC_k:

$$\begin{aligned} &\text{If } \mathrm{EDC}_k = \mathrm{UDC} \;\&\; \mathrm{DC}_k < t \leq \mathrm{DC}_k + \mathrm{TOS}, \text{go long} \\ &\text{If } \mathrm{EDC}_k = \mathrm{UDC} \;\&\; \mathrm{TOS} < t \leq \mathrm{DC}_{k+1}, \text{go short} \\ &\text{If } \mathrm{EDC}_k = \mathrm{DDC} \;\&\; \mathrm{DC}_k < t \leq \mathrm{DC}_k + \mathrm{TOS}, \text{go short} \\ &\text{If } \mathrm{EDC}_k = \mathrm{DDC} \;\&\; \mathrm{TOS} < t \leq \mathrm{DC}_{k+1}, \text{go long,} \end{aligned}$$

i.e. the investor reverts his/her position at the anticipated OS point such as points A and C in Fig. 1.

C. Risk estimation

As previously discussed, the DC approach allows to summarize the movement of financial markets' prices and derive trading strategies from them. Under the DC approach, there has been some new risk measures put forward to better understand the summaries of the financial markets price movements. Different risk and volatility measures have recently been defined for directional change events. The time that it takes to complete a trend, number of DCs and the magnitude of price change in each trend together are shown to provide additional perspectives on volatility and risk dynamics [8].

A conventional measure to quantify the risk of an asset is the Value-at-risk (VaR). This measure is used widely, for example by large banks [12]. It is a single number for the senior management to express and summaries the total market risk of a portfolio with financial assets. VaR measures the worst expected loss over a given horizon under normal market conditions at a given confidence level. In this paper, we study the use of the well studied and understood Value-at-Risk measure [13], [3], as a way to provide information about the DC approach.

The VaR value is calculated over a time horizon h at a significance level c. It indicates the maximum loss that a portfolio of assets will suffer over a horizon h with a confidence of c. Assume that a portfolio has value W_t at time t. Let r denote the one period percentage return of the portfolio. If $f(r)$ is the probability density function of the returns, define r_v such that

$$1 - c = \int_{-\infty}^{r_v} f(r) dr. \tag{30}$$

Given $Pr(r < r_v) = c$ the Value-at-Risk V_t of the portfolio at time t is then defined as

$$V_t = -r_v W_t. \tag{31}$$

In this paper we propose the use of fuzzy GARCH models to calculate intraday Value-at-Risk, by using the output of a FGARCH(l,q,p) model given by (6) to approximate $f(r)$, given 5 minute tick data. The use of a fuzzy GARCH model does not place any restrictions on the output density $f(r)$, allowing for skewed or bimodal densities, as well as time varying mean of returns [5]. Intraday VaR was introduced in [14] and it results from applying VaR techniques to intraday returns and further studied in [15], [16], [17] In this paper we validate the use of the FGARCH(l,q,p) model for intra day Value-at-Risk performing a failure test of unconditional coverage test using the Kupiec test [18] and independence test using Christoffersen's Markov test [19].

Kupiec confidence regions are defined through the tail point of the log-likelihood ratio LR_{uc}

$$LR_{uc} = 2\ln\left(\left(\frac{1-I/T}{c}\right)^{T-I}\left(\frac{I/T}{1-c}\right)^{I}\right), \tag{32}$$

where I is the number of exceptions and T is the total number of observations. It is considered that an exception $I_t(c)$ has occurred when $r_{t+1} < \mathrm{VaR}_t(c)$. This ratio is shown to be asymptotically χ^2-distributed, with 1 degree of freedom, under the null hypothesis that the VaR model is valid [18]. The Kupiec test statistic is two sided. The VaR model is rejected both when there are too few exceptions, as well as when there are too many exceptions.

Christoffersen developed a Markov test [19] to examine whether the likelihood of a VaR violation depends on the occurrence of a VaR violation on the previous time period. If the VaR measure accurately reflects the underlying portfolio risk then violation of VaR should be independent of adjacent violations. The idea behind this test is that clustered violations represent a signal of model misspecification.

The Christoffersen's Markov test is carried out by recording violations of the VaR on adjacent days, such that if I_t is a first-order Markov process the one-step ahead transition probabilities $\Pr(I_{t+1}|I_t)$ are given by

$$\Pr(I_{t+1}|I_t) = \begin{pmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{pmatrix} \tag{33}$$

where π_{ij} is the transition $\Pr(I_{t+1}=j|I_t=i)$. Under the null hypothesis, the violations have a constant conditional mean which implies that $\pi_{01}=\pi_{11}=c$.

IV. Application to 5-minute HSBA data

We apply the directional change analysis and the proposed directional change IVaR estimation approach to HSBA prices, i.e. HSBC Holdings PLC returns observed on 5-minute intervals for the period between 05/07/17 8:01 and 25/07/17 15:20. These data, at a different month and on a tick-by-tick basis are also analyzed in [8]. The dataset contains 7185 data points.

We use an FGARCH(2,1,1) model as defined by (1) and (2). The purpose in this analysis is to forecast the distribution of returns and the IVaR values particularly during the overshoot periods. For this purpose, we alter the FGARCH model in section II such that the outcome variable are TOS period ahead returns:

$$R_l : \text{If } \mathbf{x} \text{ is } F_l \text{ then } y^l_{t+\mathrm{TOS}} \mid x_t, h^l_t \sim N(\mu^l, h^l_t)\,, \tag{34}$$

$$\text{with } h^l_t = \alpha^l_0 + \sum_{i=1}^{q}\alpha^l_i y^2_{t-i} + \sum_{j=1}^{p}\beta^l_j h_{t-j}\,, \tag{35}$$

where TOS is the nearest integer predicted time between a confirmed directional change and an overshoot event obtained from the directional change summary of the data. In other words, we first apply the directional change methodology in section III to the data, obtain DC summaries, including the average time for an overshoot event after a confirmed directional change. The predicted TOS in turn depends on the threshold variable θ used to obtain the DC summaries. The FGARCH(2,1,1) models are estimated using maximum likelihood, using (11) as detailed in Section II.

TABLE I
Kupiec test. Bold face indicates rejection. Non-rejection regions 1659<I(0.01)<1676, 1582<I(0.05)<1618, 1491<I(0.10)<1541, 144<I(0.90)<170, 67<I(0.95)<103 and 9<I(0.99)<26

	$I(0.01)$	$I(0.05)$	$I(0.10)$	$I(0.90)$	$I(0.95)$	$I(0.99)$
FGARCH(2,1,1)	1665	1598	1544	146	88	25

TABLE II
Christoffersen Markov test

	π_{ij}	0.01	0.05	0.10	0.90	0.95	0.99
FGARCH(2,1,1)	π_{01}	0.000	0.030	0.060	0.920	0.950	0.980
	π_{11}	0.012	0.053	0.085	0.884	0.932	1.000

A. FGARCH for IVaR estimation

In this paper we propose the use of the FGARCH(l,q,p) model for intra day Value-at-Risk. To validate this use, we performed a failure test of unconditional coverage test using the Kupiec test in (32) and independence test using Christoffersen's Markov test in (33) for both tails of the estimated conditional density, since both are relevant in investment strategies. In this section we divide the original dataset. 5500 data points were used to estimate the model, while the remaining 1685 were used for out-of-sample validation. Table I and Table II show the results of the unconditional coverage and independence test, respectively, for the FGARCH(2,1,1) model for the out-of-sample data. We consider that an investor using the DC approach can change his position at any point. The risk model under consideration provides the maximum risk that an investor is exposed to, at the next period, since this is the first available period to change his position. This is also the IVaR included in the DC summarization presented in Section IV-B.

Table I shows that the validity of the model is never rejected for out-of-sample. Regarding the independence test presented in Table II, the one-step ahead transition probabilities π_{01} and

TABLE III
SUMMARY OF DIRECTIONAL CHANGES FOR 5 MINUTE HSBA DATA AND IVAR(α) ESTIMATES FOR DC AND OS EVENTS WITH $\alpha = 0.05$.

θ	TDC	TOS	TOS/TDC	#DC	#UDC	#DDC	IVaR$_{DC}(\alpha)$	IVaR$_{OS}(\alpha)$
0.1	5.15	19.69	3.82	324	200	124	-0.139	-0.139
0.05	3.62	11.38	3.15	872	513	359	-0.044	-0.044
0.01	1.31	3.46	2.65	2946	1828	1118	-0.044	-0.044

TABLE IV
PERCENTAGE OF IVAR(α) VIOLATIONS FOR DIFFERENT α AND θ FOR 5-MINUTE HSBA RETURNS FOR PREDICTED OS PERIODS.

α	0.01	0.05	0.10	0.90	0.95	0.99
$\theta = 0.1$	0.001	0.006	0.016	0.984	0.994	0.997
$\theta = 0.05$	0.059	0.116	0.211	0.785	0.885	0.945
$\theta = 0.01$	0.078	0.128	0.206	0.796	0.889	0.935

π_{11} are not too distant from the theoretical values for all values of c. These results indicate that the FGARCH(2,1,1) model can be successfully applied for IVaR estimation, for the HSBA data under study.

B. DC properties for HSBA data

We calculate and present directional changes using three different threshold values, $\theta = \{0.1, 0.5, 0.01\}$, leading to three different directional change intrinsic event series. Our aim is to illustrate the calculation of IVaR values for any threshold value of a directional change. One can, in addition, optimize parameter θ given a loss function as in [8], [10] and calculate IVaR levels given this optimal parameter value.

Table III presents summary statistics of the directional changes for different threshold percentages, θ including the average length of a DC event (TDC), average length of an OS event (TOS), average ratio of OS event length over average ratio of DC event length (TOS/TDC), number of DC events (#DC), number of upward DC events (#UDC) and number of downward DC events (#DDC) as defined by equations (23)–(28) in Section III-A. As expected, the number of total, upward and downward directional changes (#DC, #UDC and #DDC) are inversely related to the threshold percentage θ, i.e. for smaller threshold values we find a larger number of directional changes. Similarly, average timing of DC events (TDC, TOS, TOS/TDC) decrease with relatively small threshold values. The number of upward directional changes (#UDC) are higher than the number of downward directional changes (#DDC), indicating a slight asymmetry in directional changes for all threshold values we consider.

One of the most important values for financial forecasting of DC events is the ratio of an overshoot event to a directional change (TOS/TDC) reported in Table III since this number can be used to predict the next directional change in the prices [10], hence lead to profits if investments are performed accordingly. For exchange rate data, the average ratio of TOS/TDC is typically around the value of 2 [7]. The values we find for 5-minute HSBA data are slightly above this ratio, particularly for the large threshold value of $\theta = 0.1$.

C. IVaR calculations for predicted overshoot events

In this subsection we present the IVaR values for predicted overshoot events. These IVaR values are particularly interesting for an investor who wants to predict the changes in the price direction and invest according to the directional changes, such as points A and B in Fig. 1.

For each threshold value θ, we calculate the predicted OS points according to TOS/TDC given in Table III, also as defined in Section III-B. We then use a FGARCH(2,1,1) model to directly predict the return distribution, hence the IVaR levels at the predicted OS points. The estimated IVaR values at the 5% level, together with the return series are given in Fig. 2 for different thresholds θ. As expected, IVaR levels are lower in periods of extreme returns, such as those around observation 2000. However, IVaR values show differences between the three figures in Fig. 2. We conclude that the selection of the threshold parameter in directional change calculation affects the IVaR values substantially. In addition, we note that the exact IVaR values are sensitive to the number of rules in the FGARCH model. For $L = 3$, the obtained IVaR values are slightly different than those reported in Fig. 2, although the general conclusions in the remaining sections remain the same. We do not report results for the FGARCH(3,1,1) model due to space limitations.

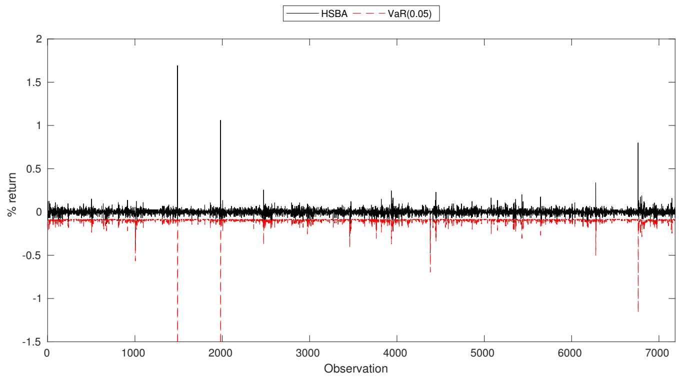
In addition, we consider the effect of θ on the accuracy of IVaR predictions. Table IV reports the percentage of IVaR violations in the dataset at different levels of α, and different values of θ. In a perfectly accurate risk prediction, α and the reported percentage of observations would be the same. For $\theta = 0.1$, percentage of IVaR violations, i.e. percentage of data points below the predicted IVaR values are much lower than α for $\alpha = \{0.01, 0.05\}$. Notice that $\theta = 0.1$ corresponds to a high threshold for directional changes where a relatively small number of DC and OS events occur, but the magnitude of these events will typically be high. This implies that the risk estimates are relatively conservative, especially for low levels of α when the considered DS and OS events are more extreme. On the other hand, for lower threshold values, $\theta = \{0.05, 0.01\}$, actual IVaR violations are higher than α for $\alpha = \{0.01, 0.05, 0.10\}$. In this case, the IVaR estimates seem to relatively underestimate the risk compared to the high threshold values. Hence the IVaR values, and their accuracy, depend on the selected directional change threshold. This has important implications for the optimization of θ in the literature. Risk estimation can be improved by the incorporating risk accuracy in the loss function of the existing optimization methods for θ [8], [10].

D. Return and IVaR values for DC based investments

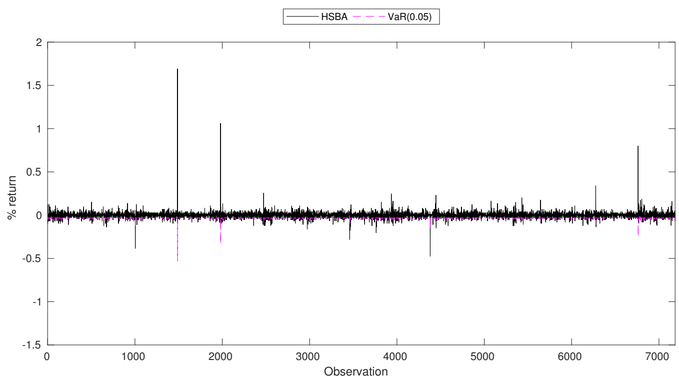
In this section we apply FGARCH method to calculate IVaR values for two investment decisions which can be performed based on directional changes. These investment decisions correspond to the 'strategies' DC1 and DC2 and the baseline strategy MW, defined in Section III-B.

The top panel in Table V presents the summary of the returns and 5% IVaR levels for these strategies during the full

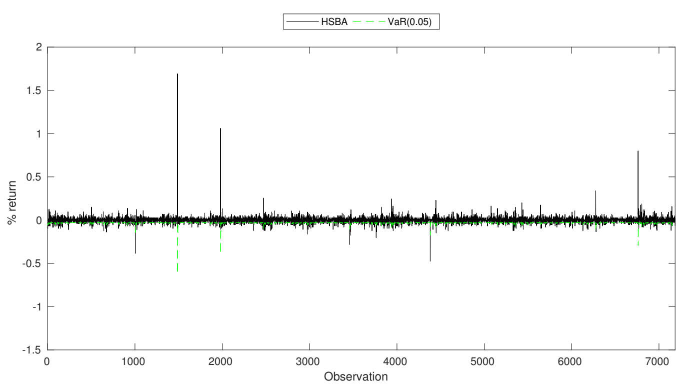
Fig. 2. Returns and IVaR(0.05) estimates from the FGARCH(2,1,1) model for HSBA OS events, directional changes computed with different threshold values.



(a) DC calculated with $\theta = 0.1$



(b) DC calculated with $\theta = 0.05$



(c) DC calculated with $\theta = 0.01$

sample (estimation and validation sample), for DC calculations with three threshold levels $\theta = \{0.1, 0.05, 0.01\}$. Both DC1 and DC2 have higher mean returns than the MW returns for all threshold values we consider, hence using the directional changes for buying or selling this stock is beneficial for mean returns. The over-performance of DC strategies is also clear when we consider out-of-sample mean returns, reported in the third panel of Table V. The over-performance of DC1 and DC2 can further be improved by optimizing the threshold parameter θ as in [8], [10].

The second panel in Table V presents the 5% IVaR estimates

TABLE V
RETURN AND RISK PROPERTIES FOR DIFFERENT INVESTMENT STRATEGIES, FOR 5-MINUTE HSBA RETURNS, ALL TIME PERIODS.

	$\theta = 0.1$		$\theta = 0.05$		$\theta = 0.01$		
	DC1	DC2	DC1	DC2	DC1	DC2	MW
Full Sample Returns							
min	-0.7998	-0.4751	-0.4751	-1.0618	-1.0618	-0.7998	-1.6922
mean	0.0010	0.0014	0.0034	0.0030	0.0095	0.0100	-0.0011
max	1.6922	1.6922	1.6922	1.6922	1.6922	1.6922	0.7998
std.dev.	0.0422	0.0421	0.0420	0.0421	0.0411	0.0410	0.0422
Full Sample IVaR(0.05)							
min	-2.4598	-2.4598	-0.5302	-0.5302	-0.6208	-0.6208	-0.6222
mean	-0.0979	-0.0979	-0.0312	-0.0312	-0.0313	-0.0314	-0.0315
max	-0.0822	-0.0822	-0.0291	-0.0291	-0.0264	-0.0264	-0.0264
std.dev.	0.0453	0.0453	0.0083	0.0083	0.0126	0.0126	0.0127
Out of Sample Returns							
min	-0.7998	-0.1356	-0.1746	-0.7998	-0.1063	-0.7998	-0.3392
mean	0.0008	0.0020	0.0034	0.0027	0.0105	0.0086	-0.0005
max	0.3392	0.7998	0.7998	0.3392	0.7998	0.3392	0.7998
std.dev.	0.0386	0.0385	0.0384	0.0385	0.0371	0.0376	0.0386
Out of Sample IVaR(0.05)							
min	-1.1651	-1.1656	-0.2521	-0.2521	-0.2943	-0.2957	-0.2957
mean	-0.0976	-0.0976	-0.0312	-0.0312	-0.0312	-0.0313	-0.0312
max	-0.0822	-0.0822	-0.0291	-0.0291	-0.0264	-0.0264	-0.0264
std.dev.	0.0364	0.0364	0.0065	0.0065	0.0100	0.0100	0.0101

for the investment strategies for the full sample where we report the mean, minimum, maximum values as well as the standard deviation of time-varying IVaR estimates over the whole sample. Similarly, the bottom panel in Table V summarizes the 5% IVaR estimates for the validation sample. Mean, minimum and maximum IVaR estimates are particularly low for DC1 and DC2 strategies with $\theta = 0.1$ compared to the other strategies. This result is in line with our finding of relatively conservative risk estimates obtained with high directional change thresholds, given in section IV-C. For lower threshold values, $\theta = \{0.05, 0.01\}$, however, the reported IVaR summaries are similar between both strategies, DC1 and DC2, and the baseline strategy MW. DC1 and DC2 strategies for these threshold values lead to higher returns, but a similar IVaR implied risk compared to the MW values. Hence the return over-performance of DC1 and DC2 does not lead to increased risk, particularly for relatively low threshold values.

For the two strategies, DC1 and DC2, DC events and predicted OS events are particularly important points as the investment strategy is adjusted according to these points. More specifically, DC1 is based on the DC events while DC2 is based on the DC events and the predicted OS events. We therefore analyze the return and risk properties of the strategies particularly for the collection of DC and OS events, named as 'DC periods' and 'OS periods', over the sample period. Return and 5% IVaR summaries for these periods are shown in Table VI. Since the OS and DC periods depend on θ, we also report the risk and return properties of the MW estimates at these periods for each value of θ.

The top two panels in Table VI present the results for DC events. The bottom two panels in Table VI present the results for predicted OS events. Regardless of the threshold

TABLE VI
RETURN AND RISK PROPERTIES FOR DIFFERENT INVESTMENT STRATEGIES, FOR 5-MINUTE HSBA RETURNS, FOR DC AND OS PERIODS.

	$\theta = 0.1$			$\theta = 0.05$			$\theta = 0.01$		
	DC1	DC2	MW	DC1	DC2	MW	DC1	DC2	MW
Returns during DC periods									
min	0.000	0.000	-0.202	0.000	0.000	-0.230	0.000	0.000	-0.230
mean	0.027	0.027	-0.000	0.026	0.026	-0.001	0.023	0.023	-0.001
max	0.202	0.202	0.107	0.230	0.230	0.151	0.256	0.256	0.256
std.dev.	0.024	0.024	0.036	0.024	0.024	0.036	0.023	0.023	0.033
IVaR(0.05) during DC periods									
min	-0.727	-0.727	-0.727	-0.252	-0.252	-0.252	-0.391	-0.391	-0.391
mean	-0.112	-0.112	-0.112	-0.032	-0.032	-0.032	-0.032	-0.032	-0.032
max	-0.083	-0.083	-0.083	-0.029	-0.029	-0.029	-0.027	-0.027	-0.027
std.dev.	0.053	0.053	0.053	0.009	0.009	0.009	0.011	0.011	0.011
Returns during predicted OS periods									
min	-0.110	-0.282	-0.108	-0.282	-0.282	-0.149	-1.062	-0.161	-1.062
mean	0.008	0.007	0.007	0.005	0.006	0.001	0.011	0.012	-0.000
max	0.282	0.161	0.282	0.151	0.151	0.282	0.246	1.062	0.282
std.dev.	0.039	0.039	0.039	0.036	0.036	0.036	0.036	0.036	0.038
IVaR(0.05) during predicted OS periods									
min	-1.162	-1.162	-1.161	-0.334	-0.333	-0.333	-0.406	-0.406	-0.407
mean	-0.117	-0.117	-0.117	-0.032	-0.032	-0.032	-0.032	-0.032	-0.032
max	-0.083	-0.083	-0.083	-0.029	-0.029	-0.029	-0.027	-0.027	-0.027
std.dev.	0.076	0.076	0.076	0.011	0.011	0.011	0.013	0.013	0.013

choice, both directional change strategies yield to equal or higher mean returns and equal or lower variation (std. dev.) in returns compared to the MW strategy in DC periods as well as OS periods. In addition, IVaR levels are similar across DC1, DC2 and MW strategies given a value of θ in both DC and OS periods. Hence the over-performance of DC1 and DC2 over MW holds during the DC and OS events. In addition, comparing the DC and OS return values with those in Table V, we conclude that both DC and OS periods are also characterized by higher returns compared to the overall sample. DC and OS events are thus important to predict and use in investment decisions to improve returns.

V. CONCLUSIONS

This paper proposes a FGARCH model for intraday Value-at-Risk (IVaR) estimation and for assessing risk properties in time series represented as directional change events. We apply the proposed methodology to 5-minute HSBA data where we show that the FGARCH model enables the prediction of intra-day market risk for returns represented by their directional changes. For these data, we present the IVaR estimates obtained from the proposed method, accuracy of the IVaR estimates in capturing time-varying IVaR, and the sensitivity of the results to the choice of the threshold parameter in directional changes. In addition, we show that the proposed methodology can be used to predict market risk for investment strategies based on directional changes. We evaluate the risk properties of two directional change based trading strategies and a baseline moving window strategy for the HSBA data. Our results show that the directional change based strategies over-perform the baseline strategy in mean returns without deterioration in the risk properties measured by IVaR levels, regardless of the threshold choice for directional changes. The proposed methodology is applicable to other intra-day data frequencies and it is generalizable to other risk metrics. In future work, we will continue to study different metrics for risk estimation of DC approach, to provide both profit and risk summaries. Furthermore, we plan to improve the accuracy of the risk estimates by optimizing the threshold parameter for directional changes.

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