Linear Programming Problems

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SIMPLEX METHOD
(MATRIX FORM)
MATRIX ALGEBRA BASICS
REVISED SIMPLEX

Matrices and Matrix Operations

A matrix is a rectangular array of numbers. For example

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$$

is a 3x2 matrix (matrices are denoted by **Boldface Capital Letters**)

In more general terms,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \|a_{ij}\|$$

is an $m \times n$ matrix, where $a_{11}, \ldots, a_{mn}$ represent the numbers that are the elements of this matrix; $\|a_{ij}\|$ is shorthand notation for identifying the matrix whose element in row i and column j is a_{ij} for every $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Matrix Operations

Let $\mathbf{A} = \|a_{ij}\|$ and $\mathbf{B} = \|b_{ij}\|$ be two matrices having the same number of rows and the same number of columns.

Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *equal* ($\mathbf{A} = \mathbf{B}$) if and only if *all* the corresponding elements are equal ($a_{ij} = b_{ij}$ for all i and j).

<u>To multiply a matrix by a number (denote this number by k)</u>

$$k\mathbf{A} = \|ka_{ij}\|$$

For example,

$$3\begin{bmatrix} 1 & \frac{1}{3} & 2 \\ 5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 6 \\ 15 & 0 & -9 \end{bmatrix}$$

<u>To add two matrices **A** and **B**</u>

$$\mathbf{A} + \mathbf{B} = \|a_{ij} + b_{ij}\|$$

For example,

$$\begin{bmatrix} 5 & 3 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 7 \end{bmatrix}$$

Subtraction of two matrices

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B},$$

so that

$$\mathbf{A} - \mathbf{B} = \|a_{ij} - b_{ij}\|.$$

For example,

$$\begin{bmatrix} 5 & 3 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -2 & 5 \end{bmatrix}$$

Matrix multiplication

Matrix multiplication **AB** is defined if and only if the *number of columns* of **A** equals the *number of rows* of **B**.

Thus, if **A** is an $m \times n$ matrix and **B** is an $n \times s$ matrix, then their product is

$$\mathbf{AB} = \left\| \sum_{k=1}^{n} a_{ik} b_{kj} \right\|,$$

where this product is an $m \times s$ matrix. However, if **A** is an $m \times n$ matrix and **B** is an $r \times s$ matrix, where $n \neq r$, then **AB** is not defined.

To illustrate matrix multiplication,

$$\begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1(3)+2(2) & 1(1)+2(5) \\ 4(3)+0(2) & 4(1)+0(5) \\ 2(3)+3(2) & 2(1)+3(5) \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 12 & 4 \\ 12 & 17 \end{bmatrix}.$$

On the other hand, if one attempts to mutliply these matrices in the reverse order, the resulting product

$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 2 & 3 \end{bmatrix}$$

is not even defined.

Even when both **AB** and **BA** are defined,

$$\mathbf{AB} \neq \mathbf{BA}$$

in general.

<u>Matrix division is not defined</u>

Matrix operations satisfy the following laws.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}),$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC},$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C},$$

The relative sizes of these matrices are such that the indicated operations are defined.

Transpose operations

This operation involves nothing more than interchanging the rows and columns of the matrix.

Thus, for any matrix $\mathbf{A} = \|a_{ij}\|$, its transpose $\mathbf{A}^T$ is

$$\mathbf{A}^T = \|a_{ji}\|.$$

For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \\ 4 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 2 & 1 & 4 \\ 5 & 3 & 0 \end{bmatrix}.$$

Special kinds of matrices

Identity matrix

The identity matrix **I** is a square matrix whose elements are 0s except for 1s along the main diagonal. (A square matrix is one in which the number of rows is equal to the number of columns).

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{IA} = \mathbf{A} = \mathbf{AI},$$

where **I** is assigned the appropriate number of rows and columns in each case for the multiplication operation to be defined.

Null matrix

The null matrix **0** is a matrix of any size whose elements are all 0s .

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Therefore, for any matrix **A**,

$$\mathbf{A} + \mathbf{0} = \mathbf{A}, \qquad \mathbf{A} - \mathbf{A} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{0A} = \mathbf{0} = \mathbf{A0},$$

where **0** is the appropriate size in each case for the operations to be defined.

Inverse of matrix

(*a*) If $\mathbf{A}$ is nonsingular, there is a unique nonsingular matrix $\mathbf{A}^{-1}$, called the **inverse** of $\mathbf{A}$, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

(*b*) If $\mathbf{A}$ is nonsingular and $\mathbf{B}$ is a matrix for which either $\mathbf{AB} = \mathbf{I}$ or $\mathbf{BA} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$.

(*c*) Only nonsingular matrices have inverses.

To illustrate matrix inverses, consider the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}.$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Vectors

A special kind of matrix that plays an important role in matrix theory is the kind that has either a *single row* or a *single column*. Such matrices are often referred to as **vectors**. Thus

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]$$

is a **row vector**, and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(We use **boldface lowercase** letters to represent vectors)

is a **column vector**.

A **null vector 0** is either a row vector or a column vector whose elements are *all* 0s, that is,

$$\mathbf{0} = [0, 0, \ldots, 0] \qquad \text{or} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

One reason vectors play an important role in matrix theory is that any $m \times n$ matrix can be partitioned into either m row vectors or n column vectors, and important properties of the matrix can be analyzed in terms of these vectors.

Partitioning of matrices

Up to this point, matrices have been rectangular arrays of elements, each of which is a number. However, the notation and results are also valid if each element is itself a matrix.

For example, the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

may be written as

$$\mathbf{A} = [\mathbf{C}_1 \quad \mathbf{C}_2 \quad \mathbf{C}_3] \quad \text{where} \quad \mathbf{C}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \mathbf{C}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$\text{and} \quad \mathbf{C}_3 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

or as

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} \quad \text{where} \quad \mathbf{R}_1 = [a_{11} \quad a_{12} \quad a_{13}] \quad \text{and} \quad \mathbf{R}_2 = [a_{21} \quad a_{22} \quad a_{23}]$$

or as

$$\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2]$$

where

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

or where

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

The process of dividing a matrix into smaller matrices, or submatrices, is called partitioning and is usually denoted by a dotted line. The four partitions described would be denoted respectively as follows:

$$\left[\begin{array}{c:c:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]; \qquad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hdashline a_{21} & a_{22} & a_{23} \end{array}\right]$$

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c:cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]$$

Matrix operations can then be performed with matrices whose elements are matrices, provided the rules of operation are valid for the given matrix and for the resulting submatrices.

Example : Calculate **AB**, given

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution. Partition the two matrices:

$$\mathbf{A} = \left[\begin{array}{c|cc} 6 & 0 & 1 \\ 2 & 1 & 0 \end{array}\right] = [\mathbf{A}_1 \quad \mathbf{A}_2] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc} 4 & 3 \\ \hline 0 & 1 \\ 1 & 0 \end{array}\right] = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = [4 \quad 3], \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$\mathbf{AB} = [\mathbf{A}_1 \quad \mathbf{A}_2] \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2$$

and

$$\mathbf{AB} = \left[\begin{bmatrix} 6 \\ 2 \end{bmatrix} [4 \quad 3] + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] = \left[\begin{bmatrix} 24 & 18 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$\mathbf{AB} = \begin{bmatrix} 25 & 18 \\ 8 & 7 \end{bmatrix}$$

Matrix Form of Linear Programming

Original Form of the Model

Maximize $\quad Z = 3x_1 + 5x_2,$

subject to

$$
\begin{aligned}
x_1 \quad\quad\quad &\le 4 \\
2x_2 &\le 12 \\
3x_1 + 2x_2 &\le 18
\end{aligned}
$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0.$$

Augmented Form of the Model

Maximize $\quad Z = 3x_1 + 5x_2,$

subject to

$$
\begin{array}{lrrrrrl}
(1) & x_1 & & +x_3 & & & = 4 \\
(2) & & 2x_2 & & +x_4 & & = 12 \\
(3) & 3x_1 & +2x_2 & & & +x_5 & = 18
\end{array}
$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, 3, 4, 5.$$

Maximize $\quad Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \le \mathbf{b} \qquad \text{and} \qquad \mathbf{x} \ge \mathbf{0},$$

Maximize $\quad Z = \mathbf{cx},$

subject to

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \ge \mathbf{0}$$

where $\mathbf{c}$ is the row vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$, $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors and $\mathbf{A}$ is the matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Maximize $\quad Z = \mathbf{cx}$,

subject to

$\mathbf{Ax} \leq \mathbf{b} \quad$ and $\quad \mathbf{x} \geq \mathbf{0}$,

Maximize $\quad Z = \mathbf{cx}$,

subject to

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq \mathbf{0}$$

where $\mathbf{c}$ is the row vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$, $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors and $\mathbf{A}$ is the matrix

where $\mathbf{I}$ is the $m \times m$ identity matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Maximize $\quad Z = 3x_1 + 5x_2$,

subject to

$$\begin{aligned} (1) \quad & x_1 + x_3 = 4 \\ (2) \quad & 2x_2 + x_4 = 12 \\ (3) \quad & 3x_1 + 2x_2 + x_5 = 18 \end{aligned}$$

and

$x_j \geq 0, \quad$ for $j = 1, 2, 3, 4, 5$.

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Solving for a Basic Feasible Solution

For initialization,

Maximize $Z = \mathbf{cx}$,

subject to

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \text{ and } \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq \mathbf{0}$$

For any iteration,

Maximize $Z = \boldsymbol{c_B}\mathbf{x_B} + \boldsymbol{c_N}\mathbf{x_N}$

subject to

$$[\mathbf{B}, \mathbf{N}] \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} = \mathbf{b} \text{ and } \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} \geq \mathbf{0}$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & \cdots & N_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ N_{m1} & N_{m2} & \cdots & N_{mn} \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix} \quad \mathbf{x}_\mathrm{N} = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}$$

Solving for a Basic Feasible Solution

For initialization,

$$\text{Maximize} \quad Z = \mathbf{cx},$$
subject to
$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix}\mathbf{x} \\ \mathbf{x}_s\end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix}\mathbf{x} \\ \mathbf{x}_s\end{bmatrix} \geq \mathbf{0}$$

$$\mathbf{x}_B = \mathbf{x} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{I}^{-1}\mathbf{b} = \mathbf{c}_B \mathbf{b}$$

For any iteration,

$$\text{Maximize} \quad Z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N$$
subject to
$$[\mathbf{B}, \mathbf{N}]\begin{bmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{bmatrix} \geq \mathbf{0}$$

$$Z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N = \mathbf{c}_B \mathbf{x}_B = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$
$$\mathbf{B}\mathbf{x}_B = \mathbf{b} - \mathbf{N}\mathbf{x}_N$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & \cdots & N_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ N_{m1} & N_{m2} & \cdots & N_{mn} \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}$$

Example

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\mathbf{x}_B = \mathbf{x} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{I}^{-1}\mathbf{b} = \mathbf{c}_B \mathbf{b}$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

Iteration 0

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \text{ so } \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

$$\mathbf{c}_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

Iteration 1

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$\text{so } \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}, \mathbf{c}_B = [0, 5, 0], \text{ so } Z = [0, 5, 0]\begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.$$

Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \mathbf{c}_B = [0, 5, 3], Z = [0, 5, 3]\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

Matrix Form of the Set of Equations in the Simplex Tableau

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

For the original set of equations, the matrix form is

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.$$

For any iteration,

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix}$$

Example

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc:ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right], \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

For Iteration 2

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad \boxed{\mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}},$$

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}},$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \boxed{[0, \tfrac{3}{2}, 1]},$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = \boxed{[0, 0]} \quad \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}}, \quad \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \boxed{36.}$$

$$\begin{bmatrix} 1 & \boxed{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}} & \boxed{\mathbf{c}_B\mathbf{B}^{-1}} \\ 0 & \boxed{\mathbf{B}^{-1}\mathbf{A}} & \boxed{\mathbf{B}^{-1}} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}} \\ \boxed{\mathbf{B}^{-1}\mathbf{b}} \end{bmatrix}. \quad \rightarrow \quad \left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ \hline 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix}$$

Summary of the Revised Simplex Method

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	(1, 2, . . . , m)	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

1. Initialization (Iteration 0)

$$\mathbf{c}=[3,5],\ [\mathbf{A},\mathbf{I}]=\left[\begin{array}{cc|ccc}1&0&1&0&0\\0&2&0&1&0\\3&2&0&0&1\end{array}\right]$$

$$\mathbf{b}=\begin{bmatrix}4\\12\\18\end{bmatrix},\ \mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix},\ \mathbf{x}_s=\begin{bmatrix}x_3\\x_4\\x_5\end{bmatrix}$$

$$\mathbf{x}_B=\begin{bmatrix}x_3\\x_4\\x_5\end{bmatrix},\ \mathbf{B}=\mathbf{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\mathbf{B}^{-1}$$

$$\mathbf{x}_B=\begin{bmatrix}x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}4\\12\\18\end{bmatrix}=\begin{bmatrix}4\\12\\18\end{bmatrix}$$

$$\mathbf{c}_B=[0,0,0],\quad \text{so}\quad Z=[0,0,0]\begin{bmatrix}4\\12\\18\end{bmatrix}=0$$

Optimality test:

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}=[0,0,0]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\\3&2\end{bmatrix}-[3,5]=[-3,-5]$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

2. Iteration 1

Step 1: Determine the entering basic variable

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 0, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} - [3, 5] = [-3, -5]$$

$$-c_2 = -5 < -3 = -c_1$$

So x_2 is chosen to be the entering variable.

Step 2: Determine the leaving basic variable

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \text{—} & 0 \\ \text{—} & 2 \\ \text{—} & 2 \end{bmatrix} = \begin{bmatrix} \text{—} & 0 \\ \text{—} & 2 \\ \text{—} & 2 \end{bmatrix}$$

$$a_{12} = 0 \qquad a_{22} = 2 \qquad a_{32} = 2$$

$$b_2/a_{22} = \tfrac{12}{2} \qquad b_3/a_{32} = \tfrac{18}{2}$$

So the number of the pivot row r =2

Thus, x_4 is chosen to be the entering variable.

Step 3: Determine the new BF solution

The new set of basic variables is $\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}$

To obtain the new $\mathbf{B}^{-1}$, $\boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a_{12}}{a_{22}} \\ \dfrac{1}{a_{22}} \\ -\dfrac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{2} \\ -1 \end{bmatrix}$, $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1},$$

So the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

Optimality test:

The nonbasic variables are x_1 and x_4.

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix} - [3, —] = [-3, —],$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 0]\begin{bmatrix} — & 0 & — \\ — & \frac{1}{2} & — \\ — & -1 & — \end{bmatrix} = [—, \tfrac{5}{2}, —],$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

3. Iteration 2

Step 1: Determine the entering basic variable

$\boldsymbol{x_1}$ is chosen to be the entering variable.

Step 2: Determine the leaving basic variable

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix} = \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix}. \qquad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Coefficient of: Original Variables	Coefficient of: Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) $(1, 2, \ldots, m)$	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	**0** $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) $(1, 2, \ldots, m)$	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

The ratio 4/1 > 6/3 indicate that x5 is the leaving basic variable

Step 3: Determine the new BF solution

The new set of basic variables is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a'_{11}}{a'_{31}} \\ -\dfrac{a'_{21}}{a'_{31}} \\ \dfrac{1}{a'_{31}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{3} \\ 0 \\ \dfrac{1}{3} \end{bmatrix}. \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore, the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}.$$

Optimality test:

The nonbasic variables are x_4 and x_5.

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} — & \frac{1}{3} & -\frac{1}{3} \\ — & \frac{1}{2} & 0 \\ — & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [—, \tfrac{3}{2}, 1].$$

Relationship between the initial and final simplex tableaux

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Initial Tableau

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$

Final Tableau

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*].$

Other rows: $\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*].$

Combined: $\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right].$

$\mathbf{z}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}$ $\quad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$ $\quad Z^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$

$\mathbf{A}^* = \mathbf{B}^{-1}\mathbf{A}$ $\quad \mathbf{S}^* = \mathbf{B}^{-1}$ $\quad \mathbf{b}^* = \mathbf{B}^{-1}\mathbf{b}$

$$\text{(1)} \quad \mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \mid \mathbf{y}^* \mid \mathbf{y}^*\mathbf{b}] = [\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \mid \mathbf{c}_B\mathbf{B}^{-1} \mid \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}]$$

$$\text{(2)} \quad \mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \mid \mathbf{S}^* \mid \mathbf{S}^*\mathbf{b}] = [\mathbf{B}^{-1}\mathbf{A} \mid \mathbf{B}^{-1} \mid \mathbf{B}^{-1}\mathbf{b}]$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Initial Tableau

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0]$.

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}]$.

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right]$.

For iteration 1:

$$\mathbf{S^*} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{y^*} = [0, \ \tfrac{5}{2}, \ 0]$$

$$\mathbf{t^*} = \mathbf{t} + \mathbf{y^*T} = [-3, \ -5 \mid 0, \ 0, \ 0 \mid 0] + [0, \ \tfrac{5}{2}, \ 0] \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [-3, \ 0, \ \boxed{0, \ \tfrac{5}{2}, \ 0}, \ 30]$$

$$\mathbf{T^*} = \mathbf{S^*T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 3 & 0 & 0 & -1 & 1 & 6 \end{array}\right]$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Initial Tableau

Row 0: $\mathbf{t} = [-3, \ -5 \mid 0, \ 0, \ 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $$\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$$

Combined: $$\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$$

For iteration 2:

$$\mathbf{S}^* = \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad \mathbf{y}^* = [0, \ \tfrac{3}{2}, \ 1]$$

$$\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [-3, \ -5 \mid 0, \ 0, \ 0 \mid 0] + [0, \ \tfrac{3}{2}, \ 1]\left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [0, \ 0, \ \boxed{0, \ \tfrac{3}{2}, \ 1}, \ 36]$$

$$\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right]$$

DUAL SIMPLEX

ques

Primal
Maximize $z = 5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$ $-x_1 + 5x_2 \geq 3$ $4x_1 + 7x_2 \leq 8$ x_1 unrestricted, $x_2 \geq 0$

Question 2: Find the dual of given primal below:

maximize Z = 3x1 + 2x2

subject to

2x1 + 4x2 ≤ 22

-x1 + 4x2 ≤ 10

4x1 - 2x2 ≤14

x1 - 3x2 ≤ 1

x1, x2 ≥ 0

Primal	Primal in equation form	Dual variables
	Substitute $x_1 = x_1^+ - x_1^-$	
Maximize $z = 5x_1 + 6x_2$	Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$	
subject to	subject to	
$x_1 + 2x_2 = 5$	$x_1^- - x_1^+ + 2x_2 = 5$	y_1
$-x_1 + 5x_2 \geq 3$	$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$	y_2
$4x_1 + 7x_2 \leq 8$	$4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$	y_3
x_1 unrestricted, $x_2 \geq 0$	$x_1^-, x_1^+, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } z = 5y_1 + 3y_2 + 8y_3$$

subject to

$$\left.\begin{aligned} y_1 - y_2 + 4y_3 &\geq 5 \\ -y_1 + y_2 - 4y_3 &\geq -5 \end{aligned}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$\left.\begin{aligned} -y_2 &\geq 0 \\ y_3 &\geq 0 \\ y_1, y_2, y_3 &\text{ unrestricted} \end{aligned}\right\} \Rightarrow (y_1 \text{ unrestricted}, y_2 \leq 0, y_3 \geq 0)$$

The first and second constraints are replaced by an equation. The general rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produces an unrestricted dual variable, as the first primal constraint demonstrates.

Minimize $w = 22y_1 + 10y_2 + 14y_3 + 1y_4$

(Dual variables y_1, y_2, y_3, y_4)

Subject to

$2y_1 - y_2 + 4y_3 + y_4 \geq 3$

$4y_1 + 4y_2 - 2y_3 - 3y_4 \geq 2$, $y_1, y_2, y_3, y_4 \geq 0$

Rules of Constructing Dual problem

Maximization problem	Minimization problem
Constraints	Variables
$\geq$	≤ 0
$\leq$	≥ 0
$=$	unrestricted
≥ 0	$\geq$
≤ 0	$\leq$
Variable	Constraints.
Unrestricted	$=$

$y_1 + y_2 + y_3 + y_4 \geq 0$

y_1, y_2, y_3, y_4

Duality Theory and Sensitivity Analysis

Duality Theory

Primal Problem

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad \text{for } i = 1, 2, \dots, m$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \dots, n.$$

Dual Problem

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \qquad \text{for } j = 1, 2, \dots, n$$

and

$$y_i \ge 0, \qquad \text{for } i = 1, 2, \dots, m.$$

Primal Problem

Maximize $\quad Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \le \mathbf{b}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

Dual Problem

Minimize $\quad y_0 = \mathbf{yb},$

subject to

$$\mathbf{yA} \ge \mathbf{c}$$

and

$$\mathbf{y} \ge \mathbf{0}.$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

$$\mathbf{y} = \mathbf{c}_B \mathbf{B}^{-1}$$

TABLE 6.1 **Primal and dual problems for the Wyndor Glass Co. example**

Primal Problem in Algebraic Form

Maximize $\quad Z = 3x_1 + 5x_2,$

subject to

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

and $\quad x_1 \geq 0, \quad x_2 \geq 0.$

Dual Problem

Dual Problem in Algebraic Form

Minimize $\quad W = 4y_1 + 12y_2 + 18y_3,$

subject to

$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$

and

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

Primal Problem in Matrix Form

Maximize $\quad Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Dual Problem in Matrix Form

Minimize $\quad W = [y_1, y_2, y_3]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

subject to

$$[y_1, y_2, y_3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \geq [3, 5]$$

and

$$[y_1, y_2, y_3] \geq [0, 0, 0].$$

TABLE 6.2 **Primal-dual table for linear programming, illustrated by the Wyndor Glass Co. example**

(a) General Case

			Primal Problem					
			Coefficient of:				Right Side	
			x_1	x_2	⋯	x_n		
Dual Problem	Coefficient of:	y_1	a_{11}	a_{12}	⋯	a_{1n}	$\leq b_1$	Coefficients for Objective Function (Minimize)
		y_2	a_{21}	a_{22}	⋯	a_{2n}	$\leq b_2$	
		⋮					⋮	
		y_m	a_{m1}	a_{m2}	⋯	a_{mn}	$\leq b_m$	
		Right Side	VI c_1	VI c_2	⋯	VI c_n		
			Coefficients for Objective Function (Maximize)					

(b) Wyndor Glass Co. Example

	x_1	x_2	
y_1	1	0	≤ 4
y_2	0	2	≤ 12
y_3	3	2	≤ 18
	VI	VI	
	3	5	

Initial Tableau

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0]$.

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}]$.

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right]$.

Final Tableau

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*]$.

Other rows: $\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*]$.

Combined: $\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right]$.

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

$$\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$$

$$\mathbf{z}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}$$

$$\mathbf{z} = \mathbf{y}\mathbf{A},$$

so $$z_j = \sum_{i=1}^{m} a_{ij}y_i,$$

$$z_j - c_j = \sum_{i=1}^{m} a_{ij}y_i - c_j$$

for $j = 1, 2, \ldots, n$.

Dual Problem

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$\sum_{i=1}^{m} a_{ij}y_i \geq c_j, \quad$ for $j = 1, 2, \ldots, n$

and

$y_i \geq 0, \quad$ for $i = 1, 2, \ldots, m$.

$z_j - c_j$ is the surplus variable for the functional constraints in the dual problem.

TABLE 6.4 Notation for entries in row 0 of a simplex tableau

Iteration	Basic Variable	Eq.	Coefficient of:									Right Side
			Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of:						Right Side
			Z	x_1	x_2	x_3	x_4	x_5	
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

TABLE 6.5 Row 0 and corresponding dual solution for each iteration for the Wyndor Glass Co. example

Iteration	Primal Problem	Dual Problem					
	Row 0	y_1	y_2	y_3	$z_1 - c_1$	$z_2 - c_2$	W
0	$[-3, \ -5 \ \vdots \ 0, \ 0, \ 0 \ \vdots \ 0]$	0	0	0	−3	−5	0
1	$[-3, \ 0 \ \vdots \ 0, \ \frac{5}{2}, \ 0 \ \vdots \ 30]$	0	$\frac{5}{2}$	0	−3	0	30
2	$[\ 0, \ 0 \ \vdots \ 0, \ \frac{3}{2}, \ 1 \ \vdots \ 36]$	0	$\frac{3}{2}$	1	0	0	36

If a solution for the primal problem and its corresponding solution for the dual problem are both feasible, the value of the objective function is optimal.

If a solution for the primal problem is feasible and the value of the objective function is not optimal (for this example, not maximum), the corresponding dual solution is infeasible.

Summary of Primal-Dual Relationships

Weak duality property: If $\mathbf{x}$ is a feasible solution for the primal problem and $\mathbf{y}$ is a feasible solution for the dual problem, then

$$\mathbf{cx} \leq \mathbf{yb}.$$

Strong duality property: If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

Complementary-solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution $\mathbf{x}$ for the primal problem and a **complementary solution** $\mathbf{y}$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx} = \mathbf{yb}.$$

If $\mathbf{x}$ is *not optimal* for the primal problem, then $\mathbf{y}$ is *not feasible* for the dual problem.

Summary of Primal-Dual Relationships

Complementary optimal solutions property: At the final iteration, the simplex method simultaneously identifies an optimal solution $\mathbf{x}^*$ for the primal problem and a **complementary optimal solution** $\mathbf{y}^*$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{c}\mathbf{x}^* = \mathbf{y}^*\mathbf{b}.$$

The y_i^* are the shadow prices for the primal problem.

Symmetry property: For *any* primal problem and its dual problem, all relationships between them must be *symmetric* because the dual of this dual problem is this primal problem.

Duality theorem: The following are the only possible relationships between the primal and dual problems.

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions*.
3. If one problem has *no feasible solutions*, then the other problem has either *no feasible solutions* or an *unbounded* objective function.

Complementary basic solutions property: Each *basic* solution in the *primal problem* has a **complementary basic solution** in the *dual problem*, where their respective objective function values (Z and y_0) are equal. Given row 0 of the simplex tableau for the primal basic solution, the complementary dual basic

TABLE 6.4 Notation for entries in row 0 of a simplex tableau

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	Right Side
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

Complementary slackness property: Given the association between variables in Table 6.7, the variables in the primal basic solution and the complementary dual basic solution satisfy the **complementary slackness** relationship shown in Table 6.8. Furthermore, this relationship is a symmetric one, so that these two basic solutions are complementary to each other.

TABLE 6.7 Association between variables in primal and dual problems

	Primal Variable	**Associated Dual Variable**
Any problem	(Decision variable) x_j (Slack variable) x_{n+i}	$z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$ y_i (decision variable) $i = 1, 2, \ldots, m$
Wyndor problem	Decision variables: x_1 x_2 Slack variables: x_3 x_4 x_5	$z_1 - c_1$ (surplus variables) $z_2 - c_2$ y_1 (decision variables) y_2 y_3

TABLE 6.8 Complementary slackness relationship for complementary basic solutions

Primal Variable	**Associated Dual Variable**	
Basic	Nonbasic	(m variables)
Nonbasic	Basic	(n variables)

Primal Problem

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$
\begin{aligned}
&(1) \quad x_1 + x_3 = 4 \\
&(2) \quad 2x_2 + x_4 = 12 \\
&(3) \quad 3x_1 + 2x_2 + x_5 = 18
\end{aligned}
$$

and

$x_j \geq 0, \quad$ for $j = 1, 2, 3, 4, 5.$

Dual Problem

Minimize $W = 4y_1 + 12y_2 + 18y_3$,

subject to

$$
\begin{aligned}
y_1 + 3y_3 - 3 - z_1 - c_1 &= 0 \\
2y_2 + 2y_3 - 5 - z_2 - c_2 &= 0
\end{aligned}
$$

and

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ z_1 - c_1 \geq 0,\ z_2 - c_2 \geq 0$

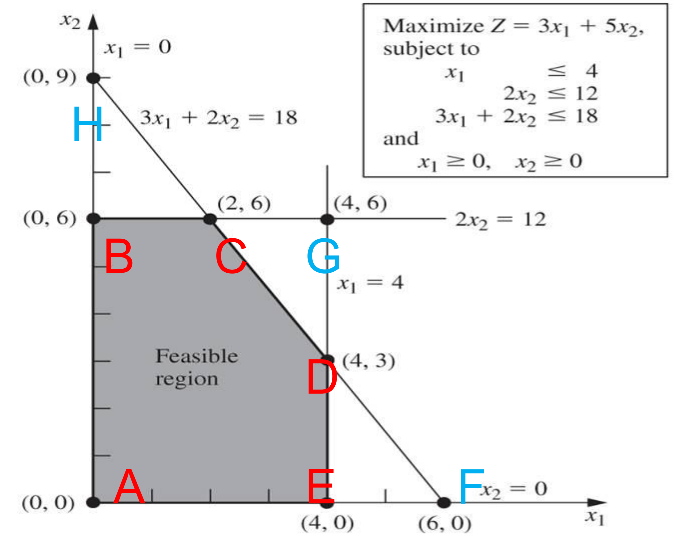


Table 6.9 **Complementary Basic Solutions for the Wyndor Glass Co. Example**

		Primal Problem			*Dual Problem*	
	No.	Basic Solution	Feasible?	$Z = y_0$	Feasible?	Basic Solution
A	1	(0, 0, 4, 12, 18)	Yes	0	No	(0, 0, 0, −3, −5)
E	2	(4, 0, 0, 12, 6)	Yes	12	No	(3, 0, 0, 0, −5)
F	3	(6, 0, −2, 12, 0)	No	18	No	(0, 0, 1, 0, −3)
D	4	(4, 3, 0, 6, 0)	Yes	27	No	$(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$
B	5	(0, 6, 4, 0, 6)	Yes	30	No	$(0, \frac{5}{2}, 0, -3, 0)$
C	6	(2, 6, 2, 0, 0)	Yes	36	Yes	$(0, \frac{3}{2}, 1, 0, 0)$
G	7	(4, 6, 0, 0, −6)	No	42	Yes	$(3, \frac{5}{2}, 0, 0, 0)$
H	8	(0, 9, 4, −6, 0)	No	45	Yes	$(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$

Complementary optimal basic solutions property: Each *optimal* basic solution in the *primal problem* has a **complementary optimal basic solution** in the dual problem, where their respective objective function values (Z and y_0) are equal. Given row 0 of the simplex tableau for the optimal primal solution, the complementary optimal dual solution $(\mathbf{y}^*, \mathbf{z}^* - \mathbf{c})$ is found as shown in Table 6.4.

TABLE 6.4 Notation for entries in row 0 of a simplex tableau

Iteration	Basic Variable	Eq.	Coefficient of:									Right Side
			Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	
Any	Z	(0)	1	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	y_1	y_2	$\cdots$	y_m	W

TABLE 6.10 Classification of basic solutions

		Satisfies Condition for Optimality?	
		Yes	**No**
Feasible?	**Yes**	Optimal	Suboptimal
	No	Superoptimal	Neither feasible nor superoptimal

Neither feasible nor superoptimal

Neither feasible nor superoptimal

Table 6.9 Complementary Basic Solutions for the Wyndor Glass Co. Example

	Primal Problem			*Dual Problem*	
No.	Basic Solution	Feasible?	$Z = y_0$	Feasible?	Basic Solution
1	(0, 0, 4, 12, 18)	Yes	0	No	(0, 0, 0, −3, −5)
2	(4, 0, 0, 12, 6)	Yes	12	No	(3, 0, 0, 0, −5)
3	(6, 0, −2, 12, 0)	No	18	No	(0, 0, 1, 0, −3)
4	(4, 3, 0, 6, 0)	Yes	27	No	$(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$
5	(0, 6, 4, 0, 6)	Yes	30	No	$(0, \frac{5}{2}, 0, -3, 0)$
6	(2, 6, 2, 0, 0)	Yes	36	Yes	$(0, \frac{3}{2}, 1, 0, 0)$
7	(4, 6, 0, 0, −6)	No	42	Yes	$(3, \frac{5}{2}, 0, 0, 0)$
8	(0, 9, 4, −6, 0)	No	45	Yes	$(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$

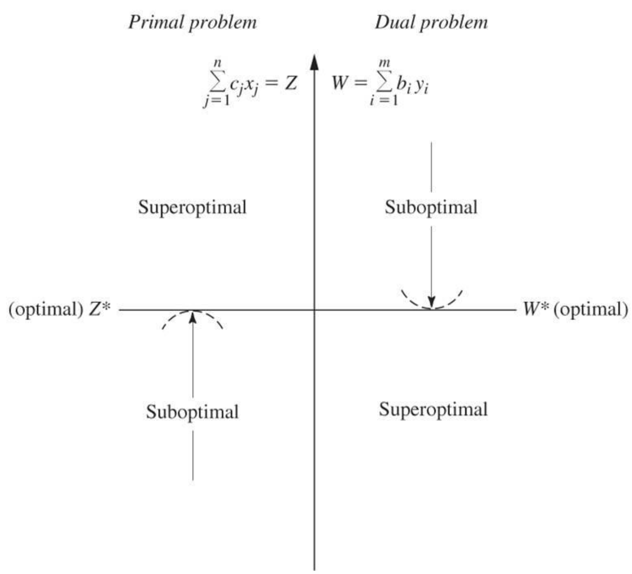
Suboptimal **Superptimal**

Optimal **Optimal**

Superoptimal **Suboptimal**

■ **TABLE 6.11** Relationships between complementary basic solutions

Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions: Primal Feasible?	Both Basic Solutions: Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No



Adapting to Other Primal Forms

Primal Problem

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

Dual Problem

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \ge 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

TABLE 6.12 Conversions to standard form for linear programming models

Nonstandard Form	Equivalent Standard Form
Minimize $\quad Z$	Maximize $\quad (-Z)$
$\sum_{j=1}^{n} a_{ij} x_j \ge b_i$	$-\sum_{j=1}^{n} a_{ij} x_j \le -b_i$
$\sum_{j=1}^{n} a_{ij} x_j = b_i$	$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad$ and $\quad -\sum_{j=1}^{n} a_{ij} x_j \le -b_i$
x_j unconstrained in sign	$x_j^+ - x_j^-, \quad x_j^+ \ge 0, \quad x_j^- \ge 0$

■ **TABLE 6.13 Constructing the dual of the dual problem**

Dual Problem	*Converted to Standard Form*
Minimize $W = \mathbf{yb}$, subject to $\mathbf{yA} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.	Maximize $(-W) = -\mathbf{yb}$, subject to $-\mathbf{yA} \leq -\mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.
Converted to Standard Form	*Its Dual Problem*
Maximize $Z = \mathbf{cx}$, subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.	Minimize $(-Z) = -\mathbf{cx}$, subject to $-\mathbf{Ax} \geq -\mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

TABLE 6.14 Corresponding primal-dual forms

Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
Maximize Z (or W)		Minimize W (or Z)
Constraint i:		Variable y_i (or x_i):
$\leq$ form	$\longleftrightarrow$	$y_i \geq 0$
$=$ form	$\longleftrightarrow$	Unconstrained
$\geq$ form	$\longleftrightarrow$	$y_i' \leq 0$
Variable x_j (or y_j):		Constraint j:
$x_j \geq 0$	$\longleftrightarrow$	$\geq$ form
Unconstrained	$\longleftrightarrow$	$=$ form
$x_j' \leq 0$	$\longleftrightarrow$	$\leq$ form

Primal Problem

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

Dual Problem

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geqslant 0, \quad j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \quad i = 1, 2, \cdots, m \qquad (1)$$

$$-\sum_{j=1}^{n} a_{ij} x_j \leqslant -b_i, \quad i = 1, 2, \cdots, m \qquad (2)$$

$$x_j \geqslant 0, \quad j = 1, 2, \cdots, n$$

y_i' : *dual* variables corresponding to (1)

y_i'' : dual variables corresponding to (2)

$$\min \omega = \sum_{i=1}^{m} b_i y_i' + \sum_{i=1}^{m} (-b_i y_i'')$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i' + \sum_{i=1}^{m} (-a_{ij} y_i'') \geqslant c_j, \quad j = 1, 2, \cdots, n$$

$$y_i', y_i'' \geqslant 0, \quad i = 1, 2, \cdots, m$$

$$\min \omega = \sum_{i=1}^{m} b_i (y_i' - y_i'')$$

$$\sum_{i=1}^{m} a_{ij} (y_i' - y_i'') \geqslant c_j, \quad j = 1, 2, \cdots, n$$

$$y_i = y_i' - y_i''$$

y_i: unconstrained in sign

$$\min \omega = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \geqslant c_j, \quad j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

y_i'' : dual variables corresponding to (2)

$$\min \omega = \sum_{i=1}^{m} (-b_i y_i'')$$

subject to

$$\sum_{i=1}^{m} (-a_{ij} y_i'') \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i'' \geq 0, \quad i = 1, 2, \cdots, m$$

$$y_i = -y_i''$$

$$\min \omega = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i \leq 0, \quad i = 1, 2, \cdots, m$$

TABLE 6.14 Corresponding primal-dual forms

Primal Problem (or Dual Problem)	Dual Problem (or Primal Problem)
Maximize Z (or W)	Minimize W (or Z)
Constraint i:	Variable y_i (or x_i):
$\leq$ form $\longleftrightarrow$	$y_i \geq 0$
$=$ form $\longleftrightarrow$	Unconstrained
$\geq$ form $\longleftrightarrow$	$y_i' \leq 0$
Variable x_j (or y_j):	Constraint j:
$x_j \geq 0$ $\longleftrightarrow$	$\geq$ form
Unconstrained $\longleftrightarrow$	$=$ form
$x_j' \leq 0$ $\longleftrightarrow$	$\leq$ form

Primal Problem

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

Dual Problem

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

TABLE 6.15 One primal-dual form for the radiation therapy example

Primal Problem		*Dual Problem*	
Maximize $-Z = -0.4x_1 - 0.5x_2,$		Minimize $W = 2.7y_1 + 6y_2 + 6y_3',$	
subject to		subject to	
$0.3x_1 + 0.1x_2 \leq 2.7$	$\longleftrightarrow$	$y_1 \geq 0$	(S)
$0.5x_1 + 0.5x_2 = 6$	$\longleftrightarrow$	y_2 unconstrained in sign	(O)
$0.6x_1 + 0.4x_2 \geq 6$	$\longleftrightarrow$	$y_3' \leq 0$	(B)
and		and	
$x_1 \geq 0$	$\longleftrightarrow$	$0.3y_1 + 0.5y_2 + 0.6y_3' \geq -0.4$	(S)
$x_2 \geq 0$	$\longleftrightarrow$	$0.1y_1 + 0.5y_2 + 0.4y_3' \geq -0.5$	(S)

TABLE 6.16 The other primal-dual form for the radiation therapy example

Primal Problem		*Dual Problem*
Minimize $Z = 0.4x_1 + 0.5x_2,$		Maximize $W = 2.7y_1' + 6y_2' + 6y_3,$
subject to		subject to
$0.3x_1 + 0.1x_2 \leq 2.7$	$\longleftrightarrow$	$y_1' \leq 0$
$0.5x_1 + 0.5x_2 = 6$	$\longleftrightarrow$	y_2' unconstrained in sign
$0.6x_1 + 0.4x_2 \geq 6$	$\longleftrightarrow$	$y_3 \geq 0$
and		and
$x_1 \geq 0$	$\longleftrightarrow$	$0.3y_1' + 0.5y_2' + 0.6y_3 \leq 0.4$
$x_2 \geq 0$	$\longleftrightarrow$	$0.1y_1' + 0.5y_2' + 0.4y_3 \leq 0.6$

Dual Simplex Method

TABLE 6.10 Classification of basic solutions

		Satisfies Condition for Optimality? Yes	Satisfies Condition for Optimality? No
Feasible?	**Yes**	Optimal	Suboptimal
	No	Superoptimal	Neither feasible nor superoptimal

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Table 6.9 Complementary Basic Solutions for the Wyndor Glass Co. Example

No.	Primal Problem: Basic Solution	Primal Problem: Feasible?	$Z = y_0$	Dual Problem: Feasible?	Dual Problem: Basic Solution
1	(0, 0, 4, 12, 18)	Yes	0	No	(0, 0, 0, −3, −5)
2	(4, 0, 0, 12, 6)	Yes	12	No	(3, 0, 0, 0, −5)
3	(6, 0, −2, 12, 0)	No	18	No	(0, 0, 1, 0, −3)
4	(4, 3, 0, 6, 0)	Yes	27	No	($-\frac{9}{2}$, 0, $\frac{5}{2}$, 0, 0)
5	(0, 6, 4, 0, 6)	Yes	30	No	(0, $\frac{5}{2}$, 0, −3, 0)
6	(2, 6, 2, 0, 0)	Yes	36	Yes	(0, $\frac{3}{2}$, 1, 0, 0)
7	(4, 6, 0, 0, −6)	No	42	Yes	(3, $\frac{5}{2}$, 0, 0, 0)
8	(0, 9, 4, −6, 0)	No	45	Yes	(0, 0, $\frac{5}{2}$, $\frac{9}{2}$, 0)

Suboptimal

Optimal

Superoptimal

Superptimal

Optimal

Suboptimal

Simplex Method : Keep the solution in any iteration suboptimal (not satisfying the condition for optimality, but the condition for feasibility).

Dual Simplex Method : Keep the solution in any iteration superoptimal (not satisfying the condition for feasibility, but the condition for optimality).

If a solution satisfies the condition for optimality, the coefficients in row (0) of the simplex tableau must nonnegative.

If a solution does not satisfy the condition for feasibility, one or more of the values of b in the right-side of simplex tableau must be negative.

Summary of Dual Simplex Method

Maximize $Z = -4y_1 - 12y_2 - 18y_3$

subject to

$y_1 + 3y_3 \geq 3$

$2y_2 + 2y_3 \geq 5$

$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$

Iteration	Basic Variable	Eq.	Coefficient of: Z	y_1	y_2	y_3	y_4	y_5	Right Side
0	Z	(0)	1	4	12	18	0	0	0
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_5	(2)	0	0	−2	−2	0	1	−5
1	Z	(0)	1	4	0	6	0	6	−30
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_2	(2)	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5}{2}$
2	Z	(0)	1	2	0	0	2	6	−36
	y_3	(1)	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	1
	y_2	(2)	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{3}{2}$

1. *Initialization:* After converting any functional constraints in $\geq$ form to $\leq$ form (by multiplying through both sides by -1), introduce slack variables as needed to construct a set of equations describing the problem. Find a basic solution such that the coefficients in Eq. (0) are zero for basic variables and nonnegative for nonbasic variables (so the solution is optimal if it is feasible). Go to the feasibility test.

2. *Feasibility test:* Check to see whether all the basic variables are *nonnegative.* If they are, then this solution is feasible, and therefore optimal, so stop. Otherwise, go to an iteration.
3. *Iteration:*

Step 1 Determine the *leaving basic variable:* Select the *negative* basic variable that has the largest absolute value.

Step 2 Determine the *entering basic variable:* Select the nonbasic variable whose coefficient in Eq. (0) reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to Eq. (0). This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest absolute value of the ratio of the Eq. (0) coefficient to the coefficient in that equation.

Step 3 Determine the *new basic solution:* Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination. When we set the nonbasic variables equal to zero, each basic variable (and Z) equals the new right-hand side of the one equation in which it appears (with a coefficient of $+1$). Return to the feasibility test.

Sensitivity Analysis

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

Maximize $\quad Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \le \mathbf{b}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

The simplex method already has been used to obtain an optimal solution to a linear programming model with specified values for the b_i, c_j, and a_{ij} parameters. To initiate sensitivity analysis, one or more of the parameters is changed. After the changes are made, let $\overline{b}_i$, $\overline{c}_j$, and $\overline{a}_{ij}$ denote the values of the various parameters. Thus, in matrix notation,

$$\mathbf{b} \rightarrow \overline{\mathbf{b}}, \qquad \mathbf{c} \rightarrow \overline{\mathbf{c}}, \qquad \mathbf{A} \rightarrow \overline{\mathbf{A}},$$

for the revised model.

Initial Tableau

Row 0: $\mathbf{t} = [-3, \ -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $$\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$$

Combined: $$\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$$

Final Tableau

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*].$

Other rows: $$\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*].$$

Combined: $$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right].$$

(1) $\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \mid \mathbf{y}^* \mid \mathbf{y}^*\mathbf{b}] = [\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \mid \mathbf{c}_B\mathbf{B}^{-1} \mid \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}]$

(2) $\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \mid \mathbf{S}^* \mid \mathbf{S}^*\mathbf{b}] = [\mathbf{B}^{-1}\mathbf{A} \mid \mathbf{B}^{-1} \mid \mathbf{B}^{-1}\mathbf{b}]$

$$\mathbf{S}^* = \mathbf{B}^{-1} \qquad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$$

Original Model

Maximize $\quad Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

Revised Model

Maximize $\quad Z = [4, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \mathbf{c}_B = [0, 5, 4] \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{B}^{-1}\bar{\mathbf{A}} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
New initial tableau	Z	(0)	1	−4	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	24
	x_5	(3)	0	2	2	0	0	1	18
Revised final tableau	Z	(0)	1	0	0	0	$\frac{1}{2}$	2	48
	x_3	(1)	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	7
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
	x_1	(3)	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	−3

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, \mathbf{4}] \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = [0, \tfrac{1}{2}, 2]$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \bar{\mathbf{c}} = [0, 5, 4] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [4, 5] = [0, 0]$$

$$\mathbf{B}^{-1}\bar{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ -3 \end{bmatrix} \qquad \mathbf{c}_B\mathbf{B}^{-1}\bar{\mathbf{b}} = [0, 5, 4] \begin{bmatrix} 7 \\ 12 \\ -3 \end{bmatrix} = 48$$

Case 1 : Changes in b_i

Suppose that the only changes in the current model are that one or more of the b_i parameters ($i = 1, 2, \ldots, m$) has been changed. In this case, the *only* resulting changes in the final simplex tableau are in the *right side* column.

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 **0**	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

Final Tableau

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*].$

Other rows: $\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*].$

Combined: $\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right].$

$$\mathbf{S}^* = \mathbf{B}^{-1} \qquad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$$

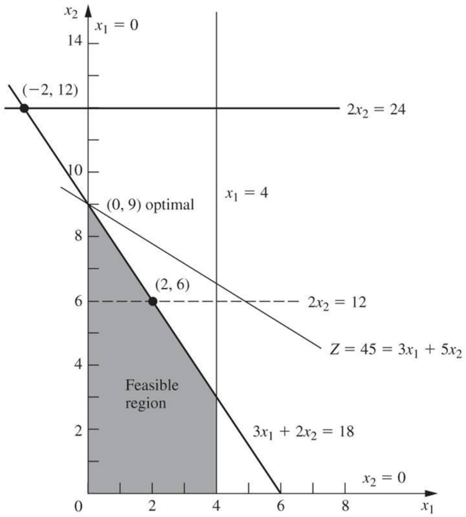
$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

Right side of final row 0: $Z^* = \mathbf{y}^*\bar{\mathbf{b}}$

Right side of final rows $1, 2, \ldots, m$: $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$

■ **TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2



$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \overline{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

$$Z^* = \mathbf{y}^*\overline{\mathbf{b}} = [0, \tfrac{3}{2}, 1] \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54$$

$$\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}, \qquad \text{so} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}$$

Incremental analysis

Equivalently, because the only change in the original model is $\Delta b_2 = 24 - 12 = 12$, incremental analysis can be used to calculate these same values more quickly.

$$\Delta Z^* = \mathbf{y}^* \, \Delta\bar{\mathbf{b}} = \mathbf{y}^* \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{y}^* \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

$$\Delta \mathbf{b}^* = \mathbf{S}^* \, \Delta\bar{b} = \mathbf{S}^* \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{S}^* \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

$$\mathbf{y}^* = [0, \tfrac{3}{2}, 1] \qquad \mathbf{S}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

$\Delta Z^* = \frac{3}{2}(12) = 18,$ so $Z^* = 36 + 18 = 54$

$\Delta b_1^* = \frac{1}{3}(12) = 4,$ so $b_1^* = 2 + 4 = 6$

$\Delta b_2^* = \frac{1}{2}(12) = 6,$ so $b_2^* = 6 + 6 = 12$

$\Delta b_3^* = -\frac{1}{3}(12) = -4,$ so $b_3^* = 2 - 4 = -2$

Therefore, the current (previously optimal) basic solution has become

$$(x_1, x_2, x_3, x_4, x_5) = (-2, 12, 6, 0, 0)$$

The dual simplex method now can be applied to find the new optimal solution.

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36
x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	12
x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	−2

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$c_1 = 3,$	$c_2 = 5$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = 2,$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = 2,$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

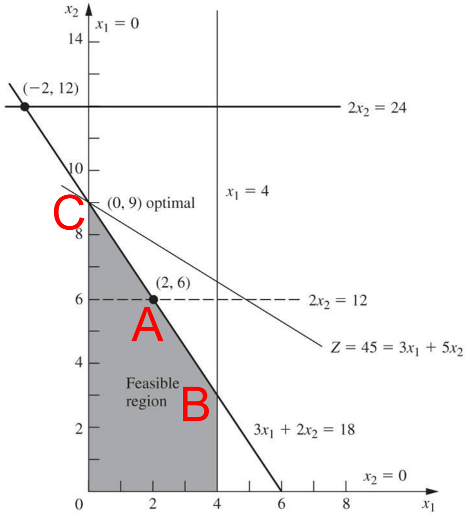
Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	−3	0	0	1	−1	6

$$\Delta b_1^* = \tfrac{1}{3}(12) = 4, \quad \text{so } b_1^* = 2 + 4 = 6$$
$$\Delta b_2^* = \tfrac{1}{2}(12) = 6, \quad \text{so } b_2^* = 6 + 6 = 12$$
$$\Delta b_3^* = -\tfrac{1}{3}(12) = -4, \quad \text{so } b_3^* = 2 - 4 = -2$$

$$b_1^* = 2 + \tfrac{1}{3}\Delta b_2,$$
$$b_2^* = 6 + \tfrac{1}{2}\Delta b_2,$$
$$b_3^* = 2 - \tfrac{1}{3}\Delta b_2,$$

The allowable range of b_i to stay feasible

$$b_1^* = 2 + \tfrac{1}{3}\Delta b_2 \geq 0 \Rightarrow \tfrac{1}{3}\Delta b_2 \geq -2 \Rightarrow \Delta b_2 \geq -6,$$
$$b_2^* = 6 + \tfrac{1}{2}\Delta b_2 \geq 0 \Rightarrow \tfrac{1}{2}\Delta b_2 \geq -6 \Rightarrow \Delta b_2 \geq -12,$$
$$b_3^* = 2 - \tfrac{1}{3}\Delta b_2 \geq 0 \Rightarrow 2 \geq \tfrac{1}{3}\Delta b_2 \Rightarrow \Delta b_2 \leq 6.$$



The solution remains feasible only if

$$-6 \leq \Delta b_2 \leq 6$$

$$b_2 = 12 + \Delta b_2 \longrightarrow 6 \leq b_2 \leq 18$$

For any b_i, its **allowable range to stay feasible** is the range of values over which the optimal BF solution (with adjusted values for the basic variables) remains feasible. (It is assumed that the change in this one b_i value is the only change in the model.) The adjusted values for the basic variables are obtained from the formula $\mathbf{b^*} = \mathbf{S^*}\overline{\mathbf{b}}$. The calculation of the allowable range to stay feasible then is based on finding the range of values of b_i such that $\mathbf{b^*} \geq \mathbf{0}$.

Case 2a : Changes in the coefficients of a nonbasic variable

Consider a particular variable x_j (fixed j) that is a nonbaic variable in the optimal solution shown by the final simplex tableau.

$\mathbf{A}_j$ is the vector of column j in the matrix **A .**

We have $c_j \longrightarrow \bar{c}_j, \qquad \mathbf{A}_j \longrightarrow \overline{\mathbf{A}}_j$ for the revised model.

$$c_1 = 3 \longrightarrow \bar{c}_1 = 4$$

$$\mathbf{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \longrightarrow \overline{\mathbf{A}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

We can observe that the changes lead to a single revised constraint for the dual problem.

$$y_1 + 3y_3 \geq 3 \longrightarrow y_1 + 2y_3 \geq 4$$

$$y_1^* = 0, \qquad y_2^* = 0, \qquad y_3^* = \tfrac{5}{2}.$$

$$0 + 2(\tfrac{5}{2}) \geq 4$$

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$c_1 = 3,$	$c_2 = 5$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = 2,$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = 2,$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	-3	0	0	1	-1	6

The allowable range of the coefficient c_i of a nonbasic variable

For any c_j, its **allowable range to stay optimal** is the range of values over which the current optimal solution (as obtained by the simplex method for the current model before c_j is changed) remains optimal. (It is assumed that the change in this one c_j is the only change in the current model.) When x_j is a nonbasic variable for this solution, the solution remains optimal as long as $z_j^* - c_j \geq 0$, where $z_j^* = \mathbf{y}^*\mathbf{A}_j$ is a constant unaffected by any change in the value of c_j. Therefore, the allowable range to stay optimal for c_j can be calculated as $c_j \leq \mathbf{y}^*\mathbf{A}_j$.

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$c_1 = 3,$	$c_2 = 5$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = 2,$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = 2,$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	-3	0	0	1	-1	6

$$c_1 \leq \mathbf{y}^*\mathbf{A}_1 = [0, \quad 0, \quad \tfrac{5}{2}]\begin{bmatrix}1\\0\\3\end{bmatrix} = 7\tfrac{1}{2},$$

$c_1 \leq 7\frac{1}{2}$ is the allowable range to stay optimal.

Case 2b : Introduction of a new variable

Maximize $\quad Z = 3x_1 + 5x_2 + 4x_{\text{new}},$

subject to

$$
\begin{aligned}
x_1 \qquad\quad + 2x_{\text{new}} &\le 4 \\
2x_2 + 3x_{\text{new}} &\le 12 \\
3x_1 + 2x_2 + \ x_{\text{new}} &\le 18
\end{aligned}
$$

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_{\text{new}} \ge 0.$$

After we introduced slack variables, the original optimal solution for this problem without x_{new} was $(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$. Is this solution, along with $x_{\text{new}} = 0$, still optimal? $\quad (y_1, y_2, y_3, z_1 - c_1, z_2 - c_2) = (0, \frac{3}{2}, 1, 0, 0)$

Table 6.9 **Complementary Basic Solutions for the Wyndor Glass Co. Example**

	Primal Problem			*Dual Problem*	
No.	Basic Solution	Feasible?	$Z = y_0$	Feasible?	Basic Solution
1	(0, 0, 4, 12, 18)	Yes	0	No	(0, 0, 0, −3, −5)
2	(4, 0, 0, 12, 6)	Yes	12	No	(3, 0, 0, 0, −5)
3	(6, 0, −2, 12, 0)	No	18	No	(0, 0, 1, 0, −3)
4	(4, 3, 0, 6, 0)	Yes	27	No	$(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$
5	(0, 6, 4, 0, 6)	Yes	30	No	$(0, \frac{5}{2}, 0, -3, 0)$
6	(2, 6, 2, 0, 0)	Yes	36	Yes	$(0, \frac{3}{2}, 1, 0, 0)$
7	(4, 6, 0, 0, −6)	No	42	Yes	$(3, \frac{5}{2}, 0, 0, 0)$
8	(0, 9, 4, −6, 0)	No	45	Yes	$(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$

$$2y_1 + 3y_2 + y_3 \ge 4$$

$$2(0) + 3(\tfrac{3}{2}) + (1) \ge 4$$

Case 3 : Changes in the coefficients of a basic variable

$$c_2 = 5 \longrightarrow \bar{c}_2 = 3, \qquad \mathbf{A}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \overline{\mathbf{A}}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$c_1 = 3,$	$\boxed{c_2 = 5}$	$(n = 2)$
$a_{11} = 1,$	$\boxed{a_{12} = 0,}$	$b_1 = 4$
$a_{21} = 0,$	$\boxed{a_{22} = 2,}$	$b_2 = 24$
$a_{31} = 3,$	$\boxed{a_{32} = 2,}$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	-3	0	0	1	-1	6

TABLE 6.24 Sensitivity analysis procedure applied to Variation 5 of the Wyndor Glass Co. model

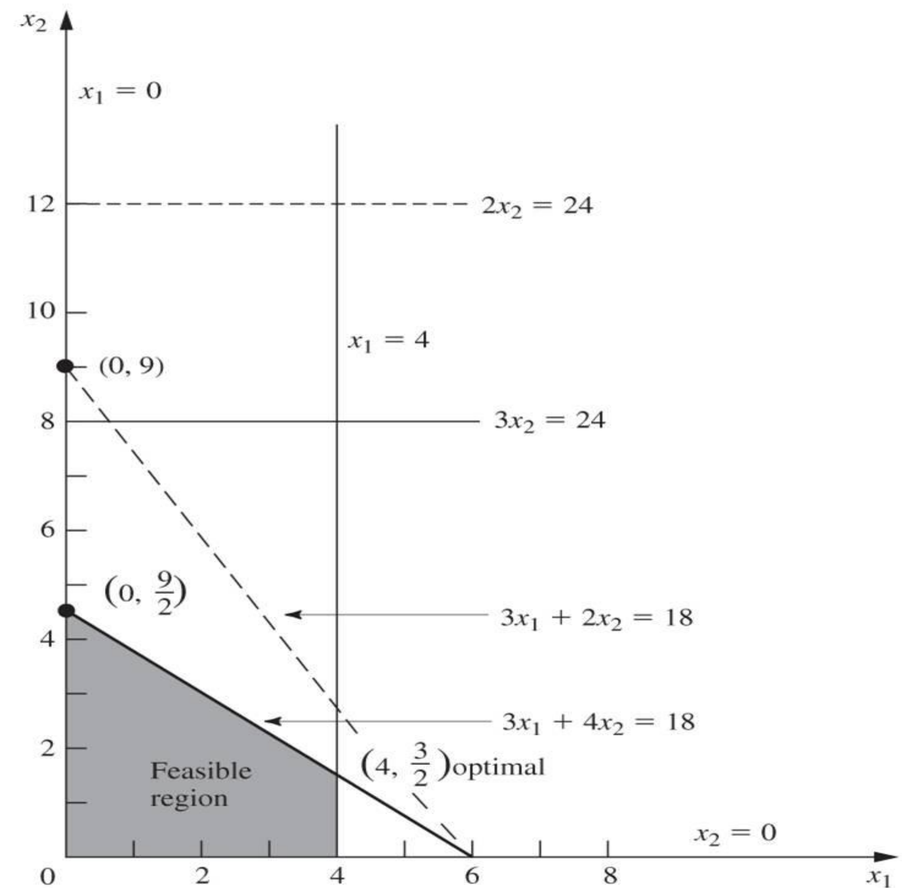
	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
	Z	(0)	1	$-\frac{3}{4}$	0	0	0	$\frac{3}{4}$	$\frac{27}{2}$
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	$\frac{3}{4}$	1	0	0	$\frac{1}{4}$	$\frac{9}{2}$
	x_4	(3)	0	$-\frac{9}{4}$	0	0	1	$-\frac{3}{4}$	$\frac{21}{2}$
New final tableau after reoptimization (only one iteration of the simplex method needed in this case)	Z	(0)	1	0	0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{33}{2}$
	x_1	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$
	x_4	(3)	0	0	0	$\frac{9}{4}$	1	$-\frac{3}{4}$	$\frac{39}{2}$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

■ **TABLE 6.24** Sensitivity analysis procedure applied to Variation 5 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
				Coefficient of:					
	Z	(0)	1	$-\frac{3}{4}$	0	0	0	$\frac{3}{4}$	$\frac{27}{2}$
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	$\frac{3}{4}$	1	0	0	$\frac{1}{4}$	$\frac{9}{2}$
	x_4	(3)	0	$-\frac{9}{4}$	0	0	1	$-\frac{3}{4}$	$\frac{21}{2}$
New final tableau after reoptimization (only one iteration of the simplex method needed in this case)	Z	(0)	1	0	0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{33}{2}$
	x_1	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$
	x_4	(3)	0	0	0	$\frac{9}{4}$	1	$-\frac{3}{4}$	$\frac{39}{2}$



The allowable range of the coefficient c_i of a basic variable

$$c_2 = 3 + \Delta c_2$$

$$\left[0, 0, \tfrac{3}{4} - \tfrac{3}{4}\Delta c_2, 0, \tfrac{3}{4} + \tfrac{1}{4}\Delta c_2 : \tfrac{33}{2} + \tfrac{3}{2}\Delta c_2\right]$$

$$\tfrac{3}{4} - \tfrac{3}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \tfrac{3}{4} \geq \tfrac{3}{4}\Delta c_2 \quad \Rightarrow \quad \Delta c_2 \leq 1.$$
$$\tfrac{3}{4} + \tfrac{1}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \tfrac{1}{4}\Delta c_2 \geq -\tfrac{3}{4} \quad \Rightarrow \quad \Delta c_2 \geq -3$$

Thus, the range of values is $-3 \leq \Delta c_2 \leq 1$.

Since $c_2 = 3 + \Delta c_2$, add 3 to this range of values, which yields

$$0 \leq c_2 \leq 4$$

as the allowable range to stay optimal for c_2.

Model Parameters

$c_1 = 3,$	$\boxed{c_2 = \mathbf{3}}$	$(n = 2)$
$a_{11} = 1,$	$a_{12} = 0,$	$b_1 = 4$
$a_{21} = 0,$	$a_{22} = \mathbf{3},$	$b_2 = 24$
$a_{31} = 3,$	$a_{32} = \mathbf{4},$	$b_3 = 18$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	0	0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{33}{2}$
x_1	(1)	0	1	0	1	0	0	4
x_2	(2)	0	0	1	$-\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$
x_4	(3)	0	0	0	$\frac{9}{4}$	1	$-\frac{3}{4}$	$\frac{39}{2}$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

Case 4 : Introduction of a new constraint

EXAMPLE: To illustrate this case, suppose that the new constraint

$$2x_1 + 3x_2 \leq 24$$

is introduced into the model

The previous optimal solution (0, 9) violates the new constraint

TABLE 6.21 Data for Variation 2 of the Wyndor Glass Co. model

Model Parameters

$$\begin{array}{lll} c_1 = 3, & c_2 = 5 & (n = 2) \\ a_{11} = 1, & a_{12} = 0, & b_1 = 4 \\ a_{21} = 0, & a_{22} = 2, & b_2 = 24 \\ a_{31} = 3, & a_{32} = 2, & b_3 = 18 \end{array}$$

Final Simplex Tableau after Reoptimization

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	45
x_3	(1)	0	1	0	1	0	0	4
x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_4	(3)	0	-3	0	0	1	-1	6

TABLE 6.25 Sensitivity analysis procedure applied to Variation 6 of the Wyndor Glass Co. model

	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Revised final tableau	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	0	45
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	9
	x_4	(3)	0	-3	0	0	1	-1	0	6
	x_6	New	0	2	3	0	0	0	1	24
Converted to proper form	Z	(0)	1	$\frac{9}{2}$	0	0	0	$\frac{5}{2}$	0	45
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	0	9
	x_4	(3)	0	-3	0	0	1	-1	0	6
	x_6	New	0	$-\frac{5}{2}$	0	0	0	$-\frac{3}{2}$	1	-3
New final tableau after reoptimization (only one iteration of dual simplex method needed in this case)	Z	(0)	1	$\frac{1}{3}$	0	0	0	0	$\frac{5}{3}$	40
	x_3	(1)	0	1	0	1	0	0	0	4
	x_2	(2)	0	$\frac{2}{3}$	1	0	0	0	$\frac{1}{3}$	8
	x_4	(3)	0	$-\frac{4}{3}$	0	0	1	0	$-\frac{2}{3}$	8
	x_5	New	0	$\frac{5}{3}$	0	0	0	1	$-\frac{2}{3}$	2

Parametric Linear Programming

Systematic Changes in the c_j Parameters

The objective function of the ordinary linear programming model is

$$Z = \sum_{j=1}^{n} c_j x_j$$

For the case where the   parameters are being changed, the objective function of the ordinary linear programming model is replaced by

$$Z(\theta) = \sum_{j=1}^{n} (c_j + \alpha_j \theta) x_j$$

where θ is a parameter and α_j are given input constants representing the relative rates at which the coefficients are to be changed.

Example:

To illustrate the solution procedure, suppose $\alpha_1=2$ and $\alpha_2 = -1$ for the original Wyndor Glass Co. problem, so that

$$Z(\theta) = (3 + 2\theta)x_1 + (5 - \theta)x_2$$

Beginning with the final simplex tableau for θ = 0, we see that its Eq. (0)

$$(0) \qquad Z + \tfrac{3}{2}x_4 + x_5 = 36$$

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Coefficient of: Original Variables	Coefficient of: Slack Variables	Right Side
0	Z	(0)	1	$-\mathbf{c}$	$\mathbf{0}$	0
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{A}$	$\mathbf{I}$	$\mathbf{b}$
Any	Z	(0)	1	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$	$\mathbf{c}_B\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$
	$\mathbf{x}_B$	$(1, 2, \ldots, m)$	$\mathbf{0}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}\mathbf{b}$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	-3	-5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	-1	1	6
2	Z	(0)	1	-2θ	θ	0	$\frac{3}{2}$	1	36
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2

If θ ≠ 0, we have

$$(0) \qquad Z - 2\theta x_1 + \theta x_2 + \tfrac{3}{2}x_4 + x_5 = 36$$

Because both x_1 and x_2 are basic variables, they both need to be eliminated algebraically from Eq. (0)

$$
\begin{array}{ll}
 & Z - 2\theta x_1 + \theta x_2 + \frac{3}{2}x_4 + x_5 = 36 \\
 & \quad +2\theta \text{ times Eq. (3)} \\
 & \quad - \ \theta \text{ times Eq. (2)} \\
\hline
(0) & Z + (\frac{3}{2} - \frac{7}{6}\theta)x_4 + (1 + \frac{2}{3}\theta)x_5 = 36 - 2\theta.
\end{array}
$$

The optimality test says that the current BF solution will remain optimal as long as these coefficients of the nonbasic variables remain nonnegative:

$$\frac{3}{2} - \frac{7}{6}\theta \geq 0, \qquad \text{for } 0 \leq \theta \leq \frac{9}{7},$$

$$1 + \frac{2}{3}\theta \geq 0, \qquad \text{for all } \theta \geq 0.$$

Range of θ	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Optimal Solution
$0 \leq \theta \leq \frac{9}{7}$	$Z(\theta)$	(0)	1	0	0	0	$\frac{9-7\theta}{6}$	$\frac{3+2\theta}{3}$	$36 - 2\theta$	$x_4 = 0$
										$x_5 = 0$
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	$x_3 = 2$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	$x_2 = 6$
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	$x_1 = 2$

Summary of the Parametric Linear Programming Procedure for Systematic Changes the c_j Parameters

TABLE 7.2 The c_j parametric linear programming procedure applied to the Wyndor Glass Co. example

Range of θ	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Optimal Solution
	$Z(\theta)$	(0)	1	0	0	0	$\frac{9-7\theta}{6}$	$\frac{3+2\theta}{3}$	$36-2\theta$	$x_4 = 0$
										$x_5 = 0$
$0 \le \theta \le \frac{9}{7}$	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	$x_3 = 2$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	$x_2 = 6$
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	$x_1 = 2$
	$Z(\theta)$	(0)	1	0	0	$\frac{-9+7\theta}{2}$	0	$\frac{5-\theta}{2}$	$27+5\theta$	$x_3 = 0$
										$x_5 = 0$
$\frac{9}{7} \le \theta \le 5$	x_4	(1)	0	0	0	3	1	-1	6	$x_4 = 6$
	x_2	(2)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	$x_2 = 3$
	x_1	(3)	0	1	0	1	0	0	4	$x_1 = 4$
	$Z(\theta)$	(0)	1	0	$-5+\theta$	$3+2\theta$	0	0	$12+8\theta$	$x_2 = 0$
										$x_3 = 0$
$\theta \ge 5$	x_4	(1)	0	0	2	0	1	0	12	$x_4 = 12$
	x_5	(2)	0	0	2	-3	0	1	6	$x_5 = 6$
	x_1	(3)	0	1	0	1	0	0	4	$x_1 = 4$

1. Solve the problem with $\theta = 0$ by the simplex method.
2. Use the sensitivity analysis procedure (Cases 2*a* and 3, Sec. 6.7) to introduce the $\Delta c_j = \alpha_j \theta$ changes into Eq. (0).
3. Increase θ until one of the nonbasic variables has its coefficient in Eq. (0) go negative (or until θ has been increased as far as desired).
4. Use this variable as the entering basic variable for an iteration of the simplex method to find the new optimal solution. Return to step 3.

Systematic Changes in the b_i Parameters

For the case where the b_i parameters change systematically, the one modification made in the original linear programming model is that is replaced by, for $i = 1, 2, \ldots, m$, where the α_i are given input constants. Thus the problem becomes

$$\text{Maximize} \qquad Z(\theta) = \sum_{j=1}^{n} c_j x_j,$$

subject to

$$\sum_{j=1}^{n} a_{ij} \leq b_i + \alpha_i \theta \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

The goal is to identify the optimal solution as a function of θ

Example:

$$\text{Maximize} \quad Z = -4y_1 - 12y_2 - 18y_3$$

subject to

$$y_1 \quad + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

Iteration	Basic Variable	Eq.	Coefficient of: Z	y_1	y_2	y_3	y_4	y_5	Right Side
0	Z	(0)	1	4	12	18	0	0	0
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_5	(2)	0	0	−2	−2	0	1	−5
1	Z	(0)	1	4	0	6	0	6	−30
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_2	(2)	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5}{2}$
2	Z	(0)	1	2	0	0	2	6	−36
	y_3	(1)	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	1
	y_2	(2)	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{3}{2}$

Suppose that α_1=2 and α_2 = -1 so that the functional constraints become

$$y_1 \quad + 3y_3 \geq 3 + 2\theta \quad \text{or} \quad -y_1 \quad - 3y_3 \leq -3 - 2\theta$$
$$2y_2 + 2y_3 \geq 5 - \theta \quad \text{or} \quad -2y_2 - 2y_3 \leq -5 + \theta.$$

This problem with θ = 0 has already been solved in the table, so we begin with the final tableau given there.

Using the sensitivity procedure, we find that the entries in the right side column of the tableau change to the values given below

$$y_0^* = \mathbf{y}^*\overline{\mathbf{b}} = [2, 6]\begin{bmatrix} -3 - 2\theta \\ -5 + \theta \end{bmatrix} = -36 + 2\theta,$$

$$\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}} = \begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 - 2\theta \\ -5 + \theta \end{bmatrix} = \begin{bmatrix} 1 + \frac{2\theta}{3} \\ \frac{3}{2} - \frac{7\theta}{6} \end{bmatrix}$$

Therefore, the two basic variables in this tableau

$$y_3 = \frac{3 + 2\theta}{3} \qquad \text{and} \qquad y_2 = \frac{9 - 7\theta}{6}$$

Remain nonnegative for $0 \le \theta \le \frac{9}{7}$.

Summary of the Parametric Linear Programming Procedure for Systematic Changes the b_i Parameters

TABLE 7.3 The b_i parametric linear programming procedure applied to the dual of the Wyndor Glass Co. example

Range of θ	Basic Variable	Eq.	Z	y_1	y_2	y_3	y_4	y_5	Right Side	Optimal Solution
$0 \le \theta \le \frac{9}{7}$	$Z(\theta)$	(0)	1	2	0	0	2	6	$-36 + 2\theta$	$y_1 = y_4 = y_5 = 0$
	y_3	(1)	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{3 + 2\theta}{3}$	$y_3 = \frac{3 + 2\theta}{3}$
	y_2	(2)	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{9 - 7\theta}{6}$	$y_2 = \frac{9 - 7\theta}{6}$
$\frac{9}{7} \le \theta \le 5$	$Z(\theta)$	(0)	1	0	6	0	4	3	$-27 - 5\theta$	$y_2 = y_4 = y_5 = 0$
	y_3	(1)	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5 - \theta}{2}$	$y_3 = \frac{5 - \theta}{2}$
	y_1	(2)	0	1	-3	0	-1	$\frac{3}{2}$	$\frac{-9 + 7\theta}{2}$	$y_1 = \frac{-9 + 7\theta}{2}$
$\theta \ge 5$	$Z(\theta)$	(0)	1	0	12	6	4	0	$-12 - 8\theta$	$y_2 = y_3 = y_4 = 0$
	y_5	(1)	0	0	-2	-2	0	1	$-5 + \theta$	$y_5 = -5 + \theta$
	y_1	(2)	0	1	0	3	-1	0	$3 + 2\theta$	$y_1 = 3 + 2\theta$

1. Solve the problem with $\theta = 0$ by the simplex method.
2. Use the sensitivity analysis procedure (Case 1, Sec. 6.7) to introduce the $\Delta b_i = \alpha_i \theta$ changes to the *right side* column.
3. Increase θ until one of the basic variables has its value in the *right side* column go negative (or until θ has been increased as far as desired).
4. Use this variable as the leaving basic variable for an iteration of the dual simplex method to find the new optimal solution. Return to step 3.