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Assignment 4

1. (a) Let f be a function such that $\frac{1}{x} - 1 \leq f(x) \leq \frac{1}{x}$ for each $x \in \mathbb{R} \setminus \{0\}$. Find the following limit:

$$\lim_{x \to 0} x f(x)$$

   (b) Use the $\epsilon - \delta$ definition of the limit to find the limit

$$\lim_{x \to 0^+} \sqrt{x} sin(x)$$

2. (a) Let f be a function with $D_f = \mathbb{R}$ such that $\lim_{x \to 0} f(x) = 1$ and $f(x_1 + x_2) = f(x_1)f(x_2)$ for each $x_1,\ x_2$. Show that f is continuous on $\mathbb{R}$.

   (b) Let A be a subset of $\mathbb{R}$ and $x_0 \in A \setminus A^0$, show that the following function is discontinuous at x_0:

$$f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$

   (c) Let $f,\ g$ be continuous functions on $\mathbb{R}$, such that $f(x) < g(x) \implies x \in \mathbb{R} \setminus \mathbb{Q}$. Show that $f(x) \geq g(x)$ for each $x \in \mathbb{R}$.

3. (a) Let f be a function with $D_f = \mathbb{R}$ such that $f(0) + 3f(5) + 2f(7) = 0$ and $f(x) \neq 0$ for each $x \in \mathbb{R}$. Show that f is not continuous on $\mathbb{R}$.

   (b) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists $c \in [0, 1]$ such that $f^2(c) = c$.

   (c) If f is non constant and continuous on an interval $[a, b]$, then show that the range $R = \{y \in \mathbb{R} \mid y = f(x),\ x \in I\}$ of f is a closed interval.

4. (a) Suppose that $f'(0)$ exists and $f(x + y) = f(x)f(y)$ for each x & y in $\mathbb{R}$. Show that f' exists for each $x \in \mathbb{R}$.

   (b) Find the derivative of the following function at 0 and 1 (if it exists):

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

5. (a) Let $f : [a, b] \to \mathbb{R}$ with $f(a) = f(b)$. If f is continuous on $[a, b]$ and differentiable on (a, b), show that there exists distinct $x_1,\ x_2 \in (a, b)$ such that $f'(x_1) + f'(x_2) = 0$. Hint: Apply MVT to two different sub intervals of $[a, b]$.

(b) Show that

$$|ysin(y) - xsin(x)| \leq 2|y - x|$$

for each $x,\ y \in (0, 1)$. Hint: Apply MVT to a suitable function.

Solutions

1. (a) Notice that (a) can be rewritten as $1 - x \leq xf(x) \leq 1$ by multiplying the inequality by x. When the inequality approaches 0, the left hand side is 1, and the right hand side is equal to 1. By Squeeze Theorem, we have that $\lim_{x \to 0} xf(x) = 1$.

   (b) *Scratch work.* Notice that $\lim_{x \to 0^+} \sqrt{x}sin(x) = 0$. Want to show that given $|x - 0^+| < \delta$, then $|\sqrt{x}sin(x) - 0| < \epsilon$. But $|\sqrt{x}sin(x)| \leq |\sqrt{x}| < \epsilon$, since $|sin(x)| \leq 1$. Then $|\sqrt{x}| < \epsilon \implies |x| < \epsilon^2$. Pick $\delta = \epsilon^2$, then the proof follows.

   Proof. Given $\epsilon > 0$, let $\delta = \epsilon^2$. If $0 < |x - 0^+| < \epsilon^2$, then

$$|\sqrt{x}sin(x)| \leq |\sqrt{x}| < \sqrt{\epsilon^2} = \epsilon$$

□

2. (a) Let $x_n \to x_0 \implies x_n - x_0 \to 0$. Then $f(x_n - x_0) \to f(0) = 1$. Thus, $f(x_n - x_0) = f(x_n + (-x_0)) = 1$. But $f(x_n) * f(-x_0) = 1$, by property. Then $f(x_n + (-x_0)) = f(x_n)f(-x_0) \to 1 \iff f(x_n) \to \frac{1}{f(-x_0)} = \frac{f(0)}{f(-x_0)} = \frac{f(x_0 - x_0)}{f(-x_0)} = \frac{f(x_0)f(-x_0)}{f(-x_0)} = f(x_0)$. Then $f(x_n) \to f(x_0)$. Thus f is continuous since it satisfies the following:

   Theorem 1. *Let $f : D_f \to \mathbb{R}$ and $x_0 \in D_f$. T.F.A.E*
   (a) f is continous at x_0
   (b) $x_n \to x_0 \implies f(x_n) \to f(x_0)$

   (b) Notice $x_0 \in A \setminus A^0 \iff x_0 \in \partial(A) \iff x_0 \in \overline{A} \cap \overline{A^c}$. Let $x_n \in A$ and $y_n \in A^c$. Then $x_n \to x_0 \in \partial(A),\ y_n \to y_0 \in \partial(A)$. But $f(x_n) \to 1,\ f(y_n) \to 0$. Thus, f is discontinous at x_0.

   (c) Let $x \in \mathbb{R}$. Then for some $q_n \subseteq \mathbb{Q}$, we have that $q_n \to x$. Then $f(q_n) \to f(x) \implies g(q_n) \to g(x)$. Then $f(q_n) - g(q_n) \to f(x) - g(x) \geq 0$ if $x \notin \mathbb{R} \setminus \mathbb{Q}$. Thus $f(x) \geq g(x)$, since $f(x) < g(x)$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, take $t_n \subseteq \mathbb{R} \setminus \mathbb{Q} \implies t_n \to x$. Then $f(t_n) - g(t_n) \to f(x) - g(x) \geq 0$ if $x \notin \mathbb{R} \setminus \mathbb{Q}$. Then $f(x) \geq g(x)$. Thus, $f(x) \geq g(x),\ \forall x \in \mathbb{R}$.

3. (a) We use Bolzano's Theorem:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ a continuous function such that $f(a)f(b) < 0$. Then $\exists\ c \in (a,b)$ such that $f(c) = 0$.*

Suppose f is continuous in $\mathbb{R}$. Given $f(0) + 3f(5) + 2f(7) = 0$, then each $f(0), f(5), f(7)$ is 0, contradicting $f(x) \neq 0$, $\forall x \in \mathbb{R}$. Suppose at least one $f(0), f(5), f(7)$ is negative. Then for some $x_1 \in [0,7] \subseteq D_f$, we have $f(x_1) < 0$. But then we have some $x_2, x_3 \in [0,7] \subseteq D_f$ such that $f(x_2), f(x_3)$ are both greater than 0, or one is less than 0. Since f is continuous, $\exists\ c \in (0,7) \subseteq D_f$ such that $f(c) = 0$. But $f(x) \neq 0$, for each $x \in \mathbb{R}$. Thus, f is not continuous on $\mathbb{R}$ in all cases.

(b) Let $g(x) = f(x) - \sqrt{x}$. Note that $g(x)$ is continuous on $[0,1]$. Indeed, $g(0) = f(0) \geq 0$, $g(1) = f(1) - 1 \leq 0$. Using **Theorem 2**, $\exists c \in [0,1]$ such that $g(1)g(0) < 0$. Then $g(c) = 0 \implies g^2(c) = 0$. But $g^2(c) = (f(c) - \sqrt{c})^2$. Then $(f(c) - \sqrt{c})^2 = 0 \implies f(c) = \sqrt{c} \implies f^2(c) = c$, as was to be shown.

(c) We use the Max Value Theorem.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then $\exists\ m, M \in \mathbb{R}$ such that $M = sup\{f(x) \mid x \in [a,b]\}$, $m = inf\{f(x) \mid x \in [a,b]\}$.*

Since f is continuous and closed on $[a,b]$, then there exists c & $d \in [a,b]$ such that $f(c) \geq f(x) \geq f(d)$. But then R must be bounded above and below by $f(c)$ and $f(d)$, respectively. Since R is bounded and contains its boundary points, then R is closed on the interval $[f(d), f(c)]$.

4. (a) Since $f(0) = f(0+0) = f(0)f(0) = f^2(0) \implies f(0) = f^2(0) \implies f^2(0) - f(0) = f(0)(f(0) - 1) \implies f(0) = 0$ or 1. Since $f'(0)$ exists, we have

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - 1}{h}, \text{ we let } f(0) = 1 \text{ for the next step} \qquad (*) \end{aligned}$$

Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h}, \text{ by definition of our function} \\ &= \lim_{h\to 0} f(x)\frac{f(h)-1}{h} \\ &= f(x)f'(0), \text{ by (*)} \end{aligned}$$

Since f is differentiable, $f'(x)$ exists for all $x \in \mathbb{R}$.

(b) Note that if f is not continuous at 1, then f is not differentiable at 1. Take $x_n \subseteq \mathbb{R}\setminus\mathbb{Q} \to 1,\ y_n \subseteq \mathbb{Q} \to 1$. But $\lim f(x_n) \to 0,\ \lim f(y_n) \to 1$. Thus, $f(1)$ is discontinuous. So $f(1)$ is not differentiable. Similarily, take $b_n \subseteq \mathbb{R}\setminus\mathbb{Q} \to 0,\ c_n \subseteq \mathbb{Q} \to 0$. Then $\lim f(b_n) \to 0,\ \lim f(c_n) \to 0$. But we must prove $f'(0)$ exists. Indeed, if $x \in \mathbb{Q}$, then we have $f'(0) = 2(0) = 0$. If $x \in \mathbb{R}\setminus\mathbb{Q}$, then $f'(0) = 0$, since $f(x) = 0$ is a constant function. Thus, $f'(0)$ exists, but $f(1)$ is not differentiable.

5. (a) The Mean Value Theorem

Theorem 4. *If f is defined and continuous on $[a, b]$ and differentiable on (a, b), then there is at least one c in (a, b), $a < c < b$, such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Note that $a < \frac{a+b}{2} < b$. Fix $c = \frac{a+b}{2} \implies 2c - a = b \implies c - a = b - c$. By MVT, if we let $x_1 \in (a, c),\ x_2 \in (c, b)$, then

$$\begin{aligned} f'(x_1) &= \frac{f(c)-f(a)}{c-a} \\ f'(x_2) &= \frac{f(b)-f(c)}{b-c} \implies \\ f'(x_1)+f'(x_2) &= \frac{f(c)-f(a)}{c-a} + \frac{f(b)-f(c)}{b-c} \end{aligned}$$

Since $f(a) = f(b)$, $b - c = c - a$ from before, we can pick $t = b - c = c - a$. Then we have

$$f'(x_1)+f'(x_2) = \frac{f(c)-f(b)+f(b)-f(c)}{t} = 0$$

Thus, $\exists\ x_1, x_2 \in [a, b]$, $x_1 \neq x_2$, and $f'(x_1) + f'(x_2) = 0$

(b) Let $f(x) = xsin(x) \implies f'(c) = c \cdot cos(c) + sin(c),\ c \in (0,1)$. By MVT, we have

$$\begin{aligned} f'(c) &= \frac{ysin(y) - xsin(x)}{y-x} \\ c \cdot cos(c) + sin(c) &= \frac{ysin(y) - xsin(x)}{y-x} \\ (y-x)c \cdot cos(c) + sin(c) &= ysin(y) - xsin(x) \qquad (*) \end{aligned}$$

Since $c \in (0,1)$ and $|cos(c)| \leq 1,\ |sin(c)| \leq 1$, then

$$\begin{aligned} &|c \cdot cos(c) + sin(c)| \leq 2 \implies \\ &-2 \leq c \cdot cos(c) + sin(c) \leq 2(\frac{y-x}{y-x}) \implies \\ &(y-x)c \cdot cos(c) + sin(c) \leq 2(y-x) \leq 2|y-x| \implies \\ &(y-x)c \cdot cos(c) + sin(c) \leq 2|y-x| \end{aligned}$$

By (*), we have

$$\begin{aligned} ysin(y) - xsin(x) &\leq 2|y-x| \implies \\ |ysin(y) - xsin(x)| &\leq 2|y-x| \end{aligned}$$