



4 February, 2021

Revision:

max  $Z = C \cdot X$  subject to  $A \cdot X \leq b, X \geq 0$

Dual: min  $Z = b \cdot Y$  subj. to  $A^T Y \geq C, Y \geq 0$

Theorem: If  $X_0, Y_0$  are feasible sol<sup>n</sup> then  
 $C \cdot X_0 \leq b \cdot Y_0$ .

Result: We have  $b \cdot Y_0 - C \cdot X_0 = Y_0 \cdot u + X_0 \cdot v$ ,  
 $u = b - A \cdot n, v = A^T Y_0 - C$

Result: If  $C \cdot X_0 = b \cdot Y_0$  then  $Z_{\max} = C \cdot X_0 = b \cdot Y_0 = Z_{\min}$   
Optimum sol<sup>n</sup>

Result: If  $Z_{\max}$  is unbounded above, then  $Z_{\min}$  is not feasible

If  $Z_{\min}$  is unbounded ~~above~~ below, then  $Z_{\max}$  is not feasible

### The Duality Theorem

Theorem: If either (max) or (min) has a bounded optimum solution so does the other and  $Z_{\max} = Z_{\min}$

\* Optimum Coeff. for slack variables = optimum pt. for dual

↳ At optimum Sol<sup>n</sup>





3 cases:

- ① Optimum Sol<sup>n</sup>  $\rightarrow C_{no} = b_{yo}$
- ② Not Bounded  $\rightarrow$  Not feasible
- ③ No feasible sol<sup>n</sup>

Ex<sup>n</sup> Max  $Z = -5x_1 + 18x_2 + 6x_3 - x_4$   
Subject to

$$2x_1 - x_3 + 3x_4 \leq 20 \quad \rightarrow y_1$$

$$x_2 - 2x_3 - x_4 \leq 30 \quad \rightarrow y_2$$

$$-3x_1 + 6x_2 + 3x_3 + 4x_4 \leq 24 \quad \rightarrow y_3$$

$$x_i \geq 0 \quad i = 1, \dots, 4$$

Sol<sup>n</sup> Dual:

$$\text{Min } Z = 20y_1 + 30y_2 + 24y_3$$

subject to

$$2y_1 - 3y_3 \geq -5$$

$$y_2 + 6y_3 \geq 18$$

$$-y_1 - 2y_2 + 3y_3 \geq 6$$

$$3y_1 - y_2 + 4y_3 \geq -1$$

$$y_i \geq 0$$

Sol<sup>n</sup> is  $Z = 112$   
at  $(20, 3)$

\* Weal ~~need~~ not solve both, we can solve any one of them & we get the sol<sup>n</sup> of other in table only.

After solving and adding slack variables we get

$$Z_{\min} = -112$$

$$Z_{\max} = 112$$

$$\text{at } (10, 9, 0, 0)$$





3 cases:

- ① Optimum Sol<sup>n</sup>  $\rightarrow$  Cno = byo
- ② Not Bounded  $\rightarrow$  Not feasible
- ③ No feasible sol<sup>n</sup>

Exn- Max  $Z = -5x_1 + 18x_2 + 6x_3 - x_4$   
Subject to

$$2x_1 - x_3 + 3x_4 \leq 20 \quad \rightarrow y_1$$

$$x_2 - 2x_3 - x_4 \leq 30 \quad \rightarrow y_2$$

$$-3x_1 + 6x_2 + 3x_3 + 4x_4 \leq 24 \quad \rightarrow y_3$$

$$x_i \geq 0 \quad i=1, \dots, 4$$

Sol<sup>n</sup>

Dual:

$$\text{Min } Z = 20y_1 + 30y_2 + 24y_3$$

subject to

$$2y_1 - 3y_3 \geq -5$$

$$y_2 + 6y_3 \geq 18$$

$$-y_1 - 2y_2 + 3y_3 \geq 6$$

$$3y_1 - y_2 + 4y_3 \geq -1$$

$$y_i \geq 0$$

Sol<sup>n</sup> is  $Z = 112$   
at  $(20, 3)$

After solving  
and adding  
slack variables  
we get

$$Z_{\min} = -112$$

$$Z_{\max} = 112$$

$$\text{at } (10, 9, 0, 0)$$

\* Weal ~~need~~ not solve  
both, we can solve  
any one of them  
& we get the sol<sup>n</sup> of other  
in table only.





Primal Sol<sup>n</sup>

Introducing Slack Variables

$$2x_1 - x_3 + 3x_4 + x_5 = 20$$

$$x_2 - 2x_3 - x_4 + x_6 = 30$$

$$-3x_1 + 6x_2 + 3x_3 + 4x_4 + x_7 = 24$$

$$x_i \geq 0 \quad ; \quad i = 1, 2, \dots, 7$$

Sub~~st~~

$$\text{Max } Z = -5x_1 + 18x_2 + 6x_3 - x_4$$

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	b
$x_5$	2	0	-1	3	1	0	0	20
$x_6$	0	1	-2	-1	0	1	0	30
$x_7$	-3	6	3	4	0	0	1	24
Z	-5	18	6	-1	0	0	0	0
$x_5$	2	0	-1	3	1	0	0	20
$x_6$	1/2	0	-5/2	-5/3	0	1	7/6	26
$x_2$	-1/2	1	1/2	2/3	0	0	1/6	4
Z	4	0	-3	-13	0	0	-3	-22
$x_1$	1	0	-1/2	3/2	1/2	0	0	10
$x_6$	0	0	-9/4	-29/12	-1/4	1	-1/6	21
$x_2$	0	1	1/4	17/12	1/4	0	1/6	9
Z	0	0	-1	-19	-2	0	-3	-112


No more the values

$$\therefore Z = 112 \text{ at } (10, 9, 0, 0)$$

Now for Dual, sol<sup>n</sup>

$$\text{is } Z = 112 \text{ at } (2, 0, 3)$$



As we can see, we can solve any of the two (primal or dual) & can get the solution of other. 

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Dual Sol<sup>n</sup>

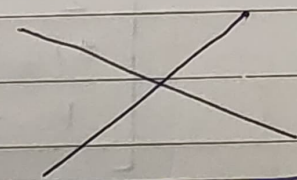
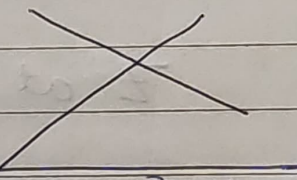
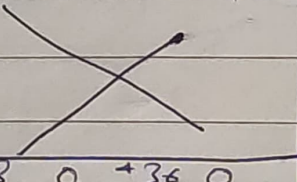
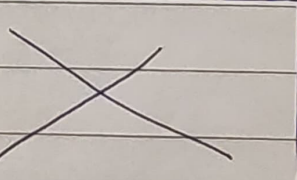
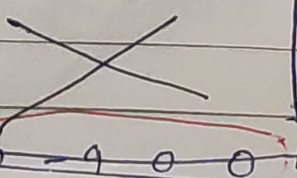
$$\text{Min } Z = 20y_1 + 30y_2 + 24y_3 \text{ sub. to}$$

$$2y_1 - 3y_3 - y_4 + A_1 = -5$$

$$y_2 + 6y_3 + y_5 + A_2 = 18$$

$$-y_1 - 2y_2 + 3y_3 - y_6 + A_3 = 6$$

$$3y_1 - y_2 + 4y_3 - y_7 + A_4 = 1$$

	$n_1$	$n_2$	$n_3$	$n_4$	$n_5$	$n_6$	$n_7$	$n_8$	$n_9$	$n_{10}$	$n_{11}$	$b$
$n_8$	2	0	-3	-1	0	0	0	1	0	0	0	-5
$n_9$	0	1	6	0	-1	0	0	0	1	0	0	18
$n_{10}$	-1	-2	3	0	0	-1	0	0	0	0	0	6
$n_{11}$	3	-1	4	0	0	0	-1	0	0	0	1	1
$Z$	20	30	24	0	0	0	0	0	0	0	0	0
$w$	-4	2	-10	1	1	1	1	0	0	0	0	20
$n_8$	17/4	-3/4	0	-1	0	0	-3/4					17/4
$n_9$	-9/2	5/2	0	0	-1	0	3/2					33/2
$n_{10}$	-13/4	-5/4	0	0	0	-1	3/4					21/4
$n_3$	3/4	-1/4	1	0	0	0	1/4					1/4
$Z$	2	36	0	0	0	0	6	0	0	0	-6	-6
$w$	7/2	-1/2	0	1	1	1	-3/2	0	0	0	5/2	-31/2
$n_8$	1	-2	0	-1	0	-1	0					1
$n_9$	2	5	0	0	-1	2	0					6
$n_7$	-13/3	+5/3	0	0	0	-4/3	1					7
$n_3$	-1/3	-2/3	1	0	0	-1/3	0					2
$Z$	28	46	0	0	0	8	0	0	0	-8	0	-18
$w$	-3	-3	0	1	1	-1	0	0	0	2	1	-7
$n_1$	1	-2	0	-1	0	-1	0					1
$n_9$	0	9	0	2	-1	4	0					4
$n_7$	0	-5/3	0	-1/3	0	-1/3	1					34/3
$n_3$	0	-4/3	1	-1/3	0	-2/3	0					7/3
$Z$	0	102	0	28	0	36	0	-28	0	+36	0	-26
$w$	0	-9	0	-2	1	-4	0	3	0	5	1	-4
$n_1$	1	0	0	-5/9	-2/9	-1/9	0					17/9
$n_2$	0	1	0	2/9	-1/9	4/9	0					4/9
$n_2$	0	0	0	-5/27	-3/27	-2/27	1					430/27
$n_3$	0	0	1	-1/27	-4/27	-2/27	0					79/27
$Z$	0	0	0	16/3	34/3	-28/3	0	-16/3	-34/3	28/3	0	-364/3
$w$	0	0	0	0	0	0	0	1	1	1	1	0
$n_1$	1	1/4	0	-1/2	-1/4	0	0					2
$n_6$	0	9/4	0	1/2	-1/4	1	0					1
$n_7$	0	29/12	0	-3/2	-17/12	0	1					17
$n_3$	0	1/6	1	0	-1/6	0	0					3
$Z$	0	21	0	10	9	0	0	-10	-9	0	0	-112





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Ex.  $X_0 = (0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3})$   
 $Y_0 = (3\frac{1}{2}, 2, 1\frac{1}{2})^T$

$$\text{Min } Z = 7x_1 + 11x_2 - 3x_3 - x_4$$

subject to  $2x_1 + 2x_2 - x_3 - 3x_4 \geq 2$   
 $-x_1 + 5x_2 - 2x_3 + x_4 \geq 12$   
 $x_1 - 4x_2 + 3x_3 + 5x_4 \geq 4$   
 $x_i \geq 0$

Sol<sup>n</sup> (1)  $X_0$  is feasible when  
 $Ax_0 \geq b$

$$\begin{bmatrix} 2 & 2 & -1 & -3 \\ -1 & 5 & -2 & 1 \\ 1 & -4 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 5\frac{2}{3} \\ 8\frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 4 \end{bmatrix} = b$$

$X_0 = \text{feasible sol}^n$

$$Z \text{ at } X_0 = C \cdot X_0 = 37$$

(2)  $A^T Y_0 \leq C \Rightarrow \begin{bmatrix} 6.5 \\ -11/3 \\ -1 \end{bmatrix} \leq C$

$Y_0 = \text{feasible}$

$$Z \text{ at } Y_0 = b \cdot Y_0 = 37$$