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Instructor: Foivos Xanthos **MTH 525 - Analysis** November 15, 2019

Assignment 3

Problem 1. a) Prove that the sequence $(-1)^n$ does not converge
b) Construct a sequence that does not contain 0 or 1 as a term, but contains subsequences converging to 0 and 1.

Solution. a) Assume that $(x_n) \to L,$ for $L \in \mathbb{R}$.
Let $\epsilon > 0$, now $\exists\ n_0 \in \mathbb{N}$ such that $\forall n \geq n_0,\ |(-1)^n - L| < \epsilon$. Pick $\epsilon = 1$.
Then $|(-1)^n - L| < 1$. In particular, this holds for all odd $n \geq n_0 \implies$
$|(-1) - L| = |1 + L| < 1$. Likewise, it holds for all even $n \geq n_0 \implies$
$|1 - L| < 1$. Then $|1 + L| + |1 - L| < 2$. But

$$\begin{aligned} 2 = |1 + 1| &= |1 + L + 1 - L| \\ &\leq |1 + L| + |1 - L| \\ &< 1 + 1 = 2 \end{aligned}$$

Then $2 < 2$, a contradiction. Thus, $(-1)^n$ diverges.

□

Solution. b) Consider

$$(x_n) = \begin{cases} 1 - (\frac{1}{10})^n & n = 2m,\ m \in \mathbb{N} \\ (\frac{1}{10})^n & n = 2m - 1 \end{cases}$$

Then $(x_n) = 0.1,\ 0.9,\ 0.01,\ 0.99,\ 0.001,\ 0.999, \ldots$ □

Problem 2. a) Let (x_n) be a convergent sequence and assume that $(x_n)(-1)^n \leq 0$ for each n. Show that $\lim(x_n) = 0$.
b) Let $(x_n),\ (y_n)$ be sequences such that $x_n y_n \in \mathbb{R}$, (x_n) is bounded, and $\lim(y_n) = 0$. Show that $\lim(x_n y_n) = 0$.

Solution. a) Notice that $x_1(-1)^1 \leq 0 \implies x_1 \geq 0$, $x_2(-1)^2 \leq 0 \implies x_2 \leq 0, \ldots$
Thus if n odd, $x_n \geq 0$ and if n even, $x_n \leq 0$. If we take a subsequence of (x_n), say (x_{2n}), them $\lim(x_{2n}) \to x \leq 0$. If we take another subsequence (x_{2n+1}), then $\lim(x_{2n+1}) \to x \geq 0$. This follows from a theorem proved in class.

Convergent Subsequence Theorem: Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R} \cup \pm\infty$.

i) If $(x_n) \to x$, then $(x_{k_n}) \to x$ for each subsequence (x_{k_n}) of (x_n).
ii) If $x_{2n} \to x$ and $x_{2n-1} \to x$, then $x_n \to x$.

In other words, every convergent sequence has all of its subseqeunces tend to the same limit as the original sequence. From our two constructed subsequences above, we have that

$$0 \leq x \leq 0 \implies \lim(x_n) = 0.$$

$\square$

Solution. b) Since (x_n) is bounded, then there exists an M such that $|x_n| \leq M$. Put $\epsilon > 0$. Then there exists an $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$

$$\begin{aligned} |x_n y_n - 0| &= |x_n||y_n| \\ &\leq M|y_n| \quad (\text{since } |x_n| \leq M) \end{aligned}$$

Since $\lim(y_n) \to 0$, $\exists n_0 \in \mathbb{N}, \epsilon > 0$ such that $\forall n \geq n_0$, $|y_n - 0| < \epsilon \to |y_n| < \epsilon$.
Pick $\epsilon = \frac{\epsilon}{M}$. Then $|y_n| \leq \frac{\epsilon}{M}$. But

$$\begin{aligned} |x_n y_n| &= |x_n||y_n| \\ &\leq M|y_n| \\ &< M\frac{\epsilon}{M} \\ &= \epsilon \end{aligned}$$

Thus, $\lim(x_n y_n) = 0$.

$\square$

Problem 3. Let (x_n) be defined as follows

$$x_{n+1} = \frac{1}{4 - x_n}, \; x_1 = 3$$

a) Show that $0 < x_{n+1} < x_n < 4$ for each n.
b) Show that $(x_n) \in \mathbb{R}$.

Solution. a) Notice $x_1 = 3,\ x_2 = 1,\ x_3 = 1/3,\ x_4 = 3/11,\ x_5 = 11/41, \ldots$
Thus, x_1 is our largest term. By induction, want to show $\forall n \in \mathbb{N},\ 0 < x_{n+1} < x_n$.
<u>Base case:</u> $x_1 = 3,\ x_2 = 1 \implies 0 < 1 \leq 3 < 4$

Induction step: Suppose it is true for $0 < x_{n+1} \leq x_n < 4$.
Want to show it's true for $0 < x_{n+2} \leq x_{n+1} < 4$. From induction step, we have that

$$\begin{aligned} x_n \geq x_{n+1} &\implies \\ -x_n \leq -x_{n+1} &\implies \\ 4 - x_n \leq -x_{n+1} + 4 &\implies \\ \frac{1}{4-x_n} \geq \frac{1}{4-x_{n+1}} & \end{aligned}$$

Thus, $0 < x_{n+1} \leq x_n < 4$ and (x_n) is decreasing

□

Solution. b) Since $x_1 = 3$ is our largest term, then (x_n) is bounded above by 3. We want to show that (x_n) is bounded below, by corollary that every bounded monotone sequence converges. Indeed, $x_1 > 0$. Assume $x_n > 0$. Since (x_n) is decreasing, $x_n \leq x_1 \to x_{n+1} = \frac{1}{4-x_n} > 0$. From induction in part a), (x_n) is bounded below by 0 $\forall n \in \mathbb{N}$. Thus, (x_n) converges. □

Problem 4. Let (x_n) be defined: $a = \lim x_{2n},\ b = \lim x_{2n-1},\ 1 = \lim x_{3n}$.
a) Show that (x_{6n}) is a subsequence of (x_{3n}) (i.e, specify appropriate k_n).
b) Find $\lim x_{6n}$ and $\lim x_{3(2n+1)}$
c) Find $\lim x_n$

Solution. a) Notice that $(x_{6n}) = (x_{3(2n)}),\ n \in \mathbb{N}$. Since $n \in \mathbb{N}$, we have that $2n \in \mathbb{N}$ as well. If we let $k_n = 2n$, then we have that (x_{6n}) is a subsequence of (x_{3n}). □

Solution. b) We use the convergent subsequence theorem stated in the solution of 2.a). From 4.a), we have that (x_{6n}) is a subsequence of (x_{3n}). Since $\lim(x_{3n}) = 1$, then $\lim(x_{6n}) = 1$, by theorem. Moreover, $(x_{3(2n+1)})$ is also a subsequence of (x_{3n}) since $(2n+1) \in \mathbb{N}$. If we let $k_n = 2n+1$, we have that $(x_{3(2n+1)})$ is a subsequence of $(x_{3n}),\ n \in \mathbb{N}$. Thus, $\lim x_{3(2n+1)} = 1$ □

Solution. c) By definition, every convergent subsequence converges to the same limit as the original sequence. Notice from 4.b), $x_{6n} = x_{2(3n)}$ is a subsequence of x_{2n} since $3n \in \mathbb{N}$ and $2(3n)$ is even for any n. But x_{6n} is also a subsequence of x_{3n}.
Since $x_{3n} \to 1$, it must be true that $x_{6n} = x_{2(3n)} \to 1 \implies x_{2n} \to 1$. Thus $a = 1$.
Similarly, $x_{3(2n+1)}$ and $x_{3(2n-1)}$ are subsequences of x_{2n-1} since $3(2n+1)$ and $3(2n-1)$ are

odd for any n. Moreover, $x_{3(2n-1)}$ is a subsequence of (x_{3n}) since $(2n-1) \in \mathbb{N}$.
Since these subsequences all converge to 1 due to $\lim(x_{3n}) = 1$, then it must be true that $b = 1$. Indeed, (x_{2n}) and (x_{2n-1}) are subsequences of (x_n).
We conclude that if $\lim(x_{2n}) \to 1$, $\lim(x_{2n-1}) \to 1$, then $\lim(x_n) \to 1$, by the convergent subsequence theorem stated in the solution of 2.a).

$\square$