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Assignment 5

Problem 1. Consider the following differential equation:

$$\frac{dx}{dt} = x(r + 2x + x^2) \tag{1}$$

Find the critical points, classify them as a function of the bifurcation parameter r, and draw the bifurcation diagram.

Solution. We solve when $(1) = 0 \implies$

$$x(x^2 + 2x + \frac{2}{2} - \frac{2}{2} + r) = 0 \implies$$
$$x((x + 2x + 1) - 1 + r) = 0 \implies$$
$$x((x+1)^2 - 1 + r) = 0$$

From here, we have $x = 0,\ x = -1 + \sqrt{1-r},$ and $x = -1 - \sqrt{1-r}$.
Case (i), if $r = 1$:

$$\frac{dx}{dt} = x(x+1)(x+1) \implies$$
$$x = 0,\ x = -1 \text{ are critical points } \implies$$
$$x = -1 \text{ is an unstable (saddle) and } x = 0 \text{ unstable}$$
(slope direction drawn on a separate sheet)

Case (ii), if $r > 1$:

$$\frac{dx}{dt} = x((x+1)^2 - 1 + r) \implies$$
$$x = 0,\ x = -1 + \gamma i,\ x = -1 - \gamma i,\ (\text{for some } \gamma \in \mathbb{R}) \text{ are critical points } \implies$$
$$x = 0 \text{ is our only valid equilibrium; } x = 0 \text{ is unstable}$$
(slope direction drawn on a separate sheet)

Case (iii), if $r < 1$:

$$\frac{dx}{dt} = x((x+1)^2 - 1 + r) \implies$$
$$x = 0,\ x = -1 + \sqrt{1-r},\ x = -1 - \sqrt{1+r} \text{ are critical points } \implies$$
$$x = -1 - \sqrt{1-r} \text{ is unstable, } x = -1 + \sqrt{1-r} \text{ is stable, } x = 0 \text{ is unstable}$$
(slope direction drawn on a separate sheet)

Bifurcation diagram drawn on a separate sheet. □

Problem 2. Consider the system of differential equations with $\mu < 1$:

$$\frac{dx}{dy} = y \tag{2}$$

$$\frac{dy}{dt} = -x + 2\mu y - x^2 y \tag{3}$$

(a) Find the eigenvalues of the linearized system at $(0, 0)$.

(b) Show that the system has a Hopf bifurcation at $\mu = 0$.

Solution. (a) We find $\mathbf{J}(0, 0)$:

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 - 2xy & 2\mu - x^2 \end{bmatrix} \implies \mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 2\mu \end{bmatrix}$$

Find eigenvalues:

$$\begin{aligned} 0 &= |\mathbf{J}(0,0) - \lambda I| \\ &= \begin{vmatrix} -\lambda & 1 \\ -1 & 2\mu - \lambda \end{vmatrix} \\ \implies \lambda_{1,2} &= \frac{2\mu \pm \sqrt{4\mu^2 - 4}}{2} \\ \implies \lambda_{1,2} &= \mu \pm \sqrt{\mu^2 - 1} \\ \implies \lambda_{1,2} &= \mu \pm i\sqrt{1 - \mu^2} \end{aligned}$$

Then $\lambda_{1,2} = \alpha(\mu) \pm i\beta(\mu) = \mu \pm i\sqrt{1 - \mu^2}$.

□

Solution. (b) From (a), we have that $\alpha(\mu) < 0$ or $\alpha(\mu) > 0$ as $\alpha(\mu)$ intersects $\mu_0 = 0$. Moreover, $\beta(\mu) = \sqrt{1 - \mu^2} \neq 0$ for any neighbourhood around $\mu_0 = 0$. By these conditions, the system has a Hopf bifurcation near $\mu = 0$, which guarantees existence of periodic orbits.

□

Problem 3. Use the given Liapunov function to determine the stability of the origin for each system below.

(a) $V(x, y) = x^2 + y^2$ with

$$\frac{dx}{dt} = -y - x^3 - xy^2 \tag{4}$$

$$\frac{dy}{dt} = x - y^3 - yx^2 \tag{5}$$

(b) $V(x, y) = x^2 + ay^2$, pick a as needed

$$\frac{dx}{dt} = -x + 4y \tag{6}$$
$$\frac{dy}{dt} = -x - y^3 \tag{7}$$

Solution. (a) Notice, V is C^1, greater than 0 on $\mathbb{R}^2\backslash\{(0,0)\}$, and equal to 0 at $(0,0)$. Thus, V is positive definite. Then

$$\begin{aligned}\frac{dV}{dt} &= \nabla V \cdot < \frac{dx}{dt}, \frac{dy}{dt} > \\ &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt} \\ &= 2x(-y - x^3 - xy^2) + 2y(x - y^3 - yx^2) \\ &= -2x^4 - 4x^2y^2 - 2y^4 \\ &= -2(x^2 + y^2)^2\end{aligned}$$

Then $\frac{dV}{dt} < 0$, since $(x^2 + y^2)^2 > 0$, on $\mathbb{R}^2\backslash\{(0,0)\}$. Thus, (0,0) is globally asymptotically stable by Liapunov's stability theorem. □

Solution. (b) Pick $a = 4$. Notice, V is C^1, greater than 0 on $\mathbb{R}^2\backslash\{(0,0)\}$, and equal to 0 at $(0,0)$. Thus, V is positive definite. Then

$$\begin{aligned}\frac{dV}{dt} &= \nabla V \cdot < \frac{dx}{dt}, \frac{dy}{dt} > \\ &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt} \\ &= 2x(-x + 4y) + 8y(-x - y^3) \\ &= -2(x^2 + 8y^4)\end{aligned}$$

Then $\frac{dV}{dt} < 0$, since $(x^2 + 8y^4) > 0$, on $\mathbb{R}^2\backslash\{(0,0)\}$. Thus, (0,0) is globally asymptotically stable by Liapunov's stability theorem. □