Operational Semantics

Part I

Jim Royer

CIS 352

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References

- Andrew Pitts' Lecture Notes on Semantics of Programming Languages
  `http://www.inf.ed.ac.uk/teaching/courses/lsi/sempl.pdf`.
  We'll be following the Pitts' notes for a while and mostly using his notation.
- Matthew Hennessy's Semantics of programming languages:
  `https://www.scss.tcd.ie/Matthew.Hennessy/splexternal2015/LectureNotes/Notes14%20copy.pdf`
  is very readable and very good.
- There are many of other good references in Hennessy's reading list:
  `https://www.scss.tcd.ie/Matthew.Hennessy/splexternal2015/reading.php`

Aexp, A little language for arithmetic expressions

Grammar

$$
\begin{aligned}
a \;&::=\; n \\
&\mid\; (a_1 + a_2) \\
&\mid\; (a_1 - a_2) \\
&\mid\; (a_1 * a_2) \\
n \;&::=\; \ldots
\end{aligned}
$$

Syntactic categories

$n \in$ **Num**	Numerals
$a \in$ **Aexp**	Arithmetic expressions

Conventions

- Metavariables: n, a, b, w, x, etc.
- We write $\ulcorner 35 \urcorner$ for the numeral 35.

Examples

- $\ulcorner 2 \urcorner$
- $(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner)$
- $(((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) - \ulcorner 9 \urcorner)$

Syntax

Concrete syntax

≈ phonemes, characters, words, tokens — the raw stuff of language

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, ...

Grammar

≈ collection of formation rules to organize parts into a whole. E.g.,

- words into noun phrases, verb phrases, ..., sentences
- key words, tokens, ... into expressions, statements, ..., programs

Abstract syntax

≈ a structure (e.g., labeled tree or data structure) showing how a "phrase" breaks down into pieces according to a specific rule.

Aexp's abstract syntax

Grammar

$$
\begin{array}{rcl}
a & ::= & n \\
  & \mid & (a_1 + a_2) \\
  & \mid & (a_1 - a_2) \\
  & \mid & (a_1 * a_2) \\
n & ::= & \ldots
\end{array}
$$

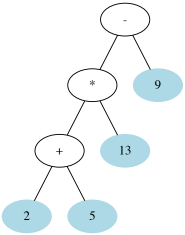
In Haskell

```
data AExp = Num Integer
          | Plus AExp AExp
          | Minus AExp AExp
          | Times AExp AExp
```

$(((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) - \ulcorner 9 \urcorner)$

```
Minus (Times (Plus (Num 2) (Num 5))
             (Num 13))
      (Num 9)
```

As a Parse Tree



What do **Aexp** expression mean? Big-step rules

$$a \ ::= \ n \mid (a_1 + a_2) \mid (a_1 - a_2) \mid (a_1 * a_2)$$

$$\textit{PLUS:}\ \frac{a_1 \Downarrow v_1 \qquad a_2 \Downarrow v_2}{(a_1 + a_2) \Downarrow v}\ (v = v_1 + v_2)$$

$$\textit{MINUS:}\ \frac{a_1 \Downarrow v_1 \qquad a_2 \Downarrow v_2}{(a_1 - a_2) \Downarrow v}\ (v = v_1 - v_2)$$

$$\textit{MULT:}\ \frac{a_1 \Downarrow v_1 \qquad a_2 \Downarrow v_2}{(a_1 * a_2) \Downarrow v}\ (v = v_1 * v_2)$$

$$\textit{NUM:}\ \frac{}{n \Downarrow v}\ (\mathcal{N}[\![n]\!] = v)$$

Notes

- $a \Downarrow v \equiv$ expression a evaluates to value v.
- $\Downarrow$ is an evaluation relation.
- Upstairs assertions are called premises.
- Downstairs assertions are called conclusions.
- Parenthetical equations on the side are called side conditions.
- $\mathcal{N}$: numerals $\to \mathbb{Z}$.
  I.e., $\mathcal{N}[\![\ulcorner -43 \urcorner]\!] = -43$.
- The $\textit{NUM}_{BSS}$ rule is an example of an axiom.

Digression: Rules, 1

General Format for Rules

$$\textit{Name:}\ \frac{\text{premise}_1 \quad \cdots \quad \text{premise}_k}{\text{conclusion}}\ (\text{side condition})$$

Example 1.

- *Modus Ponens:* $\dfrac{p \implies q \qquad p}{q}$
- *Transitivity:* $\dfrac{x \equiv y \qquad y \equiv z}{x \equiv z}$
- *PLUS:* $\dfrac{a_1 \Downarrow v_1 \qquad a_2 \Downarrow v_2}{(a_1 + a_2) \Downarrow v}\ (v = v_1 + v_2)$

Digression: Rules, 2

General Format for Rules

$$\textit{Name:}\ \frac{\text{premise}_1 \quad \cdots \quad \text{premise}_k}{\text{conclusion}}\ \text{(side condition)}$$

Definition 2.

A rule with no premises is an *axiom*.

Definition 3.

A rule is *sound* if and only if the conclusion is true whenever the premises (and side-condition—if any) are true.

Question

So an axiom is sound when . . . ?

Digression: Rules, 3

General Format for Rules

$$\textit{Name:}\ \frac{\text{premise}_1 \quad \cdots \quad \text{premise}_k}{\text{conclusion}}\ (\text{side condition})$$

Proofs from gluing together rule applications

$$\textit{Times:}\ \dfrac{\textit{Plus:}\ \dfrac{\textit{Num:}\ \dfrac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \textit{Num:}\ \dfrac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow 7}\ \left(\begin{matrix}2+5\\=7\end{matrix}\right) \qquad \textit{Num:}\ \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow 91}\ \left(\begin{matrix}7*13\\=91\end{matrix}\right)$$

Rules can also be the basis of a computation

$$\frac{\vdots}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

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$$\Downarrow\Downarrow$$

$$\frac{\dfrac{\vdots}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

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$$\downdownarrows\downdownarrows$$

$$\frac{\dfrac{\vdots}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\dfrac{\dfrac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \dfrac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

Rules can also be the basis of a computation

$$\frac{\vdots}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\dfrac{\vdots}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\dfrac{\dfrac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \dfrac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\dfrac{\dfrac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \dfrac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow 7} \qquad \dfrac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

Rules can also be the basis of a computation

$$\frac{\vdots}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\frac{\vdots}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \frac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\frac{\frac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \frac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow ??} \qquad \frac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\frac{\frac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \frac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow 7} \qquad \frac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow ??}$$

$$\downdownarrows\downdownarrows$$

$$\frac{\frac{\frac{}{\ulcorner 2 \urcorner \Downarrow 2} \qquad \frac{}{\ulcorner 5 \urcorner \Downarrow 5}}{(\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) \Downarrow 7} \qquad \frac{}{\ulcorner 13 \urcorner \Downarrow 13}}{((\ulcorner 2 \urcorner + \ulcorner 5 \urcorner) * \ulcorner 13 \urcorner) \Downarrow 91}$$

The big-step semantics in Haskell

A Haskell version of the abstract syntax

```
data Aexp  = Num Integer
           | Add Aexp Aexp
           | Sub Aexp Aexp
           | Mult Aexp Aexp
```

The big-step semantics as an evaluator function

```
aBig (Add a1 a2)  = (aBig a1) + (aBig a2)
aBig (Sub a1 a2)  = (aBig a1) - (aBig a2)
aBig (Mult a1 a2) = (aBig a1) * (aBig a2)
aBig (Num n)      = n
```

Do these rules make sense?

Theorem 4.

Suppose $e \in \mathbf{Aexp}$.
Then there is a unique integer v such that $e \Downarrow v$.

Proof (by rule induction).

CASE: *NUM*. This is immediate.

CASE: *PLUS*.
By IH, there are unique v_1 and v_2 such that $a_1 \Downarrow v_1$ and $a_2 \Downarrow v_2$.
By arithmetic, there is a unique v such that $v = v_1 + v_2$.
Hence, there is a unique v such that $a_1 + a_2 \Downarrow v$.

CASES: *MINUS* and *MULT*. These follow *mutatis mutandis*. $\square$

$$PLUS_{BSS}\colon \frac{a_1 \Downarrow v_1 \qquad a_2 \Downarrow v_2}{(a_1 + a_2) \Downarrow v} \ (v = v_1 + v_2) \quad \ldots \quad NUM_{BSS}\colon \frac{}{n \Downarrow v} \ (\mathcal{N}[\![n]\!] = v)$$

What do **Aexp** expression mean? Small-step rules

$$a \;::=\; n \mid (a_1 + a_2) \mid (a_1 - a_2) \mid (a_1 * a_2) \mid v$$

$$\textit{PLUS-1}_{SSS}: \quad \frac{a_1 \to a_1'}{(a_1 + a_2) \to (a_1' + a_2)}$$

$$\textit{PLUS-2}_{SSS}: \quad \frac{a_2 \to a_2'}{(a_1 + a_2) \to (a_1 + a_2')}$$

$$\textit{PLUS-3}_{SSS}: \quad \frac{}{(v_1 + v_2) \to v} \quad (v = v_1 + v_2)$$

$$\vdots$$

$$\textit{NUM}_{SSS}: \quad \frac{}{n \to v} \quad (\mathcal{N}[\![n]\!] = v)$$

Notes

- These are rewrite rules.
- We now allow values in expressions.
- $a \to a'$ is a transition.
- $a \to a' \equiv$ expression a evaluates (or rewrites) to a' in one-step.
- v is a terminal expression.
- The rules for $-$ and $*$ follow the same pattern as the $+$-rules.

Class exercise

Show:

$$(((\ulcorner 3 \urcorner * \ulcorner 2 \urcorner) + (\ulcorner 8 \urcorner - \ulcorner 3 \urcorner)) * (\ulcorner 5 \urcorner - \ulcorner 2 \urcorner))$$

$$\rightarrow \begin{cases} ((6 + (\ulcorner 8 \urcorner - \ulcorner 3 \urcorner)) * (\ulcorner 5 \urcorner - \ulcorner 2 \urcorner)) \\ (((\ulcorner 3 \urcorner * \ulcorner 2 \urcorner) + 5) * (\ulcorner 5 \urcorner - \ulcorner 2 \urcorner)) \\ (((\ulcorner 3 \urcorner * \ulcorner 2 \urcorner) + (\ulcorner 8 \urcorner - \ulcorner 3 \urcorner)) * 3) \end{cases}$$

Some full small-step derivations of transitions

$$MULT_1 \; \dfrac{PLUS_2 \; \dfrac{MINUS_3 \; \dfrac{}{(8-3) \to 5}}{(6+(8-3)) \to (6+5)}}{((6+(8-3)) * (5-2)) \to ((6+5) * (5-2))}$$

$$MULT_1 \; \dfrac{PLUS_3 \; \dfrac{}{(6+5) \to 11}}{((6+5) * (5-2)) \to 11 * (5-2)}$$

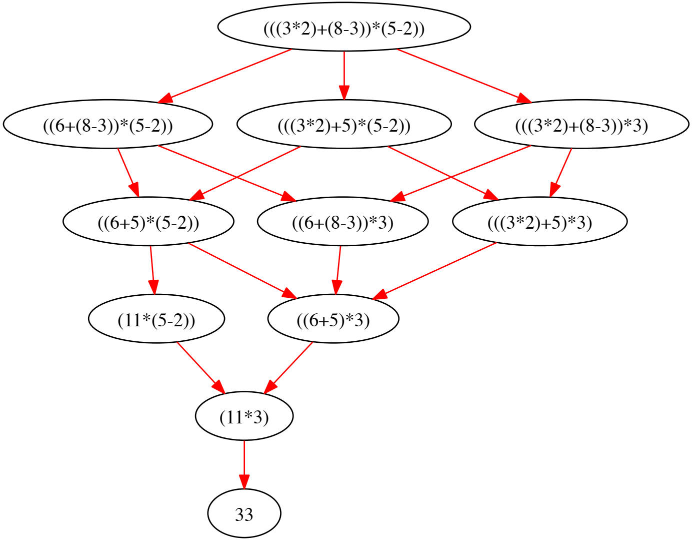
$$MULT_2 \; \dfrac{MINUS_3 \; \dfrac{}{(5-2) \to 3}}{(11 * (5-2)) \to 11 * 3}$$

$$MULT_3 \; \dfrac{}{(11 * 3) \to 33}$$

The derivations show that the steps in the transition sequence below are legal (i.e., follow from the rules).

$$((6+(8-3)) * (5-2))$$
$$\to$$
$$((6+5) * (5-2))$$
$$\to$$
$$11 * (5-2)$$
$$\to$$
$$11 * 3$$
$$\to$$
$$33$$

There is a lattice of transitions



Properties of operational semantics

Definition 5.

A transition system $(\Gamma, \rightsquigarrow, T)$ is deterministic when for all a, a_1, and a_2:

$$\text{If } a \rightsquigarrow a_1 \text{ and } a \rightsquigarrow a_2, \text{ then } a_1 = a_2.$$

Theorem 6.

*The big-step semantics for **Aexp** is deterministic.*

The proof is an easy rule induction.

Theorem 7.

*The given small-step semantics $(\textbf{Aexp} \cup \mathbb{Z}, \Rightarrow, \mathbb{Z})$ fails to be deterministic, **but** for all $a \in \textbf{Aexp}$ and $v_1, v_2 \in \mathbb{Z}$, if $a \Rightarrow^* v_1$ and $a \Rightarrow^* v_2$, then $v_1 = v_2$.*

This proof is tricky because of the nondeterminism.

Very sketchy proof-sketch

Theorem 8.

The given small-step semantics $(\mathbf{Aexp} \cup \mathbb{Z}, \Rightarrow, \mathbb{Z})$ ***fails*** *to be deterministic,* ***but*** *for all* $a \in \mathbf{Aexp}$ *and* $v_1, v_2 \in \mathbb{Z}$*, if* $a \Rightarrow^* v_1$ *and* $a \Rightarrow^* v_2$*, then* $v_1 = v_2$*.*

Proof-sketch.

- The argument is by induction on the number of operators (i.e., $+$, $-$, and $*$) occurring in a.
- **Base case:** a is a numeral, so it hasn't any operators and is a terminal expression. Hence if $a \Rightarrow^* v$, then $v = a$ is our only choice.
- **Induction step:** Suppose by induction the theorem is true for all expressions of n or fewer operators and suppose $a = a_1 + a_2$ has $n + 1$ many operators. (The arguments for $a = a_1 - a_2$ and $a = a_1 * a_2$ will be similar.)

…more

Very sketchy proof-sketch, continued

Theorem 8.

*The given small-step semantics $(\mathbf{Aexp} \cup \mathbb{Z}, \Rightarrow, \mathbb{Z})$ **fails** to be deterministic, **but** for all $a \in \mathbf{Aexp}$ and $v_1, v_2 \in \mathbb{Z}$, if $a \Rightarrow^* v_1$ and $a \Rightarrow^* v_2$, then $v_1 = v_2$.*

- The a_1 and a_2 are expressions with n or fewer operators.
- The last step in any transition sequence $a \Rightarrow^* v$ is of the form $v_1 + v_2 \Rightarrow v$ and justified by $PLUS_3$.
- In each step before the last, the final rule in the step-justification was either a $PLUS_1$ or a $PLUS_2$. [Clarify!]
- If we look at the premises of the $PLUS_1$'s, they give a small-step derivation $a_1 \Rightarrow^* v_1$. By the IH, we know that any $\Rightarrow$-reduction sequence for a_1 that ends with a value *must* produce v_1.
- Similarly, $a_2 \Rightarrow^* v_2$ is also determined.
- So, it follows that if $a \Rightarrow^* v$, we must have $v = v_1 + v_2$. □

A deterministic small-step semantics for **Aexp**

$$a \;::=\; n \mid (a_1 + a_2) \mid (a_1 - a_2) \mid (a_1 * a_2) \mid v$$

$$\textit{PLUS-1}'_{SSS}: \quad \frac{a_1 \rightarrow a_1'}{a_1 + a_2 \rightarrow a_1' + a_2}$$

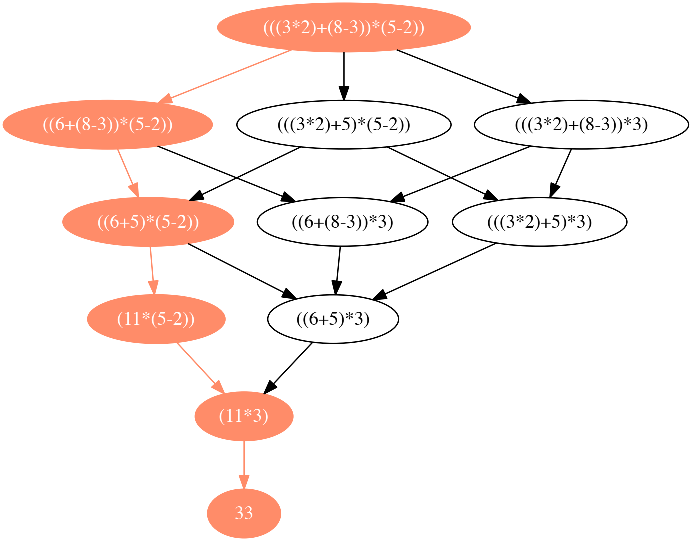
$$\textit{PLUS-2}'_{SSS}: \quad \frac{a_2 \rightarrow a_2'}{v_1 + a_2 \rightarrow v_1 + a_2'}$$

$$\textit{PLUS-3}'_{SSS}: \quad \frac{}{v_1 + v_2 \rightarrow v} \; (v = v_1 + v_2)$$

$$\vdots$$

$$\textit{NUM}_{SSS}: \quad \frac{}{n \rightarrow v} \; (\mathcal{N}[\![n]\!] = v)$$

The leftmost path through the lattice of transitions



Why multiple flavors of semantics?

They provide different views of computations.

- Big-step is good for reasoning about how the (big) pieces of things fit together.
- Small step is good at reasoning about the (small) steps of a computation fit together.
- Small step semantics is much better at modeling inherent nondeterminism (e.g., in concurrent programs).

- …and there are other flavors (e.g., denotational) for other purposes (e.g., obtaining stronger forms of soundness).