4 February, 2021

Revision:

max $Z = C \cdot X$ subject to $AX \leq b$, $X \geq 0$

Dual: min $Z = b \cdot Y$ subj. to $A^T Y \geq C$, $Y \geq 0$

Theorm: If X_0, Y_0 are feasible solⁿ then $C \cdot X_0 \leq b \cdot Y_0$.

Result: We have $bY_0 - CX_0 = Y_0 u + X_0 \cdot v$, $u = b - Ax$, $v = A^T Y_0 - C$

Result: If $CX_0 = bY_0$ then $Z_{max} = CX_0 = b \cdot Y_0 = Z_{min}$

Optimum solⁿ

Result: If Z_{max} is unbounded above, then Z_{min} is not feasible

If Z_{min} is unbounded ~~above~~ below, then Z_{max} is not feasible

The Duality Theorm

Theorm: If either (max) or (min) has a bounded optimum solution so does the other and $Z_{max} = Z_{min}$

* ~~Optimum~~ Coeff. for slack variables = optimum pt. for dual

↳ At optimum solⁿ

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3 cases:
1. Optimum Sol^n $\to$ $C_{n0} = b_{y0}$
2. Not Bounded $\longrightarrow$ Not feasible
3. No feasible sol^n

Ex- Max $Z = -5x_1 + 18x_2 + 6x_3 - x_4$

subject to

$2x_1 - x_3 + 3x_4 \le 20 \longrightarrow y_1$

$x_2 - 2x_3 - x_4 \le 30 \longrightarrow y_2$

$-3x_1 + 6x_2 + 3x_3 + 4x_4 \le 24 \longrightarrow y_3$

$x_i \geq 0 \quad i = 1, --- 4$

Sol^n Dual:

Min $Z = 20y_1 + 30y_2 + 24y_3$

subject to

$2y_1 - 3y_3 \geq -5$

$y_2 + 6y_3 \geq 18$

$-y_1 - 2y_2 + 3y_3 \geq 6$

$3y_1 - y_2 + 4y_3 \geq -1$

$y_i \geq 0$

Sol^n is $Z = 112$ at $(2, 0, 3)$

After solving and adding slack variables we get

$Z_{min} = -112$

$Z_{max} = 112$

at $(10, 9, 0, 0)$

* Need ~~need~~ not solve both, we can solve any one of them & we get the sol^n of other in table only.

3 cases:

1. Optimum Soln → $C_{x_0} = b_{y_0}$
2. Not Bounded → Not feasible
3. No feasible soln

Ex- Max $Z = -5x_1 + 18x_2 + 6x_3 - x_4$

subject to

$2x_1 - x_3 + 3x_4 \leq 20 \rightarrow y_1$

$x_2 - 2x_3 - x_4 \leq 30 \rightarrow y_2$

$-3x_1 + 6x_2 + 3x_3 + 4x_4 \leq 24 \rightarrow y_3$

$x_i \geq 0 \quad i = 1, ---4$

Soln

Dual:

Min $Z = 20y_1 + 30y_2 + 24y_3$

subject to

$2y_1 - 3y_3 \geq -5$

$y_2 + 6y_3 \geq 18$

$-y_1 - 2y_2 + 3y_3 \geq 6$

$3y_1 - y_2 + 4y_3 \geq -1$

$y_i \geq 0$

Soln is $Z = 112$
at $(2, 0, 3)$

After solving and adding slack variables we get
$Z_{min} = -112$
$Z_{max} = 112$
at $(10, 9, 0, 0)$

* Deal ~~need~~ not solve both, we can solve any one of them & we get the soln of other in table only.

Primal Solⁿ

Introducing Slack Variables

$$2x_1 - x_3 + 3x_4 + x_5 = 20$$
$$x_2 - 2x_3 - x_4 + x_6 = 30$$
$$-3x_1 + 6x_2 + 3x_3 + 4x_4 + x_7 = 24$$
$$x_i \geq 0 \; ; \; i = 1, 2, \ldots 7$$

~~subj~~

$$\text{Max } Z = -5x_1 + 18x_2 + 6x_3 - x_4$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
x_5	2	0	-1	3	1	0	0	20
x_6	0	1	-2	-1	0	1	0	30
x_7	-3	6	3	4	0	0	1	24
Z	-5	18	6	-1	0	0	0	0
x_5	2	0	-1	3	1	0	0	20
x_6	1/2	0	-5/2	-5/3	0	1	7/6	26
x_2	-1/2	1	1/2	2/3	0	0	1/6	4
Z	4	0	-3	-13	0	0	-3	-72
x_1	1	0	-1/2	3/2	1/2	0	0	10
x_6	0	0	-9/4	-29/12	-1/4	1	-1/6	21
x_2	0	1	1/4	17/12	1/4	0	1/6	9
Z	0	0	-1	-19	-2	0	-3	-112

No more +ve values

$\therefore Z = 112$ at $(10, 9, 0, 0)$

Now for Dual, solⁿ is $Z = 112$ at $(2, 0, 3)$

As we can see, we can solve any of the two (primal or dual) & can get the solution of other.

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Dual Sol^n

Min $Z = 20y_1 + 30y_2 + 24y_3$ subj. to

$2y_1 - 3y_3 - y_4 + A_1 = -5$

$y_2 + 6y_3 - y_5 + A_2 = 18$

$-y_1 - 2y_2 + 3y_3 - y_6 + A_3 = 6$

$3y_1 - y_2 + 4y_3 - y_7 + A_4 = 1$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	b
x_8	2	0	-3	-1	0	0	0	1	0	0	0	-5
x_9	0	1	6	0	-1	0	0	0	1	0	0	18
x_{10}	-1	-2	3	0	0	-1	0	0	0	1	0	6
x_{11}	3	-1	4	0	0	0	-1	0	0	0	1	1
Z	20	30	24	0	0	0	0	0	0	0	0	0
w	-4	2	-10	1	1	1	1	0	0	0	0	-20
x_8	17/4	-3/4	0	-1	0	0	-3/4					-17/4
x_9	-9/2	5/2	0	0	-1	0	3/2					33/2
x_{10}	-13/4	-5/4	0	0	0	-1	3/4					21/4
x_3	3/4	-1/4	1	0	0	0	-1/4					1/4
Z	2	36	0	0	0	0	6	0	0	0	-6	-6
w	7/2	-1/2	0	1	1	1	-3/2	0	0	0	5/2	-35/2
x_8	1	-2	0	-1	0	-1	0					1
x_9	2	5	0	0	-1	2	0					6
x_7	-13/3	-5/3	0	0	0	-4/3	1					7
x_3	-1/3	-2/3	1	0	0	-1/3	0					2
Z	28	46	0	0	0	8	0	0	0	-8	0	-48
w	-3	-3	0	1	1	-1	0	0	0	2	1	-7
x_1	1	-2	0	-1	0	-1	0					1
x_9	0	9	0	2	-1	4	0					4
x_7	0	-31/3	0	-13/3	0	-17/3	1					34/3
x_3	0	-4/3	1	-1/3	0	-2/3	0					7/3
Z	0	102	0	28	0	36	0	-28	0	36	0	-76
w	0	-9	0	-2	1	-4	0	3	0	5	1	-4
x_1	1	0	0	-5/9	-2/9	-1/9	0					17/9
x_2	0	1	0	2/9	-1/9	4/9	0					4/9
x_7	0	0	0	-55/27	-31/27	-2/27	1					430/27
x_3	0	0	1	-1/27	-4/27	-2/27	0					79/27
Z	0	0	0	16/3	34/3	-28/3	0	-16/3	-34/3	28/3	0	-364/3
w	0	0	0	0	0	0	0	1	1	1	1	0
x_1	1	1/4	0	-1/2	-1/4	0	0					2
x_6	0	9/4	0	1/2	-1/4	1	0					1
x_7	0	29/12	0	-3/2	-17/12	0	1					17
x_3	0	1/6	1	0	-1/6	0	0					3
Z	0	21	0	10	9	0	0	-10	-9	0	0	-112

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Ex $X_0 = \left(0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3}\right)^T$
$Y_0 = \left(3\frac{1}{2}, 2, 1\frac{1}{2}\right)^T$

$\text{Min } Z = 7x_1 + 11x_2 - 3x_3 - x_4$

subject to $2x_1 + 2x_2 - x_3 - 3x_4 \geq 2$
$-x_1 + 5x_2 - 2x_3 + x_4 \geq 12$
$x_1 - 4x_2 + 3x_3 + 5x_4 \geq 4$
$x_i \geq 0$

Sol^n ① X_0 is feasible when
$Ax_0 \geq b$

$$\begin{bmatrix} 2 & 2 & -1 & -3 \\ -1 & 5 & -2 & 1 \\ 1 & -4 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 5\frac{2}{3} \\ 8\frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 4 \end{bmatrix} = b$$

X_0 = feasible sol^n

Z at $X_0 = C \cdot X_0 = 37$

② $A^T Y_0 \leq C \Rightarrow \begin{bmatrix} 6.5 \\ 11/3 \\ -1 \end{bmatrix} \leq C$

Y_0 = feasible

Z at $Y_0 = b \cdot Y_0 = 37$