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Weak formulation for Boundary Value Problem

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for } 0 < x < L$$

subject to

$$u(0) = u_0 \qquad \left(a\frac{du}{dx}\right)\Bigg|_{x=L} = q_L$$

Three Steps in the development of weak form

1) Move all the eq^n to the left side, multiply the whole eq^n with a test or weight f^n w and integrate over $(0, L)$

$$0 = \int_0^L w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) - f(x)\right] dx$$

↳ weighted integral or weighted residual equivalent to original eq^n

2) Trade differentiation from u to w using integration by parts

$$0 = \int_0^L \left\{ w \left[-\frac{d}{dx}\left(a \frac{du}{dx} \right) \right] - w f(x) \right\} dx$$

$$= \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - w f(x) \right) dx$$

$$- \left[w a \frac{du}{dx} \right]_0^L$$

$$= \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - w f(x) \right) dx$$

$$- \left(w a \frac{du}{dx} \right) \Big|_{x=0} - \left(w a \frac{du}{dx} \right) \Big|_{x=L}$$

$$= \int_0^L \left(\frac{dw}{dx} a \frac{du}{dx} - w f(x) \right) dx$$

$$- (wQ)_0 - (wQ)_L$$

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3) Use the Neumann Boundary Condition

$u = u_0$ → Essential Boundary Condition

$\left(a \frac{du}{dx}\right)\Big|_{x=L} = q_L$ → Natural Boundary Condition

$\therefore w(0) = 0$ $\quad \because u(0) = u_0$

As $w(0) = 0$

and

$$q(L) = \left(a \frac{du}{dx} n_x\right)\Big|_{x=L}$$

$$= \left(a \frac{du}{dx}\right)\Big|_{x=L} = q_L$$

we get

$$0 = \int_0^L \left(a \frac{dw}{dx} \frac{du}{dx} - wq\right) dx - w(L)\, q_L$$

Weak form equivalent of original Differential Eq^n.