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The Complimentary Slackness Theorm

max $z = cx$ subj. to $Ax \leq b$, $x \geq 0$

min $Z = b \cdot y$ " " $A^T y \geq c$, $y \geq 0$

Cor-2: Let X_0, Y_0 be feasible sol^n for (max) (min) resp. then

$$bY_0 - cX_0 = (b - Ax_0) \cdot Y_0 + (A^T Y_0 - c) x_0$$

Cor-3: If $\cancel{b} Y_0 = c X_0 \Rightarrow X_0, Y_0$ are optimal sol^n

Let $\cancel{*}$ X^* & Y^* be optimal sol^n instead of X_0 & Y_0

Now,

$$\underbrace{\underset{\geq 0}{(b - AX^*)} \underset{\geq 0}{y^*}}_{\geq 0} + \underbrace{\underset{\geq 0}{(A^T y^* - c)} \underset{\geq 0}{X^*}}_{\geq 0} = 0$$

$$\Rightarrow (b - AX^*) y^* = 0$$

$$(A^T y^* - c) X^* = 0$$

Let $b - AX^* = u = (u_1, u_2 \ldots u_m) \geq 0$

$A^T y^* - c = v$

$y^* = (y_1^*, y_1^* \ldots y_m^*) \geq 0$

$$u \cdot y^* = \underset{\geq 0}{u_1 y_1^*} + \underset{\geq 0}{u_2 y_2^*} + \cdots + \underset{\geq 0}{u_m y_m^*} = 0$$

$\therefore \; u_i y_i^* = 0 \quad$ for $i = 1, \ldots, m$

→ for each i, we must have atleast one of u_i ; y_i^* is 0.

If $u_i \neq 0 \Rightarrow y_i^* = 0$

If $y_i^* \neq 0 \Rightarrow u_i = 0$

Ex. Let Dual Sol^n be $y^* = (1, 0, 3, 4)$

- 1 → first slack in Primal must be 0
- 0 → $u_2 \geq 0$
- $3, 4$ → Must be 0 (slack) in Primal

Statement : Suppose X_0 & Y_0 are feasible sol^{ns} of max & min problem respectively, then X_0 & Y_0 are optimal sol^{ns} if and only if

$$\{(b - A x_0) \cdot Y_0\} = 0$$

and $(A^T Y_0 - C) \cdot X_0 = 0$

Ex: $X^* = (7, 0, 3)$ is an optimal sol^n pt for :

min $z = 12x_1 + 5x_2 + 10x_3$ subject to

$x_1 - x_2 + 2x_3 \geq 10$

$-3x_1 + x_2 + 4x_3 \geq -9$, $7x_1 - x_2 - 5x_3 \geq 34$

$x_i \geq 0 \qquad -x_1 + 2x_2 + 3x_3 \geq 1$, $2x_1 - 3x_2 \geq -2$

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Find the optimal solⁿ point for the dual.

Solⁿ Dual:

$$\text{Max } Z = 10y_1 \cancel{+} -9y_2 + y_3 - 2y_5 + 34y_5$$

subject to

$$\cancel{12y_1 + 1y_2} \quad y_1 - 3y_2 \cancel{+} - y_3 + 2y_4 + 7y_5 \le 1$$

$$-y_1 + 2y_2 + 2y_3 - 3y_4 - y_5 \le 5$$

$$2y_1 + 4y_2 + 3y_3 \quad - 5y_5 \le 10$$

$$y_i \geq 0 \qquad i = 1, --- 5$$

$$x^* = (7, 0, 3)$$

Using Complimentary Slackness Theorm

$$x^* = (7, 0, 3)$$

$$(7 - 0 + 6) - 10 = 3$$

$$(-3\cdot 7 + 4\cdot 3) - (-9) = 0$$

$$(-7 + 9) - 1 = 1$$

$$(14) - (-2) = 16$$

$$(49 - 15) - 34 = 0$$

So, $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 16 \\ 0 \end{bmatrix}$

$$u \cdot x^* = 0 \quad \Rightarrow \quad \begin{pmatrix} 3 \\ 0 \\ 1 \\ 16 \\ 0 \end{pmatrix} \cdot x^* = 0$$

Now, where u_i is not zero, y_i^* will be zero

$\therefore\ y_1^* = 0,\ y_3^* = 0,\ y_4^* = 0$

Now, for Dual :

$X^* = (7, 0, 3)$

$v \cdot X^* = 0 \quad \Rightarrow v_1 = v_3 = 0$

$10y_1 - 9y_2 + y_3 - 2y_4 + 34y_5 = Z$

subj. to

$y_1 - 3y_2 - y_3 + 2y_4 + 7y_5 \le 12$ (=)

$\cancel{-y_1 + 2y_2 + 2y_3 - 3y_4 - y_5 \le 5}$ → No Info, neglect this eq^n

$2y_1 + 4y_2 + 3y_3 - 5y_5 \le 10$ (=)

$y_i \ge 0$

$y_1 = y_3 = y_4 = 0$

$v_1 = v_3 = 0$

$$\left.\begin{array}{l} -3y_2^* + 7y_5^* = 12 \\ 4y_2^* - 5y_5^* = 10 \end{array}\right\} \quad \begin{array}{l} y_2^* = 10 \\ y_5^* = 6 \end{array}$$

$Y^* =$ opt sol^n of Dual

$= (0, 10, 0, 0, 6)$

max Z = min Z (Duality Theorem)

$$\left.\begin{array}{ll} \text{for } Y^* \Rightarrow & Z = 114 \text{ (max)} \\ X^* & Z = 114 \text{ (min)} \end{array}\right\} \therefore Y^* \text{ is optimal}$$

Qn: $x^* = (3, 0, 1, 0)$ check whether x^* is an optimal solⁿ of

$$\max Z = 9x_1 + 3x_2 + 5x_3 + 22x_4$$

subj. to

$$2x_1 - \cancel{3}x_2 + 2x_3 + 6x_4 \leq 8$$
$$5x_1 + 3x_2 + x_3 + 2x_4 \leq 16$$
$$4x_1 + x_2 - x_3 + 3x_4 \leq 12$$
$$x_i \geq 0$$

Solⁿ

I step] Always check if its feasible or not

① $2(3) - 0 + 2(1) + 0$

$= 8 \leq 8 \Rightarrow$ True

$15 + 1 = 16 \leq 16 \Rightarrow$ True

$12 - 1 = 11 \leq 12 \Rightarrow$ True

Satisfying $\Rightarrow x^*$ is feasible solⁿ

② slack

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow$ Found by subtracting the conditions

slack $= (b - Ax_0)$

Dual: $\min Z = 8y_1 + 16y_2 + 12y_3$ subj. to

$$2y_1 + 5y_2 + 4y_3 \geq 9$$
$$-y_1 + 3y_2 + y_3 \geq 3$$
$$2y_1 + y_2 - y_3 \geq 5$$
$$6y_1 + 2y_2 + 3y_3 \geq 22$$
$$y_i \geq 0$$

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If primal has finite optimal soln then dual also has finite optimal soln. & vice-versa

So, If (P) has finite optimal soln
∴ (D) " " " "

Let $Y^* = (y_1^*, y_2^*, y_3^*)$
By Complimentary Slackness Theorm (CST)

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} Y^* = 0 \quad \Rightarrow \quad y_3^* = 0$$

$$\& \quad X^* \begin{pmatrix} \text{slack of} \\ \text{Dual} \end{pmatrix} = 0$$

$$v_1 = v_3 = 0$$

Dual

$$\min Z = 8y_1 + 16y_2 + 12y_3$$

$$2y_1 + 5y_2 + 4y_3 \geq 9 \xrightarrow{} =$$

$$-y_1 + 3y_2 + y_3 \geq 3 \longrightarrow \text{Neglect (No Info)}$$

$$2y_1 + y_2 - y_3 \geq 5 \xrightarrow{} =$$

$$6y_1 + 2y_2 + 3y_3 \geq 22$$

$$y_i \geq 0$$

$$\left.\begin{aligned} 2y_1^* + 5y_2^* &= 9 \\ 2y_1^* + y_2^* &= 5 \end{aligned}\right\} \begin{aligned} y_1^* &= 2 \\ y_2^* &= 1 \end{aligned}$$

∴ $Y^* = (2, 1, 0) \rightarrow$ for Dual

Now, check whether Y^* is optimal or not

Now, ~~for Z =~~ first we check if Y^* is feasible or not

Checking if Y^* is feasible or not

~~$2(2) + 5(1) = 9 \geq 9 \rightarrow$ True~~

~~$-2 + 3 = 1 \geq 3 \rightarrow$ True false~~

$2(2) + 5(1) = 9 \geq 9 \rightarrow$ True

$-2 + 3 = 1 \geq 3 \rightarrow$ false

Not Satisfying

$\Rightarrow Y^*$ is not feasible

$\Rightarrow X^*$ is not an optimal point

$\because$ for optimal max = min

$\therefore X^*$ is not an optimal sol^n.

Solving LPP Using Excel

Qn: Pg 17 of Copy or
Qn. 2.34 ~~of~~ on Pg 40 of book

Process

	1	2	3	4	
M_A	160	100	200	75	4200
M_B	30	35	60	80	2000
Labour	8	10	6	12	450
X	35	45	70	0	1300
Y	55	42	0	90	2600
Cost	400	575	620	590	

~~Product Y can be bought at ₹18~~

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$\min Z = 400x_1 + 575x_2 + 620x_3 + 540x_4$ subj. to
$160x_1 + 100x_2 + 200x_3 + 75x_4 \le \cancel{80}\ 4200$
$30x_1 + 35x_2 + 60x_3 + 80x_4 \le 2000$
$8x_1 + 10x_2 + 6x_3 + 2x_4 \le 450$
$35x_1 + 45x_2 + 70x_3 \ge 1300$
$55x_1 + 42x_2 + 90x_4 \ge 2600$
$x_i \ge 0$

= SUMPRODUCT( )
=

SENSITIVITY ANALYSIS

$\min Z = 2x_1 + 3x_2 \qquad \le b_1$
$= c_1x_1 + c_2x_2 \qquad \le b_2$

Price fluctuates, so model changed.
or variable

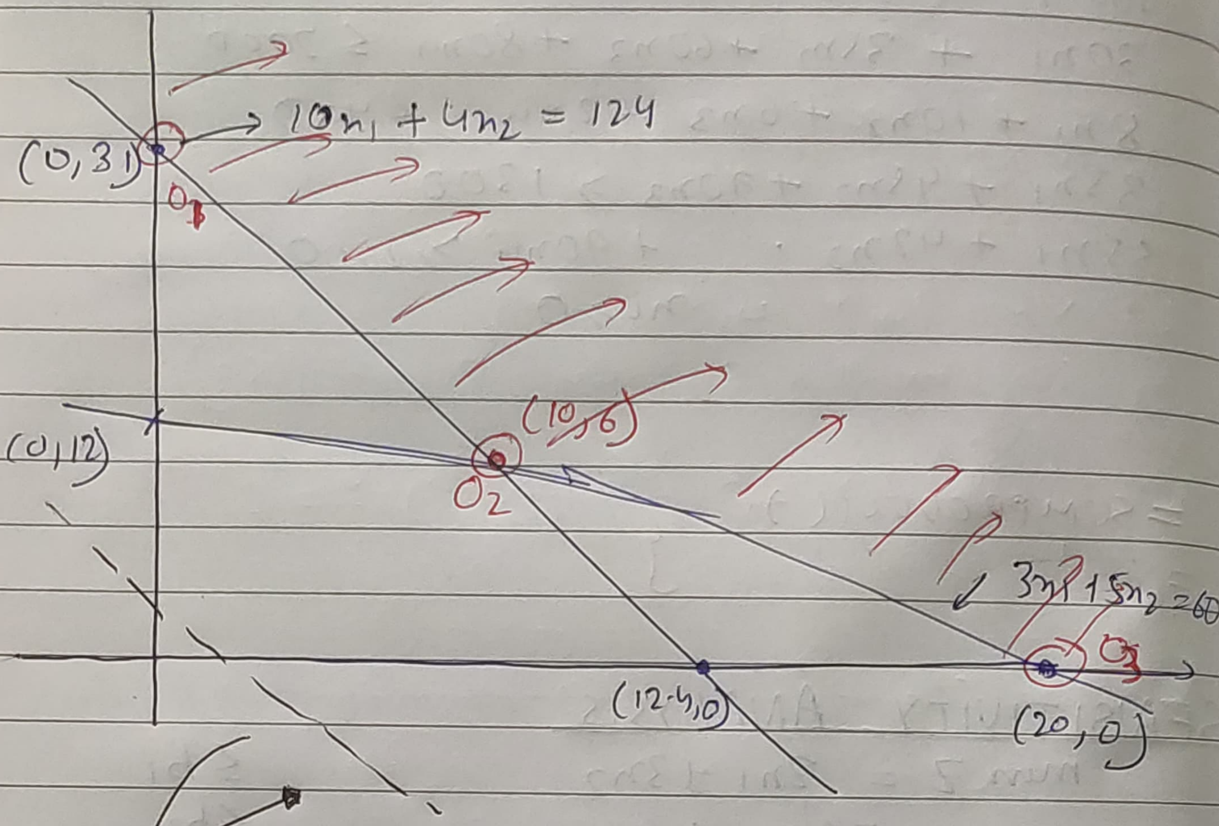
So, Sensitivity Analysis → how much to fluctuate the variables so that the same model is valid.

Qn:

	A	B	Price
Feed 1	10	3	16
Feed 2	4	5	14
Req	124	60	

$\min Z = 16x_1 + 14x_2$ subj. to
$10x_1 + 4x_2 \ge 124,\ 3x_1 + 5x_2 \ge 60$
$x_i \ge 0$

Graphically solving



Min at O_2 at (10,6)
$Z_{min} = 244$

moving of this line will be determined by slope

Z slope will determine the optimal points

Slope of 3 lines
$S_1 = -10/4$
$S_2 = -3/5$
$S_Z = -16/14 = -8/7$

$S_1 < S_Z < S_2 \Rightarrow O_2$ is optimal

Max $Z = c_1 x_1 + c_2 x_2$

Slope: $S_Z = \frac{-c_2}{c_1}$

If $S_Z \leq S_1$ (Steep slope) $\Rightarrow O_1$ is optimal

If $S_Z \geq S_2$ (Flat slope) $\Rightarrow O_3$ is optimal

$\frac{-c_1}{c_2} \geq \frac{-3}{5}$

$\Rightarrow \frac{c_1}{c_2} \leq \frac{3}{5}$

$\frac{-c_1}{c_1} \leq \frac{-10}{4}$

$\Rightarrow \frac{c_1}{c_2} \geq \frac{5}{2}$

$\Rightarrow O_1$ is optimal

If $S_1 \leq S_Z \leq S_2$

$\Rightarrow O_2$ is optimal