Paul Ycay - 500709618
Instructor: Kathleen Wilkie

MTH 630
Mathematical Biology

Ryerson University
November 12, 2019

Assignment 4

Problem 1. Consider the system of DEs where $x \geq 0$ and $y \geq 0$:

$$\frac{dx}{dt} = x(9 - 2x - y) \tag{1}$$

$$\frac{dy}{dt} = y(6 - x - 2y) \tag{2}$$

(a) Find and classify all equilibrium points.
(b) Find the nullclines.
(c) Use your work to sketch the phase diagram in the positive quadrant.

Solution. (a) Solve for equilibrium points. $(0, 0)$ is the trivial equilibrium point. Now from (1), if $0 = 9 - 2x - y$, then $y = 9 - 2x$. Then substituting in equation (2) becomes

$$\begin{aligned} &(9 - 2x)(6 - x - 2(9 - 2x)) \implies \\ &(9 - 2x)(-12 + 3x) \implies \\ &-6x^2 + 51x - 108 \end{aligned}$$

Thus, solving the above quadratic gives roots $x = 4,\ x = 9/2$. Plugging $x = 4$ into $y = 9 - 2x$ gives equilibrium point $(4, 1)$. Similarly, $x = 9/2$ yields $y = 0$. If we solve $(6 - x - 2y) = 0$ in equation (2), we get $6 - 2y = x$. Subbing into (1), we get roots $y = 1,\ y = 3$. If $y = 3$, $6 - 2(3) = 0$. Thus, our equilibrium points are $(0, 0),\ (4, 1),\ (9/2, 0),\ (0, 3)$. Now we find the Jacobian matrices at each equilibrium. We use $tr(\mathbf{J}) = \tau$, the determinant δ, and $\gamma = \tau^2 - 4\delta$ to classify the equilibrium.

$$\mathbf{J} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -4x - y + 9 & -x \\ -y & 6 - x - 4y \end{bmatrix}$$

$$\mathbf{J}(0,0) = \begin{bmatrix} 9 & 0 \\ 0 & 6 \end{bmatrix} \implies \tau = 15 > 0,\ \delta = 54 > 0,\ \gamma = 9 > 0$$

$$\mathbf{J}(4,1) = \begin{bmatrix} -8 & -4 \\ -1 & -2 \end{bmatrix} \implies \tau = -10 < 0,\ \delta = 12 > 0,\ \gamma = 52 > 0$$

$$\mathbf{J}(\frac{9}{2},0) = \begin{bmatrix} -9 & \frac{-9}{2} \\ 0 & \frac{3}{2} \end{bmatrix} \implies \tau = -15/2 < 0,\ \delta = -27/2 < 0,\ \gamma = \frac{441}{4} > 0$$

$$\mathbf{J}(0,3) = \begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix} \implies \tau = 0,\ \delta = -33 < 0,\ \gamma = 168 > 0$$

Thus, (0,0) is an *unstable node*, (4,1) is a *stable node*, ($\frac{9}{2}$,0) is a *saddle point*, and (0,3) is a *saddle point*. Now we find the *eigenvectors* at each $\mathbf{J}(\tilde{x}, \tilde{y})$ to sketch local behaviour.

For $(\mathbf{J}(0,0) - \lambda I)\mathbf{v} = 0,\ \lambda = 9,\ \lambda = 6:$

$$\mathbf{V}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies x = 0,\ y\ is\ arbitrary.$$

$$\mathbf{V}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies y = 0,\ x\ is\ arbitrary$$

For $(\mathbf{J}(4,1) - \lambda I)\mathbf{v} = 0,\ \lambda = -5 + \sqrt{13},\ \lambda = -5 - \sqrt{13}:$

$$\mathbf{V}_1 = \begin{bmatrix} 3 - \sqrt{13} \\ 1 \end{bmatrix} \implies x = -(-3 + \sqrt{13})y,\ y\ is\ arbitrary.$$

$$\mathbf{V}_2 = \begin{bmatrix} 3 + \sqrt{13} \\ 1 \end{bmatrix} \implies x = -(-3 - \sqrt{13})y,\ y\ is\ arbitrary.$$

For $(\mathbf{J}(\frac{9}{2},0) - \lambda I)\mathbf{v} = 0,\ \lambda = \frac{3}{2},\ \lambda = -9:$

$$\mathbf{V}_1 = \begin{bmatrix} -3 \\ 7 \end{bmatrix} \implies x = -\frac{3}{7}y,\ y\ is\ arbitrary.$$

$$\mathbf{V}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies y = 0,\ x\ is\ arbitrary.$$

For $(\mathbf{J}(0,3) - \lambda I)\mathbf{v} = 0,\ \lambda = 6,\ \lambda = -6:$

$$\mathbf{V}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \implies x = -4y,\ y \textit{ is arbitrary.}$$

$$\mathbf{V}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies x = 0,\ y \textit{ is arbitrary}$$

□

Solution. (b) To solve nullclines, we determine the gradient behaviour of the systems when equated at 0.

Equation (1) $= 0 \implies x = 0,\ y = 9 - 2x.$

If $x = 0$:

$$\frac{dy}{dt} = y(6 - 2y)$$

Tangent lines point *up* when $0 < y < 3$. They point *down* when $y < 0$ or $y > 3$.

If $y = 9 - 2x$:

$$\frac{dy}{dt} = -3(x-4)(2x-9)$$

Tangent lines point *down* when $x < 4$ or $x > \frac{9}{2}$. They point *up* when $4 < x < \frac{9}{2}$.

Equation (2) $= 0 \implies y = 0$ or $y = -\frac{1}{2}x + 3$.

If $y = 0$:

$$\frac{dx}{dt} = -2x(x - \frac{9}{2})$$

Tangent lines point *left* when $x < 0$ or $x > \frac{9}{2}$. They point *right* when $0 < x < \frac{9}{2}$.

If $y = -\frac{1}{2}x + 3$:

$$\frac{dx}{dt} = \frac{-3x^2}{2} + 6x$$

Tangent lines point *left* when $x < 0$ or $x > 4$. They point *right* when $0 < x < 4$. □

Solution. (c) *Sketch is provided on a separate sheet.*

□

Problem 2. Consider the system

$$\frac{dx}{dt} = y + x(x^2 + y^2 - 9) \tag{3}$$

$$\frac{dy}{dt} = -x + y(x^2 + y^2 - 9) \tag{4}$$

(a) Determine the local asymptotic stability of the origin.

(b) Switch to polar coordinates and rewrite the system in terms of r and θ.

(c) Solve the system for $r(t)$ and $\theta(t)$.

(d) Graph the phase line diagram for r and describe the solution behaviour when $0 < r < 3$ and $r > 3$.

(e) Use your above work to sketch a phase plane diagram for the original nonlinear system in the xy-plane.

Solution. (a) We determine the local asymptotic stability of the origin; that is, we are only interested at $\mathbf{J}(0,0)$.

$$\mathbf{J} = \begin{bmatrix} y^2 + 3x^2 - 9 & 1 + 2xy \\ -1 + 2yx & x^2 + 3y^2 - 9 \end{bmatrix}$$

$$\implies \mathbf{J}(0,0) = \begin{bmatrix} -9 & 1 \\ -1 & -9 \end{bmatrix} \implies \tau = -18 < 0,\ \delta = 82 > 0,\ \gamma = \tau^2 - 4\delta = -4 < 0$$

Then at the origin, since $\gamma < 0,\ \tau < 0$, we get a *stable spiral*. □

Solution. (b)
Let $x = rcos\theta,\ y = rsin\theta$. Then

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial \theta}\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = cos\theta\frac{dr}{dt} - rsin\theta\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = sin\theta\frac{dr}{dt} + rcos\theta\frac{d\theta}{dt}$$

$$(3) \implies cos\theta\frac{dr}{dt} - rsin\theta\frac{d\theta}{dt} = rsin\theta + r^3cos\theta - 9rcos\theta \tag{3*}$$
$$(4) \implies sin\theta\frac{dr}{dt} + rcos\theta\frac{d\theta}{dt} = -rcos\theta + r^3sin\theta - 9rsin\theta \tag{4*}$$

To determine $\frac{dr}{dt}$, we try $cos\theta \cdot (3^*) + sin\theta \cdot (4^*) \implies$

$$\begin{aligned}(cos^2\theta + sin^2\theta)\frac{dr}{dt} &= rsin\theta cos\theta + r^3cos^2\theta - 9rcos^2\theta \\ &- rsin\theta cos\theta + r^3sin^2\theta - 9rsin^2\theta \\ \frac{dr}{dt} &= r^3 - 9r \\ \frac{dr}{dt} &= r(r-3)(r+3) \qquad \text{(A)}\end{aligned}$$

To determine critical points, we ignore when $r < 0$. So $r = 0,\ 3$.
To determine $\frac{d\theta}{dt}$, we try $-sin\theta \cdot (3^*) + cos\theta \cdot (4^*) \implies$

$$\begin{aligned}(rsin^2\theta + rcos^2\theta)\frac{d\theta}{dt} &= -rsin^2\theta - r^3cos\theta sin\theta + 9rcos\theta sin\theta \\ &- rcos^2\theta + r^3sin\theta cos\theta - 9rsin\theta cos\theta \\ r\frac{d\theta}{dt} &= -r \\ \frac{d\theta}{dt} &= -1 \qquad \text{(B)}\end{aligned}$$

So $\theta \neq 0,\ \theta < 0$, thus θ turns *clockwise*.

□

Solution. (c) By integration, we solve for r and θ, respectively.

$$\int \frac{dr}{r(r-3)(r+3)} = \int dt$$

By partial fractions, the left hand side becomes

$$\begin{aligned}
\frac{1}{r(r-3)(r+3)} &= \frac{a}{r} + \frac{b}{r-3} + \frac{c}{r+3} \\
1 &= a(r+3)(r-3) + br(r+3) + cr(r-3) \\
\implies 1 &= \begin{cases} -9a & \text{if } r = 0 \\ 18b & \text{if } r = 3 \\ 18c & \text{if } r = -3 \end{cases} \\
\implies \frac{1}{r(r-3)(r+3)} &= \frac{-1}{9r} + \frac{1}{18(r-3)} + \frac{1}{18(r+3)}
\end{aligned}$$

Then

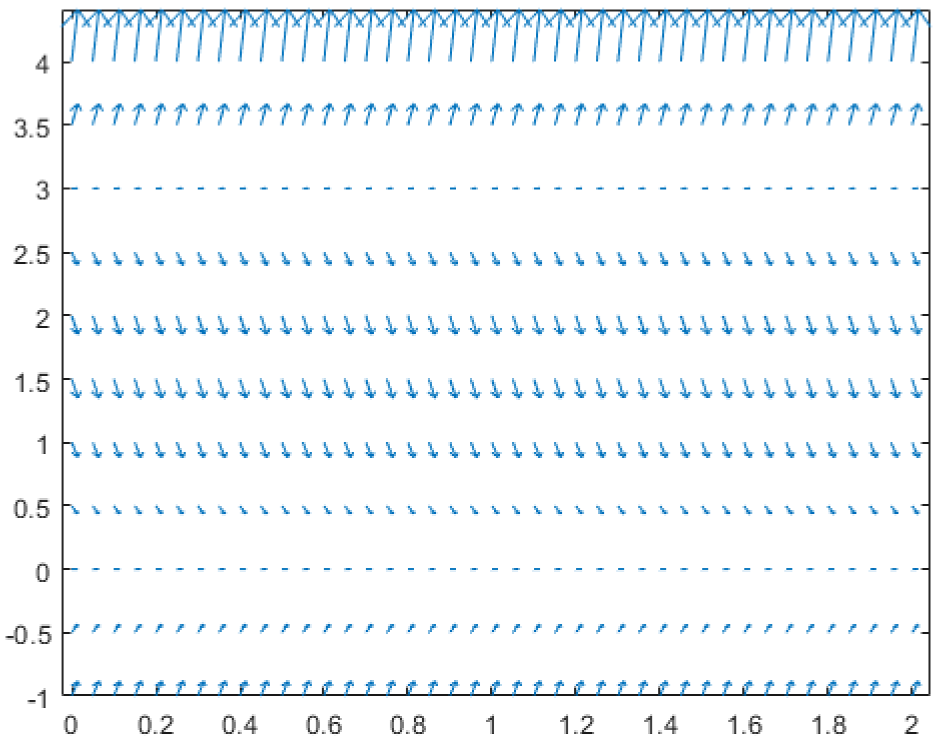
$$\begin{aligned}
\int \frac{dr}{r(r-3)(r+3)} &= \int \frac{-1}{9r} + \int \frac{1}{18(r-3)} + \int \frac{1}{18(r+3)} \\
&= \frac{-1}{9} \ln|r| + \frac{1}{18} \ln|r-3| + \frac{1}{18} \ln|r+3| \\
&= \frac{-1}{9} \ln|r| + \frac{1}{18} \ln|r^2-9|
\end{aligned}$$

$$\begin{aligned}
\implies \frac{r}{\sqrt{r^2-9}} &= e^{-9c}e^{-9t} \\
\frac{r^2}{r^2-9} &= e^{-18c}e^{-18t} \\
&= Ae^{-18t} \\
r^2 &= r^2Ae^{-18t} - 9Ae^{-18t} \\
&= \frac{9Ae^{-18t}}{Ae^{-18t}-1} \\
r^2 &= \frac{9}{1-Be^{18t}} \qquad (\text{where } B = \frac{1}{A}) \\
\implies r(t) &= \frac{3}{\sqrt{1-Be^{18t}}}, \ t \in \mathbb{R}
\end{aligned}$$

Moreover, $\theta(t) = C - t$, by integrating (B). $\square$

Solution. (d)

```
t=0:.05:2; r=-1:.5:4; % define grid of values in t and y direction
[T,R]=meshgrid(t,r); % creates 2d matrices of points in the tr-plane
dT = ones(size(T)); % dt=1 for all points
dR = R.*(R-3).*(R+3); %this is the ODE
quiver(T,R,dT,dR) % draw arrows
axis tight
hold on
```



Notice that if $r > 3$, solution curves point *upward* $\Longrightarrow$ slope is *positive*. When $0 < r < 3$, solution curves point *downward* $\Longrightarrow$ slope is *negative*; $r = 3$ is *unstable*. □

Solution. (e) *Sketch is provided on a separate sheet.* □

Problem 3. Use Bendixon's or Dulac's criteria to show that periodic solutions are not possible for the following systems:
(a)

$$\frac{dx}{dt} = x + y + x^3 - y^2 \tag{5}$$
$$\frac{dy}{dt} = -x + 2y + x^2 y + \frac{y^3}{3} \tag{6}$$

(b) with α, β, and σ are positive constants:

$$\frac{dx}{dt} = x(\alpha - x - \beta y) \tag{7}$$
$$\frac{dy}{dt} = y(\gamma - \alpha x - y) \tag{8}$$

Solution. (a) We try Bendixon's criteria:

> **(Bendixon's Criterion).** Suppose D is a simply connected open subset of $\mathbf{R}^2$. If the expression $div(f,g) \equiv \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically 0 and does not change sign in D, then there are no periodic oribts of the autonomous system in D.

Thus, $f_x + g_y = (3x^2+1) + (x^2+y^2+2) = 4x^2 + y^2 + 3$. Now notice $f_x + g_y > 0$ since $4x^2,\ y^2 > 0$ for $x, y \in \mathbb{R}$. Thus, no periodic solutions exist. □

Solution. (b) We try Dulac's criteria:

> **(Dulac's Criterion).** Suppose D is a simply connected open subset of $\mathbf{R}^2$ and $B(x,y)$ is a real valued function in D. If the expression
>
> $$div(Bf, Bg) = \frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$$
>
> is not identically 0 and does not change sign in D, then there are no periodic solutions of the autonomous system in D.

Pick $B(x,y) = \frac{1}{xy}$. Then

$$\frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y} = \frac{\partial}{\partial x}(\frac{1}{y}(\alpha - x - \beta y)) + \frac{\partial}{\partial y}(\frac{1}{x}(\gamma - \alpha x - y)) = -\frac{1}{y} - \frac{1}{x}$$

Notice that $B(x,y)$ is continuously differentiable in the positive quadrant,
$D = \{(x,y) | x > 0, y > 0\}$. Then $\nabla\cdot < Bf, Bg >$ is always strictly less than 0. Thus, no periodic orbits in D. □