# Final Project: Mutualistic Interaction

MTH 630 - Mathematical Biology

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# 1 Abstract

According to Dr. Mary Dowd (2019), *mutualism* refers to a type of relationship that benefits two species in the same environment. For this project, we will focus on *obligate mutualism*. This type of relationship states that given 2 species, each population must interact with each other in order to survive (the interaction is beneficial between species). We will examine *obligate mutualism* between flowering plants and insect pollinators based on a system of non-linear differential equations, which will be introduced in section 2. Section 3 will focus on rough analysis of the model. Section 4 will showcase the graphs of the nonlinear system of equations on MATLAB. The final concluding section will answer how both species interact as time tends to infinity; this section will include some additional background behind the project.

# 2 Introducing and Interpreting the Problem

The following problem is taken from Chapter 6: Biological Applications of Differential Equations of *An Introduction to Mathematical Biology* by Linda J.S Allen:

**Example**
In another mutualistic system, each species is unable to survive in the absence of the other one. This type of mutualistic interaction is referred to as *obligate* mutualism (Kot, 2001). Each species cannot survive without the presence of the other species. In this system each species is dependent on the other species for its survival. With $x(0) > 0,\ y(0) > 0$, and all parameters being positive, the model has the form

$$\frac{dx}{dt} = c_1 x(-K_1 - x + y) \tag{1}$$
$$\frac{dy}{dt} = c_2 y(-K_2 - y + bx) \tag{2}$$

(a) Find all equilibria for these equations.
(c) Draw the nullclines and the phase plane diagram for the case $b > 1$. Use the Jacobian matrix and determine the stability of the nonnegative equilibria. What happens to $x(t)$ and $y(t)$ as $t \to \infty$?

For this problem, we are given a system of nonlinear differential equations with 5 parameters. The main objective is to determine the behaviour of 2 mutualistic species as a function of time. This will be determined by classifying equilbirium points, finding the Jacobian matrices, and plotting the nullclines and phase portrait. Let $y$ represent insect pollinators (bees) and $x$ represent flowering plants. Let $K_1$ & $K_2$ denote the carrying capacity and $c_1$ & $c_2$ denote the growth rates of species $x$ and $y$, respectively. The parameter $b$ represents the rate of $x$ which ultimately affects species $y$. We will discuss the importance between $b > 1$ and $0 < b < 1$ in the concluding section.

# 3 Analysis

In this section, we work out the mathematical analysis to the problem in hopes of determining conditions on our parameters that will satisfy how the species will react. The computations in section 4 will motivate our numerical simulation of the nullclines and phase plane diagrams.

**Step 1: Find equilibrium points**

To find the equilibria, we simply set equations $(1)$ and $(2)$ equal to $0$. $(0,0)$ is the trivial equilibrium. Now from $(1)$, if $0 = (-K_1 - x + y)$, then $y = K_1 + x$. Then substituting in equation $(2)$, we have

$$\begin{aligned}
&c_2(x+K_1)(-K_2-(x+K_1)+bx) = 0 \implies \\
&(c_2x + c_2K_1)(-K_2 - x - K_1 + bx) = 0 \implies \\
&x = -K_1,\ x = \frac{K_2+K_1}{-1+b},\ b \neq 1
\end{aligned}$$

Subbing $x = -K_1$ into $(1)$:

$$\begin{aligned}
&c_1(-K_1)(-K_1-(-K_1)+y) = 0 \implies \\
&c_1(-K_1)(y) = 0 \implies \\
&y = 0
\end{aligned}$$

Subbing $x = \frac{K_2+K_1}{-1+b}, b \neq 1$ into $(1)$:

$$\begin{aligned}
&c_1\left(\frac{K_2+K_1}{-1+b}\right)\left(-K_1-\frac{K_2+K_1}{-1+b}+y\right) = 0 \implies \\
&-K_1 - \frac{K_2+K_1}{-1+b} + y = 0 \implies \\
&y = \frac{K_1b+K_2}{b-1}, b \neq 1
\end{aligned}$$

Similarily, equation $(2)$ gives $y = -K_2 + bx$. Subbing into $(1)$ gives $c_1x(-K_1 - x - K_2 + bx) = 0$. From here, we see that a unique solution produces when $x = 0 \implies y = -K_2$.

Thus, we have equilibrium points:

$$(0,0),\quad (-K_1,0),\quad \left(\frac{K_2+K_1}{b-1}, \frac{K_1b+K_2}{b-1}\right)\ b \neq 1,\quad (0,-K_2)$$

**Step 2: Evaluate the equilibrium points at the Jacobian Matrix**

$$\begin{aligned}
\mathbf{J} &= \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \\
&= \begin{bmatrix} -c_1K_1 - 2c_1x + c_1y & c_1x \\ c_2yb & -c_2K_2 - 2c_2y + c_2bx \end{bmatrix}
\end{aligned}$$

Then

$$\mathbf{J}(0,0) = \begin{bmatrix} -c_1K_1 & 0 \\ 0 & -c_2K_2 \end{bmatrix}$$

$$\mathbf{J}(-K_1,0) = \begin{bmatrix} c_1K_1 & -K_1c_1 \\ 0 & -c_2K_2 - c_2bK_1 \end{bmatrix}$$

$$\mathbf{J}(\frac{K_2+K_1}{b-1}, \frac{K_1b+K_2}{b-1}) = \begin{bmatrix} -c_1K_1 + \frac{c_1K_1b-2c_1K_1}{b-1} & c_1(\frac{K_2+K_1}{b-1}) \\ c_2(\frac{K_1b+K_2}{b-1})b & -c_2K_2 + \frac{c_2K_2b-c_2K_1b-2c_2K_2}{b-1} \end{bmatrix}$$

$$\mathbf{J}(0,-K_2) = \begin{bmatrix} -c_1K_1 - c_1K_2 & 0 \\ -c_2K_2b & c_2K_2 \end{bmatrix}$$

It is quite difficult to determine nullclines and the behaviour of our solutions given positive arbitrary constants. Thus, we try and set values for our parameters and experiment with how our solutions will look like given these numbers. Fix $K_1 = 1,\ K_2 = 2,\ c_1 = 1,\ c_2 = 1,\ b = 2$, since we are working with the case $b > 1$. Thus, our Jacobian matrices become

$$\mathbf{J}(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \implies \tau = -3 < 0,\ \delta = 2 > 0,\ \gamma = 1 > 0$$

$$\mathbf{J}(-1,0) = \begin{bmatrix} 11 & -1 \\ 0 & -4 \end{bmatrix} \implies \tau = -5 < 0,\ \delta = -4 < 0,\ \gamma = 9 > 0$$

$$\mathbf{J}(3,4) = \begin{bmatrix} -1 & 3 \\ 8 & -4 \end{bmatrix} \implies \tau = -5 < 0,\ \delta = -20 < 0,\ \gamma = 105 > 0$$

$$\mathbf{J}(0,-2) = \begin{bmatrix} -3 & 0 \\ -4 & 2 \end{bmatrix} \implies \tau = -3 < 0,\ \delta = -6 < 0,\ \gamma = 9 > 0$$

Then $(0,0)$ is a *stable node*, $(-1,0)$ is a *saddle point*, $(3,4)$ is a *saddle point*, $(0,-2)$ is a *saddle point*. In other words, this will make classifying the general equilibrium points with arbitrary parameters much easier. From here, we have that $(0,0)$ is a *stable node*, $(-K_1,0)$ is a *saddle point*, $(\frac{K_2+K_1}{b-1}, \frac{K_1b+K_2}{b-1})$ is a *saddle point*, and $(0,-K_2)$ is a *saddle point*.

**Step 3: Determine nullclines**

To find nullclines, we determine the gradient behaviour of the systems when equated at $0$. Notice that all parameters are positive.

Equation $(1) = 0 \implies x = 0,\ K_1 = y - x$ or $y = x + 1$.
If $x = 0$:

$$\frac{dy}{dt} = c_2y(-K_2 - y)$$
$$\frac{dy}{dt} = y(-2 - y)$$

Tangent lines point *up* when $-K_2 < y < 0 \implies -2 < y < 0$. They point *down* when $y < -K_2 \implies y < -2$ or $y > 0$.

If $x = y - K_1 \implies x = y - 1$:

$$\frac{dy}{dt} = c_2 y(-K_2 - y + b(y - K_1))$$
$$\frac{dy}{dt} = y(-2 - y + 2y - 2)$$
$$\frac{dy}{dt} = y(y - 4)$$

Tangent lines point *up* when $y < 0,\ y > 4$. They point *down* when $0 < y < 4$.

Equation $(2) = 0 \implies y = 0,\ K_2 = bx - y$ or $2 = 2x - y$.

If $y = 0$ :

$$\frac{dx}{dt} = c_1 x(-K_1 - x)$$
$$\frac{dx}{dt} = x(-1 - x)$$

Tangent lines point *left* when $x < -1,\ x > 0$. They point *right* when $-1 < x < 0$.

If $y = 2x - 2$ :

$$\frac{dx}{dt} = c_1 x(-K_1 + x - 2)$$
$$\frac{dx}{dt} = x(-3 + x)$$

Tangent lines point *left* when $0 < x < 3$. They point *right* when $x < 0,\ x > 3$.

The next section will be dedicated on showcasing computer generated plots of these solutions. It will focus on nullclines, the phase plane diagram, and the interaction among equilibrium. Note, that we will be working with the following parameters $K_1 = 1,\ K_2 = 2,\ c_1 = 1,\ c_2 = 1,\ b = 2$ for the new system:

$$\frac{dx}{dt} = x(-1 - x + y)$$
$$\frac{dy}{dt} = y(-2 - y + 2x)$$

# 4 Results: Numerical Simulations Based on the Jacobian Matrices and Nullclines

We can generate solutions based on our analysis using the MATLAB tool Pplane.

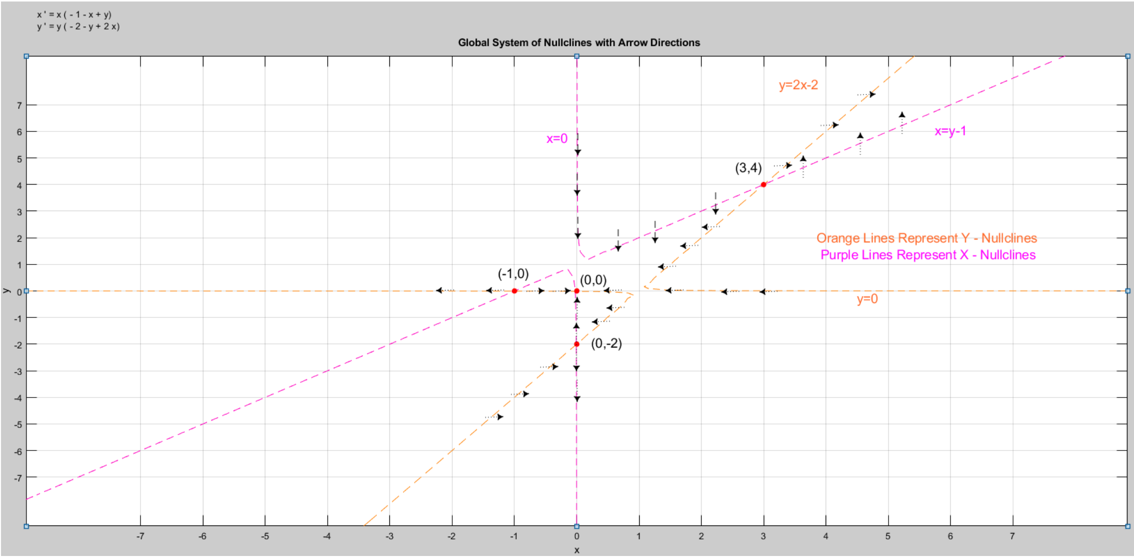


Figure 1: The Nullclines and Direction Fields for all Equilibria

This graph plots the nullclines, along with manually inputted direction curves, to illustrate the consistency between our analysis and with computer software. Purple lines represent the $x$ - nullclines, while the orange lines generate the $y$ - nullclines. Since we are dealing with a realistic problem of mutualism, we are only interested in the 1st quadrant.

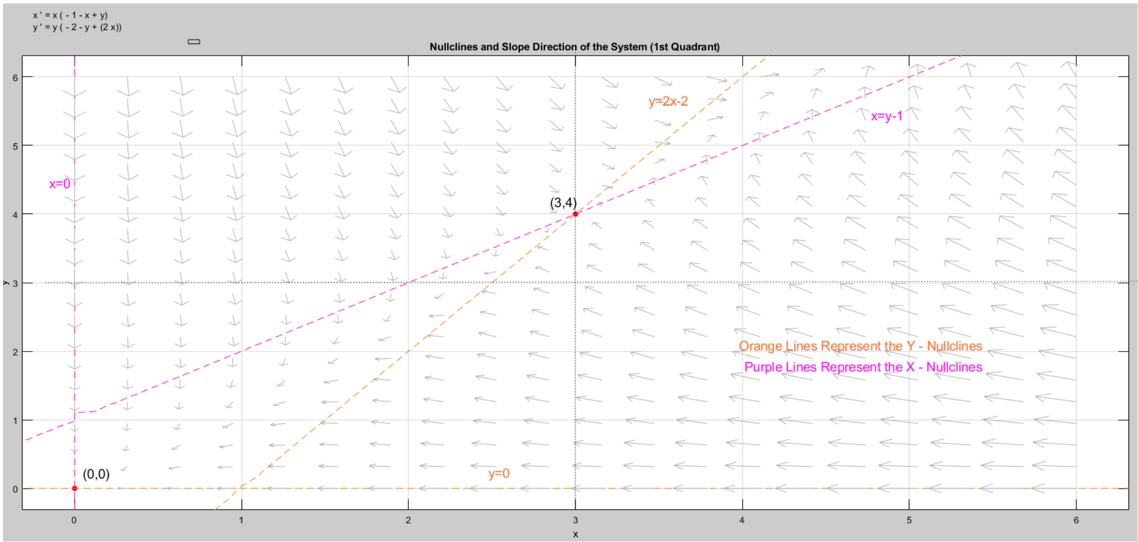


Figure 2: The Nullclines and Vector Curves for Quadrant 1

Let us zoom in on each equilibrium and determine what their stability is.

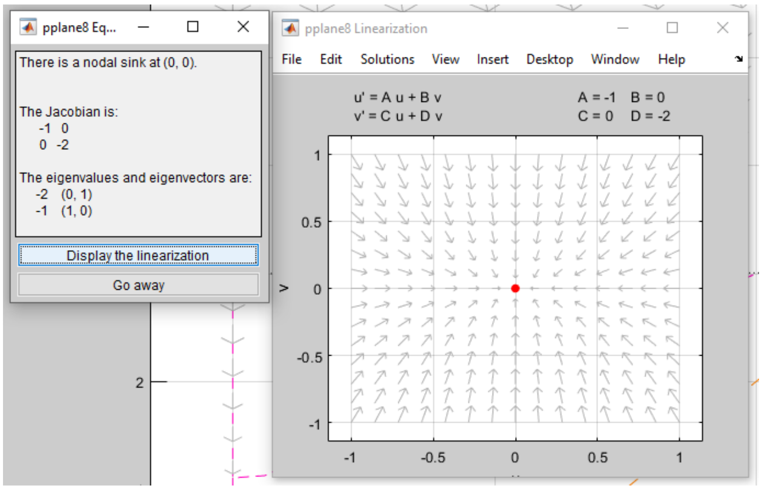


Figure 3: Stability of the $(0, 0)$ equilibrium

The equilibrium $(0, 0)$ is a nodal sink; this determines that population $x$ dies when population $y$ does, and vice versa. This is valid since mutualistic species depend on each other.

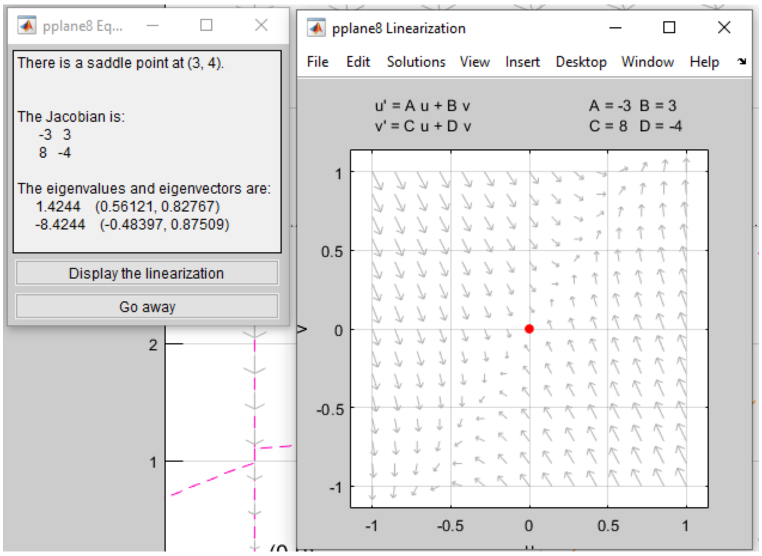


Figure 4: Stability of the $(3, 4)$ equilibrium

The equilibrium $(3, 4)$ is a saddle point. This implies that population $x$ and $y$ will tend to extinction whenever $x < 3$ & $y < 4$. However, both populations will co-exist when $x > 3$ & $y > 4$.

# 5 Conclusion & Final Remarks: What Happens to Both Species as $t \to \infty$?

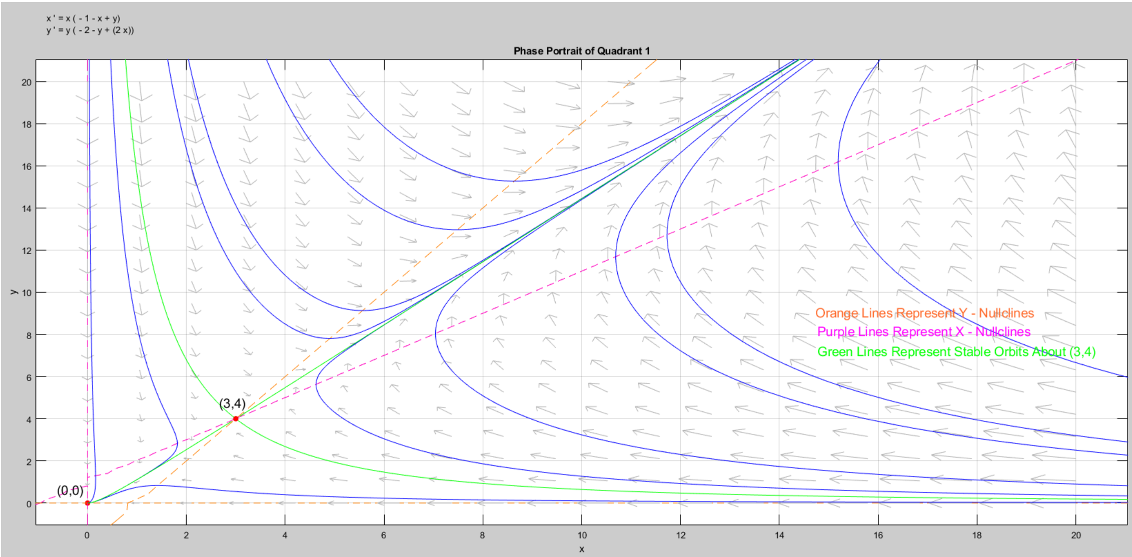


Figure 5: Final Phase Portrait with Solution Curves

From the stability of the equilibrium points and from the following phase portrait above, we can reason a couple facts about the relationship between insect pollinators and flowering plants. The saddle node $(3, 4)$ is important in determining how solutions behave around this point. At the equilibrium $(0, 0)$, the pollinators will die off whenever the growth rate of the flowering plant is less than 3, no matter how big the pollinator rate gets. The same can be said about the rate of the flowering plants, which will tend to 0 whenever the growth rate of the pollinators is less than $4$. When *both* the flowering plant rate and the pollinator rate are above 3 and 4, respectively, then we can conclude that the species will co-exist as $t \to \infty$. In the general sense, the species will co-exists when both the plant growth rate and pollinator rate are above $\frac{K_2+K_1}{b-1}$ & $\frac{K_1 b+K_2}{b-1}$, respectively. After some experimentation on MATLAB, fixing the same parameters on the system and just taking values of $b$ between $0$ and $1$, it seems that there exists only one equilibrium point in the positive quadrant, namely the stable node $(0, 0)$. If $b > 1$, a stable node around the origin and a saddle point will exist within the positive quadrant. The green lines in the figure represent a stable orbit intersecting both equilibrium. As $x$ and $y$ are taken arbitrarily large, solutions seem to tend about the line $y = \frac{4}{3}x$, since $(0, 0)$ and $(3, 4)$ are intersecting points.

Much of the work here has been inspired by Chapter 7.7: Lotka-Volterra Models from *Chaos: An Introduction to Dynamical Systems* (Alligood et. al, 1996). In the following example, the book provides a similar system of nonlinear differential equations.

We begin with two competing species. Because of the finiteness of resources, the reproduction rate per individual is adversely affected by high levels of its own species and the other species with which it is in competition. Denoting the two populations by $x$ and $y$, the reproduction rate per individual is

$$\frac{\dot{x}}{x} = a(1-x) - by, \tag{7.46}$$

where the carrying capacity of population $x$ is chosen to be 1 (say, by adjusting our units). A similar equation holds for the second population $y$, so that we have the **competing species** system of ordinary differential equations

$$\begin{aligned} \dot{x} &= ax(1-x) - bxy \\ \dot{y} &= cy(1-y) - dxy \end{aligned} \tag{7.47}$$

where $a, b, c$, and $d$ are positive constants. The first equation says the population of species $x$ grows according to a logistic law in the absence of species $y$ (i.e., when $y = 0$). In addition, the rate of growth of $x$ is negatively proportional to $xy$, representing competition between members of $x$ and members of $y$. The larger the population $y$, the smaller the growth rate of $x$. The second equation similarly describes the rate of growth for population $y$.

Figure 6: A Competing Species Model from *Chaos: An Introduction to Dynamical Systems*

The next two figures will demonstrate how solutions will behave when the strictly positive parameters $a,\ b,\ c,\ d$ are changed.

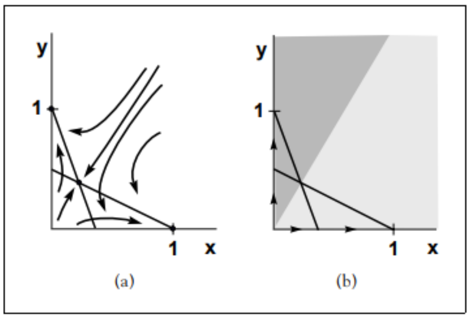


**Figure 7.20 Competing species, extinction.**
(a) The phase plane shows attracting equilibria at $(1,0)$ and $(0,1)$, and a third, unstable equilibrium at which the species coexist. (b) The basin of $(0,1)$ is shaded, while the basin of $(1,0)$ is the unshaded region. One or the other species will die out.

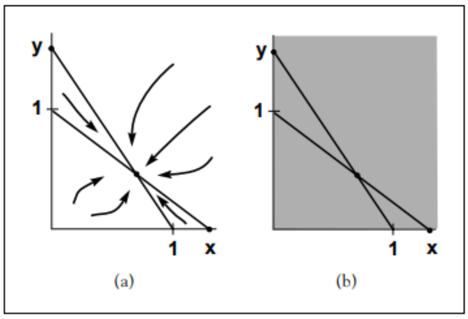


**Figure 7.21 Competing species, coexistence.**
(a) The phase plane shows an attracting equilibrium in which both species survive. The $x$-nullcline $y = \frac{3}{2} - \frac{3}{2}x$ has smaller $x$-intercept than the $y$-nullcline $y = 1 - \frac{3}{4}x$. According to Exercise T7.18, the equilibrium $(\frac{2}{3}, \frac{1}{2})$ is asymptotically stable. (b) All initial conditions with $x > 0$ and $y > 0$ are in the basin of this equilibrium.

Figure 7: Extinction vs. Coexistence of the Same System

The figure on the left has fixed parameters $a = 1,\ b = 2,\ c = 1,\ d = 3$; these values will generate extinction for one of the species in either the $x$ or $y$ intercepts. The figure on the right has fixed parameters $a = 3,\ b = 2,\ c = 4,\ d = 3$; these values will generate coexistence about the equilibrium $(\frac{2}{3}, \frac{1}{2})$.

## References

[1] Kathleen T. Alligood, Tim D. Sauer, and James A. Yorke. *Chaos: An Introduction to Dynamical Systems*. Springer-Verlag, New York, 1996.

[2] Mary Dowd. Mutualism (Biology): Definition, Types, Facts Examples. `https://sciencing.com/mutualism-biology-definition-types-facts-examples-13719234.html`. Sciencing, 2019.

[3] Nancy Chen. Dfield/Pplane User Manual. Harvard, 2004.