Bayes' Rule Probability Homework

Problem 1: Coin Flipping

You have two coins: Coin 1 is a fair coin with equal probability of heads and tails, and Coin 2 has a 75% probability of heads and a 25% probability of tails. You choose a coin at random and flip it. If it lands heads, what is the probability that you chose Coin 1?

Solution:

Let H be the event of flipping heads, and let C_i be the event of choosing Coin i. We want to find $P(C_1|H)$. Using Bayes' theorem:

$$\begin{aligned} P(C_1|H) &= \frac{P(H|C_1)P(C_1)}{P(H)} \\ &= \frac{P(H|C_1)P(C_1)}{P(H|C_1)P(C_1) + P(H|C_2)P(C_2)} \\ &= \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.75 \times 0.5} \\ &= \frac{0.25}{0.25 + 0.375} \\ &= \frac{0.25}{0.625} \\ &= \frac{2}{5}. \end{aligned}$$

Problem 2: Medical Testing

A certain disease has a prevalence rate of 1% in the population. A diagnostic test for this disease has a 98% true positive rate (sensitivity) and a 97% true negative rate (specificity). If a person tests positive for the disease, what is the probability that they actually have the disease?

Solution:

Let D be the event of having the disease, and let T^+ be the event of testing positive. We want to find $P(D|T^+)$. Using Bayes' theorem:

$$\begin{aligned} P(D|T^+) &= \frac{P(T^+|D)P(D)}{P(T^+)} \\ &= \frac{P(T^+|D)P(D)}{P(T^+|D)P(D) + P(T^+|D^c)P(D^c)} \\ &= \frac{0.98 \times 0.01}{0.98 \times 0.01 + (1 - 0.97) \times (1 - 0.01)} \\ &= \frac{0.0098}{0.0098 + 0.03 \times 0.99} \\ &\approx 0.2458. \end{aligned}$$

Problem 3: Email Spam Filtering

An email spam filter has a 99% success rate in detecting spam emails and a 95% success rate in correctly identifying legitimate emails. If 20% of all emails are spam, what is the probability that an email flagged as spam is actually spam?

Solution:

Let S be the event of an email being spam, and let F be the event of an email being flagged as spam. We want to find $P(S|F)$.

$$\begin{aligned}
P(S|F) &= \frac{P(F|S)P(S)}{P(F)} \\
&= \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|S^c)P(S^c)} \\
&= \frac{0.99 \times 0.2}{0.99 \times 0.2 + (1 - 0.95) \times (1 - 0.2)} \\
&= \frac{0.198}{0.198 + 0.05 \times 0.8} \\
&= \frac{0.198}{0.198 + 0.04} \\
&= \frac{0.198}{0.238} \\
&\approx 0.8319.
\end{aligned}$$

Problem 4: Manufacturing Defects

A factory has two production lines (Line 1 and Line 2) manufacturing the same product. Line 1 produces 70% of the products, while Line 2 produces 30% of the products. Line 1 has a defect rate of 4%, and Line 2 has a defect rate of 8%. If a product is found to be defective, what is the probability it was produced by Line 1?

Solution:

Let D be the event of a product being defective, and let L_i be the event of a product being produced by Line i. We want to find $P(L_1|D)$. Using Bayes' theorem:

$$\begin{aligned}
P(L_1|D) &= \frac{P(D|L_1)P(L_1)}{P(D)} \\
&= \frac{P(D|L_1)P(L_1)}{P(D|L_1)P(L_1) + P(D|L_2)P(L_2)} \\
&= \frac{0.04 \times 0.7}{0.04 \times 0.7 + 0.08 \times 0.3} \\
&= \frac{0.028}{0.028 + 0.024} \\
&= \frac{0.028}{0.052} \\
&\approx 0.5385.
\end{aligned}$$

Problem 5: Internet Connectivity

Two internet service providers (ISP 1 and ISP 2) offer internet connectivity in a neighborhood. ISP 1 serves 80% of the households, while ISP 2 serves the remaining 20%. The connection failure rates for ISP 1 and ISP 2 are 1% and 3%, respectively. If a household experiences a connection failure, what is the probability that they are using ISP 1?

Solution:

Let F be the event of a connection failure, and let I_i be the event of using ISP i. We want to find $P(I_1|F)$. Using Bayes' theorem:

$$\begin{aligned} P(I_1|F) &= \frac{P(F|I_1)P(I_1)}{P(F)} \\ &= \frac{P(F|I_1)P(I_1)}{P(F|I_1)P(I_1) + P(F|I_2)P(I_2)} \\ &= \frac{0.01 \times 0.8}{0.01 \times 0.8 + 0.03 \times 0.2} \\ &= \frac{0.008}{0.008 + 0.006} \\ &= \frac{0.008}{0.014} \\ &\approx 0.5714. \end{aligned}$$