15 February, 2021

| $A$ | $b$ |
|---|---|
| $c$ | $z_0$ |

(I)

$\longrightarrow$

| $A^*$ | $b^*$ |
|---|---|
| $c^*$ | $z_0^*$ |

(F)

$$B = [A^{J_1} \mid A^{J_2} \mid \cdots \mid A^{J_n}]$$
$$C_B = [c_{j_1}, c_{j_2}, \cdots c_{j_n}]$$

$$A^* = B^{-1}A$$
$$b^* = B^{-1}b$$
$$c^* = c - C_B A^*$$
$$z_0^* = z_0 - C_B b^*$$

Ex: Min $Z = 5x_1 + 3x_4 - 2x_5 + 21$

subject to

$$-6x_1 + x_3 - 2x_4 + 2x_5 = 6$$
$$-3x_1 + x_2 + 6x_4 + 3x_5 = 12$$
$$x_j \geq 0$$

Soln

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| $x_3$ | $-6$ | $0$ | $1$ | $-2$ | $2$ | $6$ |
| $x_2$ | $3$ | $1$ | $0$ | $6$ | $3$ | $15$ |
| $z$ | $5$ | $0$ | $0$ | $3$ | $-2$ | $-21$ |

Direct Final Table

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_5$ | $0$ | $1/2$ | $-1/4$ | $7/2$ | $1$ | $6$ |
| $x_1$ | $1$ | $1/6$ | $-1/4$ | $3/2$ | $0$ | $1$ |
| | $0$ | $1/6$ | $3/4$ | $5/2$ | $0$ | $-14$ |

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$B = [A^5, A^1] = \begin{bmatrix} 2 & -6 \\ 3 & -3 \end{bmatrix}$

$c_B = \begin{bmatrix} -2 & 5 \end{bmatrix}$

$A = \begin{bmatrix} -6 & 0 & 1 & -2 & 2 \\ -3 & 1 & 0 & 6 & 3 \end{bmatrix}$

$A^* = \begin{bmatrix} 0 & 1/2 & -1/4 & 7/2 & 1 \\ 1 & 1/6 & -1/4 & 3/2 & 0 \end{bmatrix}$

$b^* = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ $\qquad b = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$

$z_0 = -21$ $\qquad z_0^* = -14$

$B^{-1}A = \frac{1}{12} \begin{bmatrix} -3 & 6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -6 & 0 & 1 & -2 & 2 \\ -3 & 1 & 0 & 6 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1/2 & -1/4 & 7/2 & 1 \\ 1 & 1/6 & -1/4 & 3/2 & 0 \end{bmatrix}$

$= A^*$

lly $B^{-1} b = \frac{1}{12} \begin{bmatrix} -3 & 6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 15 \end{bmatrix} = b^*$

$c - c_B A^* = \begin{bmatrix} 5 & 0 & 0 & 3 & -2 \end{bmatrix}$

$- \begin{bmatrix} -2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & -1/4 & 7/2 & 1 \\ 1 & 1/6 & -1/4 & 3/2 & 0 \end{bmatrix}$

$= c^*$

lly, $z_0^* = z_0 - c_B b^*$

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## Change In Objective $f^n$

$A^* = B^{-1}A$

$b^* = B^{-1}b$

$c^* = c - c_B A^*$

$Z_0^* = Z_0 - c_B b^*$

$x^*$ = old LP problem

If one of $c_j$ is changed how optimum sol$^n$ will change?

$c^* = c - c_B A^* \geq 0$

then $Z_0^*$ is optimum

- Only $c_B$ will affect $Z_0^*$

Ex: $\max Z = 11x_1 + 4x_2 + x_3 + 15x_4$

subject to $3x_1 + x_2 + 2x_3 + 4x_4 \leq 28$

$8x_1 + 2x_2 - x_3 + 7x_4 \leq 50$

$x_i \geq 0$

Sol$^n$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $b$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 3 | 1 | 2 | 4 | 1 | 0 | 28 | Initial |
| $x_6$ | 8 | 2 | -1 | 7 | 0 | 1 | 50 | |
| | -11 | -4 | -1 | -15 | 0 | 0 | 0 | |
| | 1 | | | | | | | |
| | 1 | | | | | | | |
| $x_4$ | -2 | 0 | 5 | 1 | 2 | -1 | 6 | Final |
| $x_2$ | 11 | 1 | -18 | 0 | -7 | 4 | 4 | |
| | 3 | 0 | 2 | 0 | 2 | 1 | 106 | |

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$c_1 = -11$ is changed
$x_1$ is not a basic variable
$\Rightarrow$ $C_B$ is not changing
$\therefore$ Optimal value is not changing
$\therefore$ $z_0^*$ will remain same.

~~c is indirectly affecting~~ $z_0^*$. If a negative value comes in $c^*$ $\because$ of change in $c_i$, then the objective $f^n$ can further be optimized.

If there is no negative value in $c^*$ $\because$ of change in $c$.

$c_1 = -11$ is changed
This will change $c_1^*$ in (F)

$$c_1^* = c_1 - C_B \cdot A[1]$$

$$= c_1 - \begin{bmatrix} -15 & -4 \end{bmatrix} \begin{bmatrix} -2 \\ 11 \end{bmatrix}$$

$$c_1^* = c_1 + 14$$

or $c_1^* \geq 0$

$c_1 + 14 \geq 0$

$\underbrace{c_1 \geq -14}$

As long as this is satisfying $c^* \geq 0$. & $\therefore$ we don't need to pivot further.

$\rightarrow$ If change $\leq 3$
$\Rightarrow$ $x^*$ is unchanged $\Rightarrow$ $z_0^*$ is unchanged.
Otherwise more pivot steps are needed.

~~Hly for $x_3$.~~

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$c_4 \longrightarrow$ into $c_4 + \lambda$ ($x_4$ is Basic)
$C_B$ is changed
$C_B = [-15 \quad -4]$ So

$c^* = c - C_B A^*$
$= [-11 \quad -4 \quad -1 \quad -15-\lambda \quad 0 \quad 0] - [-15-\lambda \quad -4] A^*$

$= [-11 \quad -4 \quad -1 \quad -15 \quad 0 \quad 0] - [-15 \quad -4] A^*$

$+ [0 \quad 0 \quad 0 \quad -\lambda \quad 0 \quad 0] + [\lambda \quad 0] A^*$

$= [3 \quad 0 \quad 2 \quad 0 \quad 2 \quad 1] + [-2\lambda \quad 0 \quad 5\lambda \quad 0 \quad 2\lambda \quad -\lambda]$

$c^* = [3-2\lambda \quad 0 \quad 2+5\lambda \quad 0 \quad 2+2\lambda \quad 1-\lambda]$

$c^* \geq 0 \longrightarrow$ No Entering Variable (No need of pivoting)
$c^* < 0 \rightarrow$ Need to pivot further

for $c^* \geq 0$
$3 - 2\lambda \geq 0 \Rightarrow \lambda \leq 1.5$
$2 + 5\lambda \geq 0 \Rightarrow \lambda \geq -0.4$
$2 + 2\lambda \geq 0 \Rightarrow \lambda \geq -1$
$1 - \lambda \geq 0 \Rightarrow \lambda \leq 1$

$\therefore -0.4 \leq \lambda \leq 1$

If this satisfies optimal point same, $X^* = (0, 4, 0, 6)$
$z_0^* = c^* X^*$
$= 106 + 6a$

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$Z_0^* = C^* X^* = 106 + 16a$

or $Z_0^* = Z_0 - C_B b^*$

$c^* < 0 \rightarrow$ pivot

If only $c_i$ is changed to $c_i + a$

(a) If $x_i$ is not basic in final table

→ then $Z_0^*$ will be same as long as $c_i^* \geq 0$

→ Find the allowed range of change by $c_i^* = c_i - C_B A^*(i)$

→ If $c_i^* < 0 \rightarrow$ pivot to find the optimum soln

(b) If $x_i$ is Basic variable

→ $Z_0^*$ will be changed

→ allowed range by $c^* = c - C_B A^* \geq 0$

$Z_0^* = C^* X^*$

## Adding A New Variable

Min $Z = C \cdot X + C_{n+1} X_{n+1}$

subject to $[A, A^{n+1}] \begin{bmatrix} X \\ X_{n+1} \end{bmatrix} = b$

If $X_{n+1} = 0$ then $\begin{bmatrix} X^* \\ 0 \end{bmatrix}$ is a basic feasible $sol^n$

Is our $sol^n$ optimal?

$$\begin{array}{c|c} A \; A^{n+1} & b \\ \hline C \; C_{n+1} & Z_0 \end{array} \longrightarrow \begin{array}{c|c} A^* \; A^{*n+1} & b^* \\ \hline C^* \; C^*_{n+1} & Z_0^* \end{array}$$

If $c^*_{n+1} \geq 0 \rightarrow \begin{bmatrix} X^* \\ 0 \end{bmatrix}$ is optimum.

$$c^*_{n+1} = c_{n+1} - c_B B^{-1} A^{(n+1)}$$

If $c^*_{n+1}$, we need more pivoting.

Added a new variable

Same Example

Ex: $\text{Max } Z = 11x_1 + 4x_2 + x_3 + 15x_4 + 12x_7$

subject to

$$3x_1 + x_2 + 2x_3 + 4x_4 + 12x_7 \leq 28$$
$$8x_1 + 2x_2 - x_3 + 7x_4 + 5x_7 \leq 50$$
$$x_i \geq 0$$

Add $x_5, x_6$ as slack variables

Soln

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | b |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 3 | 1 | 2 | 4 | 1 | 0 | 3 | 28 |
| $x_6$ | 8 | 2 | -1 | 7 | 0 | 1 | 5 | 50 |
| | -11 | -4 | -1 | -15 | 0 | 0 | 0 | 0 |
| | 1 | | | | | | | |
| | 1 | | | | | | | |
| $x_4$ | -2 | 0 | 5 | 1 | 2 | -1 | -1 | 6 |
| $x_2$ | 11 | 1 | -18 | 0 | -7 | 4 | 1 | 4 |
| | 3 | 0 | 2 | 0 | 2 | 1 | 1 | 106 |

calculated using formulas

$$B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$$

$$c_B = [-15 \quad -4]$$

$$A^{*(7)} = B^{-1} A^{(7)}$$

$$= \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$C_7^* = C_7 - C_B A^{*(7)}$$

$$= -12 - [-15 \ -4] \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= -12 - (-11)$$

$$= -1 < 0$$

$\begin{bmatrix} X^* \\ 0 \end{bmatrix}$ is not optimum as $x_7$ can be entered as it is negative.

Initial Table

(Final Table)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | -2 | 0 | 5 | 1 | 2 | 1 | 1 | 6 |
| $x_2$ | 11 | 1 | -18 | 0 | -7 | -4 | -1 | 4 |
| | 3 | 0 | 2 | 0 | 2 | 1 | -1 | |
| $x_7$ | -2 | 0 | 5 | 1 | 2 | -1 | 1 | 6 |
| $x_2$ | 9 | 1 | -13 | 1 | -5 | 3 | 0 | 10 |
| | 1 | 0 | 7 | 1 | 4 | 0 | 0 | 112 |

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Consider the Dual :

If we add a new variable in primal

↳ Then ~~add~~ a new constraint will be added

Adding $x_7$ → New constraint in Dual

$$3y_1 + 5y_2 \geq 12$$

$Y^* = (2, 1)$ → From primal (Slack variables)

Is our $Y^*$ feasible for new constraint ?

↳ $11 \not\geq 12$ (No) → $Y^*$ Not optimal & not feaible

$x_7$ will enter

No

$x_7$ will enter

($Y^*$ not optimal)

Yes

(optimal Soln → $Y^*$)