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Assignment - 1

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Ques - Verify

(i) $A^* = B^{-1} A$

(ii) $b^* = B^{-1} b$

(iii) $C - C_B B^{-1} A = C^*$

(iv) $Z_0^* = Z_0 - C_B B^{-1} b$

Soln

The initial tableau of an LPP model in standard form can be represented as

A	b
C	Z_0

IIly, final tableau of an LPP model can be represented as

A^*	b^*
C^*	Z_0^*

$AX = b$ can be denoted in vector eqn as

$$\sum_{j=1}^{n} x_j A^{(j)} = b \qquad \left\{ \begin{array}{l} A^{(j)} \text{ denote the} \\ j^{th} \text{ column of } A \end{array} \right.$$

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Ily, $\sum_{j=1}^{n} x_j^* A^*(j) = b^*$

Now,
we will prove $b^* = B^{-1} b$

$AX = b$ & $A^* X = b^*$ are equivalent & have solution set.

Let $X^* = [x_1^*, x_2^*, x_3^* \cdots x_n^*]^T$
where

$$x_j^* = \begin{cases} b_k^*, & x_j \text{ is the basic variable in } k^{th} \text{ Eq}^n \text{ of final tableau} \\ 0 & \text{Non-Basic Variable} \end{cases}$$

So, X^* is the basic feasible soln of the final tableau.

Now, first we define B
$B = [A(j(1)), A(j(2)) \cdots A(j(m))]$
for the basic variables
$x_{j(1)}, x_{j(2)} \cdots x_{j(n)}$

Now, as X^* is the basic feasible soln
and $A^* X = b^*$
$\therefore$ $AX^* = b$ → In vector notation it is written as

$$\sum_{k=1}^{m} A(j(k))\, b_k^* = b$$

$$\Rightarrow \underbrace{\sum_{k=1}^{m} A(j(k))}_{B} \underbrace{\sum_{k=1}^{m} b_k^*}_{b^*} = b$$

$\therefore Bb^* = b$

$\therefore b^* = B^{-1}b$

<u>Now, proving $A^*(j) = B^{-1}A(j)$</u>

In the above proof, we considered an initial column vector b. $[X] \rightarrow$

So, Let the initial column vector be $A(j)$.

$\therefore b^*$ would simply be $A^*(j)$, the result of sequence of pivot steps on $A(j)$.

$\downarrow$

$AX = A(j)$

$\therefore$ We get $A^*(j) = B^{-1}A(j)$

<u>Now, proving $z_0^* = z_0 - c_B B^{-1} b$</u>

So Now, some notations,

$c_B = [c_{j(1)}, c_{j(2)} \ldots c_{j(m)}]$ for the basic variables $x_{j(1)}, \ldots x_{j(m)}$.

So, the initial problem is to minimize z with $z = -z_0 + c_1x_1 + c_2x_2 + \ldots + c_nx_n$ $= F(X)$

subject to $AX = b$ $X \geq 0$

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After pivot operations this problem is equivalent to minimizing

$$Z = -Z_0^* + c_1^* x_1 + c_2^* x_2 + \dots + c_n^* x_n$$

subject to $A^* X = b^*$ $\quad X \geq 0$

$= G(X)$

Now, $F(X)$ & $G(X) \rightarrow$ have same value

let ~~the~~ the sol^n of system of constraints be defined as U

$\therefore F(U) = G(U)$

$$\therefore -Z_0 + C_B b^* = -Z_0^*$$

$$\Rightarrow Z_0^* = Z_0 - C_B b^*$$

Now, proving $c^* = c - C_B A^*$

Now in above problem let b is $A^{(j)}$ and initial constant Z_0 is c_j

$\therefore b^*$ would be $A^{*(j)}$ and the resulting Z_0^* would be c_j^*

$\Rightarrow c_j^* = c_j - C_B A^{*(j)}$ for the model of minimizing Z with

$$Z = -c_j + c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to $AX = A^{(j)}$, $X \geq 0$

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$\therefore$ We have the following results :

- $A^* = B^{-1}A$
- $b^* = B^{-1}b$
- $C^* = C - C_B B^{-1} A = C - C_B A^*$
- $Z_0^* = Z_0 - C_B B^{-1} b = Z_0 - C_B b^*$

$\therefore$ The final tableau can be written as

$B^{-1}A$	$B^{-1}b$
$C - C_B B^{-1} A$	$Z_0 - C_B B^{-1} b$

Let's verify these results using an example :

Minimize $Z = x_1 + 4x_2 + 3x_3$

subject to

$$x_1 + 2x_2 + x_4 = 20$$
$$2x_1 + x_2 + x_3 = 10$$
$$-x_1 + 4x_2 - 2x_3 + 3x_4 = 40$$
$$x_i \geq 0 \text{ for } i = 1 - 4$$

Now, we need to introduce Artificial Variables

$$x_1 + 2x_2 + x_4 + x_5 = 20$$
$$2x_1 + x_2 + x_3 + 1x_6 = 10$$
$$-x_1 + 4x_2 - 2x_3 + 3x_4 + 1x_7 = 40$$
$$x_i \geq 0$$

x_5, x_6, x_7 are A.V.

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Tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
x_5	1	2	0	1	1	0	0	20
x_6	2	1	1	0	0	1	0	10
x_7	-1	4	-2	3	0	0	1	40
Z	1	4	3	0	0	0	0	0
w	-2	-7	1	-4	0	0	0	-70
x_2	1/2	1	0	1/2	1/2	0	0	10
x_6	3/2	0	1	-1/2	-1/2	1	0	0
x_7	-3	0	-2	1	-2	0	1	0
Z	-1	0	3	-2	-2	0	0	-40
w	3/2	0	1	-1/2	7/2	0	0	0
x_2	2	1	1	0	0	1	0	10
x_4	-3	0	-2	1	1	-2	0	0
x_7	0	0	0	0	-3	2	1	0
Z	-7	0	-1	0	0	-4	0	-40
w	0	0	0	0	4	-1	0	0
x_1	1	1/2	1/2	0	0	1/2	0	5
x_4	0	3/2	-1/2	1	1	-1/2	0	15
x_7	0	0	0	0	-3	2	1	0
Z	0	7/2	5/2	0	0	-1/2	0	-5
w	0	0	0	0	4	-1	0	0

$Z = 5$ at $(5, 0, 0, 15)$

Now, we will verify the results.

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$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 4 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 3/2 & -1/2 & 1 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -3 & 2 & 1 \end{bmatrix}$$

$$b^* = \begin{bmatrix} 5 \\ 15 \\ 0 \end{bmatrix} \qquad z_0 = 0, \quad z_0^* = -5$$

$$c = \begin{bmatrix} 1 & 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$c^* = \begin{bmatrix} 0 & 7/2 & 5/2 & 0 & 0 & -1/2 & 0 \end{bmatrix}$$

Now,

$$B = [A^1, A^4, A^7] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

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$$B^{-1} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1/2 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

Now, I result is $A^* = B^{-1}A$

$$\therefore B^{-1}A = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1/2 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 4 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 3/2 & -1/2 & 1 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -3 & 2 & 1 \end{bmatrix}$$

$= A^*$

$$\boxed{\therefore B^{-1}A = A^*}$$

Now, $b^* = B^{-1}b$

$$B^{-1}b = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1/2 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 0 \end{bmatrix} = b^*$$

$$\boxed{\therefore B^{-1}b = b^*}$$

Now, $c^* = c - c_B B^{-1} A$

$$= \begin{bmatrix} 1 & 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & -1/2 & 0 \\ -3 & 2 & 1 \end{bmatrix}^{-1} A$$

$$= \begin{bmatrix} 1 & 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 3/2 & -1/2 & 1 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & -3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 7/2 & 5/2 & 0 & 0 & -1/2 & 0 \end{bmatrix}$$

$$= C^*$$

$$\therefore \boxed{C^* = C - C_B A^*}$$

Now, proving $Z_0^* = Z_0 - C_B b^*$

$$Z_0 - C_B b^* = 0 - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 15 \\ 6 \end{bmatrix}$$

$$= -5$$

$$= Z_0^*$$

$$\therefore \boxed{Z_0^* = Z_0 - C_B b^*}$$