Topic 10: Mutualistic Interaction

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MTH 630: Mathematical Biology
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What is Mutualistic Interaction?

- According to Dr. Mary Dowd (2019), *mutualism* refers to a type of relationship that benefits two species in the same environment
- We will focus on a problem of *obligate mutualism*: 2 populations, must interact with each other in order to survive (this interaction is beneficial between species).
- We will examine *obligate mutualism* between flowering plants and insect pollinators based on a system of non-linear differential equations

Introducing and Interpreting the Problem

Example

In another mutualistic system, each species is unable to survive in the absence of the other one. This type of mutualistic interaction is referred to as *obligate* mutualism (Kot, 2001). Each species cannot survive without the presence of the other species. In this system each species is dependent on the other species for its survival. With $x(0) > 0$, $y(0) > 0$, and all parameters being positive, the model has the form

$$\frac{dx}{dt} = c_1 x(-K_1 - x + y)$$
$$\frac{dy}{dt} = c_2 y(-K_2 - y + bx)$$

(a) Find all equilibria for these equations.
(c) Draw the nullclines and the phase plane diagram for the case $b > 1$. Use the Jacobian matrix and determine the stability of the nonnegative equilibria. What happens to $x(t)$ and $y(t)$ as $t \to \infty$?

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- Main objective is to determine the behaviour of 2 mutualistic species as a function of time.
- This will be determined by classifying equilbirium points, finding the Jacobian matrices, and plotting the nullclines and phase portrait.
- Let y represent insect pollinators (bees) and x represent flowering plants. Let K_1 & K_2 denote the carrying capacity and c_1 & c_2 denote the growth rates of species x and y, respectively.
- The parameter b is some constant, which will affect how solutions will be presented. We will discuss the importance between $b > 1$ and $0 < b < 1$ later.

Analysis: Find Equilibria

We have

$$\frac{dx}{dt} = c_1 x(-K_1 - x + y) \tag{1}$$
$$\frac{dy}{dt} = c_2 y(-K_2 - y + bx) \tag{2}$$

Set equations (1) and (2) equal to 0. $(0,0)$ is the trivial equilibrium. Now from (1), if $0 = (-K_1 - x + y)$, then $y = K_1 + x$. Then substituting in equation (2), we have

$$(c_2 x + c_2 K_1)(-K_2 - x - K_1 + bx) = 0 \implies$$
$$x = -K_1, \; x = \frac{K_2 + K_1}{-1 + b}, \; b \neq 1$$

Subbing $x = -K_1$ into (1):

$$c_1(-K_1)(-K_1 - (-K_1) + y) = 0 \implies$$
$$c_1(-K_1)(y) = 0 \implies$$
$$y = 0$$

Analysis: Find Equilibria

Subbing $x = \frac{K_2+K_1}{-1+b}, b \neq 1$ into (1):

$$
\begin{aligned}
&c_1(\frac{K_2+K_1}{-1+b})(-K_1 - \frac{K_2+K_1}{-1+b} + y) = 0 \implies \\
&-K_1 - \frac{K_2+K_1}{-1+b} + y = 0 \implies \\
&y = \frac{K_1 b + K_2}{b-1},\ \ b \neq 1
\end{aligned}
$$

Similarily, equation (2) gives $y = -K_2 + bx$. Subbing into (1) gives $c_1 x(-K_1 - x - K_2 + bx) = 0$. From here, we see that a unique solution produces when $x = 0 \implies y = -K_2$.

Thus, we have equilibrium points:

$$
(0,0),\ \ (-K_1,0),\ \ (\frac{K_2+K_1}{b-1}, \frac{K_1 b + K_2}{b-1})\ \ b \neq 1,\ \ (0,-K_2)
$$

Analysis: Evaluate the Equilibrium Points of the Jacobian Matrices

$$\mathbf{J} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -c_1K_1 - 2c_1x + c_1y & c_1x \\ c_2yb & -c_2K_2 - 2c_2y + c_2bx \end{bmatrix}$$

$$\mathbf{J}(0,0) = \begin{bmatrix} -c_1K_1 & 0 \\ 0 & -c_2K_2 \end{bmatrix} \implies \lambda_{1,2} < 0 : \quad \textit{Stable Node}$$

$$\mathbf{J}(-K_1,0) = \begin{bmatrix} c_1K_1 & -K_1c_1 \\ 0 & -c_2K_2 - c_2bK_1 \end{bmatrix}$$

$$\mathbf{J}\left(\frac{K_2+K_1}{b-1}, \frac{K_1b+K_2}{b-1}\right) = \begin{bmatrix} -c_1K_1 + \frac{c_1K_1b - 2c_1K_1}{b-1} & c_1\left(\frac{K_2+K_1}{b-1}\right) \\ c_2\left(\frac{K_1b+K_2}{b-1}\right)b & -c_2K_2 + \frac{c_2K_2b - c_2K_1b - 2c_2K_2}{b-1} \end{bmatrix}$$

$$\mathbf{J}(0,-K_2) = \begin{bmatrix} -c_1K_1 - c_1K_2 & 0 \\ -c_2K_2b & c_2K_2 \end{bmatrix}$$

Analysis: Determine Behaviour of Equilibria Using Fixed Parameters

Fix $K_1 = 1,\ K_2 = 2,\ c_1 = 1,\ c_2 = 1,\ b = 2$, since we are working with the case $b > 1$.

$$\mathbf{J}(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \implies \tau = -3 < 0,\ \delta = 2 > 0,\ \gamma = 1 > 0 :\ \textit{A Stable Point}$$

$$\mathbf{J}(-1,0) = \begin{bmatrix} 1 & -1 \\ 0 & -4 \end{bmatrix} \implies \tau = -5 < 0,\ \delta = -4 < 0,\ \gamma = 9 > 0 :\ \textit{A Saddle Point}$$

$$\mathbf{J}(3,4) = \begin{bmatrix} -1 & 3 \\ 8 & -4 \end{bmatrix} \implies \tau = -5 < 0,\ \delta = -20 < 0,\ \gamma = 105 > 0 :\ \textit{A Saddle Point}$$

$$\mathbf{J}(0,-2) = \begin{bmatrix} -3 & 0 \\ -4 & 2 \end{bmatrix} \implies \tau = -3 < 0,\ \delta = -6 < 0,\ \gamma = 25 > 0 :\ \textit{A Saddle Point}$$

Then $(0,0)$ is a *stable node*, $(-K_1, 0)$ is a *saddle point*, $(\frac{K_2+K_1}{b-1}, \frac{K_1 b+K_2}{b-1})$ is a *saddle point*, and $(0, -K_2)$ is a *saddle point*

Analysis: Determine Nullclines

Notice that all parameters are positive: $K_1 = 1,\ K_2 = 2,\ c_1 = 1,\ c_2 = 1,\ b = 2$.
Equation (1) $= 0 \implies x = 0,\ K_1 = y - x$ or $y = x + 1$.
If $x = 0$:

$$\frac{dy}{dt} = c_2 y(-K_2 - y)$$
$$\frac{dy}{dt} = y(-2 - y)$$

Tangent lines point *up* when $-K_2 < y < 0 \implies -2 < y < 0$. They point *down* when $y < -K_2 \implies y < -2$ or $y > 0$.
If $x = y - K_1 \implies x = y - 1$:

$$\frac{dy}{dt} = c_2 y(-K_2 - y + b(y - K_1))$$
$$\frac{dy}{dt} = y(-2 - y + 2y - 2)$$
$$\frac{dy}{dt} = y(y - 4)$$

Tangent lines point *up* when $y < 0,\ y > 4$. They point *down* when $0 < y < 4$.

Analysis: Determine Nullclines

Equation (2) $= 0 \implies y = 0,\ K_2 = bx - y$ or $2 = 2x - y$.
If $y = 0$:

$$\frac{dx}{dt} = c_1 x(-K_1 - x)$$
$$\frac{dx}{dt} = x(-1 - x)$$

Tangent lines point *left* when $x < -1,\ x > 0$. They point *right* when $-1 < x < 0$.
If $y = 2x - 2$:

$$\frac{dx}{dt} = c_1 x(-K_1 + x - 2)$$
$$\frac{dx}{dt} = x(-3 + x)$$

Tangent lines point *left* when $0 < x < 3$. They point *right* when $x < 0,\ x > 3$.

Results: Numerical Simulations Based on the Jacobian Matrices and Nullclines

We can generate solutions based on our analysis using the MATLAB tool Pplane.

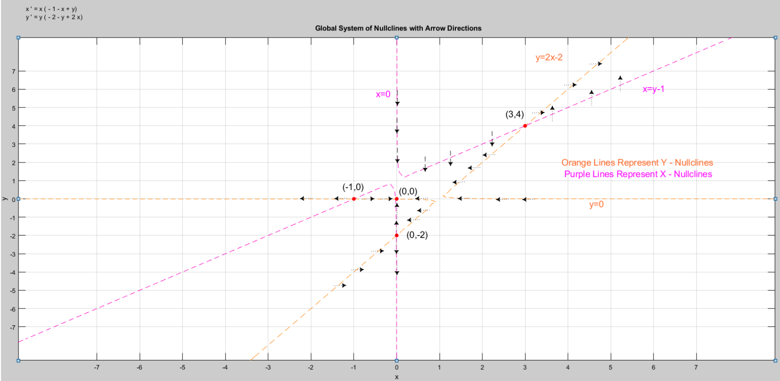

Figure: The Nullclines and Direction Fields for all Equilibria

This graph plots nullclines, with manually inputted direction curves, to illustrate the consistency between our analysis and with software. Purple lines represent the x - nullclines, while the orange lines generate the y - nullclines.

Results: Numerical Simulations Based on the Jacobian Matrices and Nullclines

Since we are dealing with a realistic problem of mutualism, we are only interested in the 1st quadrant.

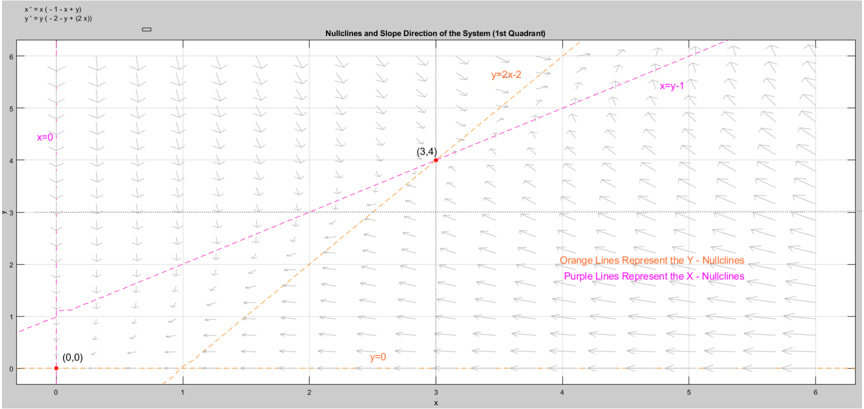

Figure: The Nullclines and Vector Curves for Quadrant 1

Results: Numerical Simulations Based on the Jacobian Matrices and Nullclines

Let us zoom in on each equilibrium and determine what their stability is.

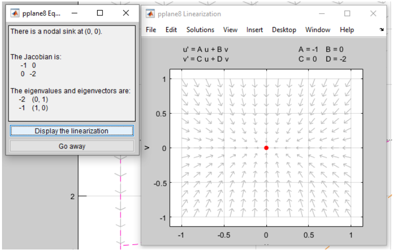

Figure: Stability of the $(0, 0)$ equilibrium

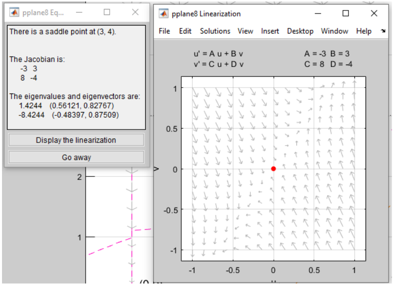

Figure: Stability of the $(3, 4)$ equilibrium

- $(0, 0)$ is a *nodal sink*; population x dies when population y does, and vice versa. This is valid since mutualistic species depend on each other.
- $(3, 4) = (\frac{K_2+K_1}{b-1}, \frac{K_1 b+K_2}{b-1})$ is a *saddle point*. Population x and y will tend to extinction whenever $x < 3$ & $y < 4$.
- Both populations will co-exist when $x > 3$ & $y > 4$.

Conclusion: What Happens to Both Species as $t \to \infty$?

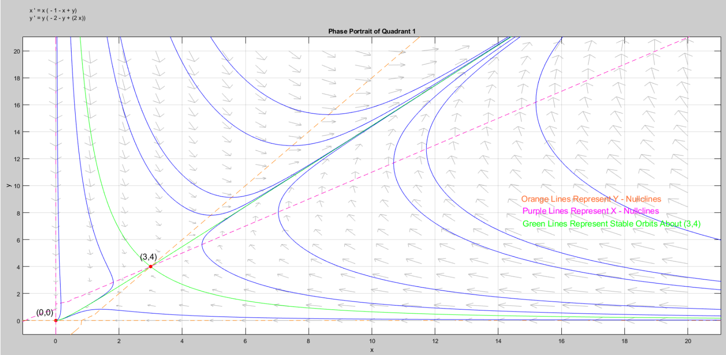

Figure: Final Phase Portrait with Solution Curves

- The saddle node $(3,4) = (\frac{K_2+K_1}{b-1}, \frac{K_1 b+K_2}{b-1})$ is important in determining how solutions behave.
- At $(0,0)$, pollinators will die off whenever growth rate of the flowering plant is less than 3.
- At $(0,0)$, rate of flowering plants will tend to 0 whenever growth rate of pollinators is less than 4.
- When *both* the flowering plant rate and the pollinator rate are above 3 and 4, respectively, we can conclude that the species will co-exist as $t \to \infty$.

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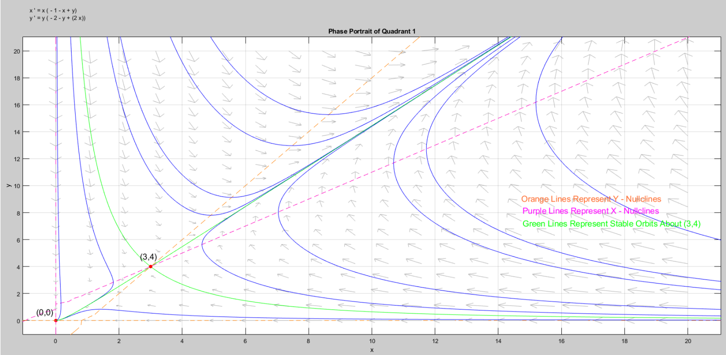

Figure: Final Phase Portrait with Solution Curves

- Green lines in figure represent a stable orbit intersecting both equilibrium.
- As x and y are taken arbitrarily large, solutions seem to tend about the line $y = \frac{4}{3}x$, since $(0, 0)$ and $(3, 4)$ are intersecting points.
- Experimenting on MATLAB, fixing the same parameters on the system, and taking values of b between 0 and 1, it seems that there exists only one equilibrium point in the positive quadrant, the stable node $(0, 0)$.
- If $b > 1$, a stable node around the origin and a saddle point will exist within the positive quadrant.

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