

## Planar Ellipse Rolling on a Curve

A planar ellipse  $E$  moves in a plane fixed in a Newtonian reference frame  $N$  in such a way that it is always in contact with a curve  $C$  fixed in  $N$ ,  $y = f(x)$ , as illustrated in Fig. 1. The object of this exercise is to relate  $dx/dt$  to the angular speed of  $E$  in  $N$  when  $E$  rolls on  $C$ .

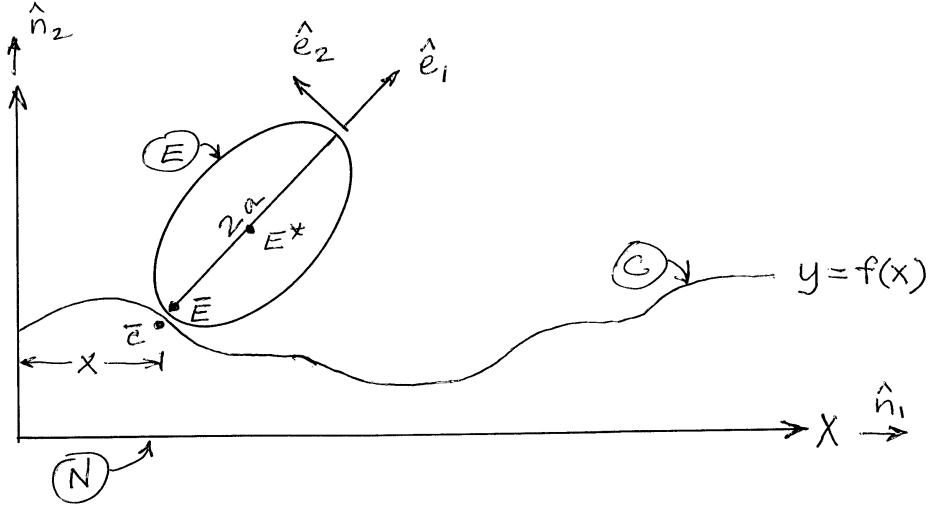


Figure 1: Ellipse in Contact with Curve

Two perpendicular unit vectors fixed in  $N$  are denoted by  $\hat{\mathbf{n}}_1$  and  $\hat{\mathbf{n}}_2$ ; they span the plane in which  $E$  moves. Two perpendicular unit vectors fixed in  $E$  are labeled  $\hat{\mathbf{e}}_1$  and  $\hat{\mathbf{e}}_2$ , as shown in Fig. 1. If  $q$  is the angle between  $\hat{\mathbf{n}}_1$  and  $\hat{\mathbf{e}}_1$  (and the angle between  $\hat{\mathbf{n}}_2$  and  $\hat{\mathbf{e}}_2$ ), the two sets of unit vectors are related by the expressions

$$\hat{\mathbf{e}}_1 = \cos q \hat{\mathbf{n}}_1 + \sin q \hat{\mathbf{n}}_2 \stackrel{\triangle}{=} C_q \hat{\mathbf{n}}_1 + S_q \hat{\mathbf{n}}_2 \quad (1)$$

$$\hat{\mathbf{e}}_2 = -\sin q \hat{\mathbf{n}}_1 + \cos q \hat{\mathbf{n}}_2 \stackrel{\triangle}{=} -S_q \hat{\mathbf{n}}_1 + C_q \hat{\mathbf{n}}_2 \quad (2)$$

where  $C_q$  and  $S_q$  are shorthand for  $\cos q$  and  $\sin q$ , respectively.

### Position of the Contact Point on the Ellipse

The locus of points on the periphery of  $E$  is defined by the equation

$$f(e_1, e_2) = \left(\frac{e_1}{a}\right)^2 + \left(\frac{e_2}{b}\right)^2 - 1 = 0 \quad (3)$$

where displacements  $e_1$  and  $e_2$  are measured in the directions of  $\hat{\mathbf{e}}_1$  and  $\hat{\mathbf{e}}_2$ , respectively, from the center  $E^*$  of the ellipse, and  $a$  and  $b$  are the semidiameters of the ellipse. Proceeding as in Ref. [1] (in connection with an ellipsoidal rattleback), the coordinates of the point  $\bar{E}$  of

$E$  that is in contact with  $C$  are found by noting that the normal to  $E$  at  $\bar{E}$  is parallel to  $\hat{\mathbf{N}}$ , the unit normal to  $C$  at  $\bar{C}$ , the point of  $C$  that is in contact with  $E$ . Because  $\nabla f(e_1, e_2)$ , the gradient to the curved surface of  $E$ , is parallel to the normal of  $E$ , one can write

$$\nabla f(e_1, e_2) = 2\lambda \hat{\mathbf{N}} \quad (4)$$

where  $\lambda$  is a quantity that will be determined presently.

The gradient is easily found from Eq. (3) to be

$$\nabla f(e_1, e_2) = \frac{2e_1}{a^2} \hat{\mathbf{e}}_1 + \frac{2e_2}{b^2} \hat{\mathbf{e}}_2 \quad (5)$$

It is worth noting that this is an outward normal; that is, it is directed away from  $E^*$ . The normal to  $C$  at  $\bar{C}$  is obtained by parameterizing the position vector to  $\bar{C}$  as a vector function of  $x$ , as illustrated in Fig. 2.

$$\mathbf{r}(x) = x\hat{\mathbf{n}}_1 + f(x)\hat{\mathbf{n}}_2 \quad (6)$$

A vector tangent to the curve at  $\bar{C}$  is then given by

$$\mathbf{T} = \frac{d\mathbf{r}}{dx} = 1\hat{\mathbf{n}}_1 + \frac{df}{dx}\hat{\mathbf{n}}_2 \triangleq \hat{\mathbf{n}}_1 + f'\hat{\mathbf{n}}_2 \quad (7)$$

where  $f'$  is short for  $df/dx$ . The unit tangent vector is given by

$$\hat{\mathbf{T}} = \frac{\mathbf{T}}{|\mathbf{T}|} = \frac{\hat{\mathbf{n}}_1 + f'\hat{\mathbf{n}}_2}{\sqrt{1 + (f')^2}} \quad (8)$$

A unit vector normal to the plane in which  $E$  moves is given by  $\hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ . The unit normal to  $C$  at  $\bar{C}$  is then

$$\hat{\mathbf{N}} = \hat{\mathbf{n}}_3 \times \hat{\mathbf{T}} = \frac{\hat{\mathbf{n}}_2 - f'\hat{\mathbf{n}}_1}{\sqrt{1 + (f')^2}} \quad (9)$$

It is worth noting that the normal is equal to  $\hat{\mathbf{n}}_2$  when the curve is a horizontal line; that is,

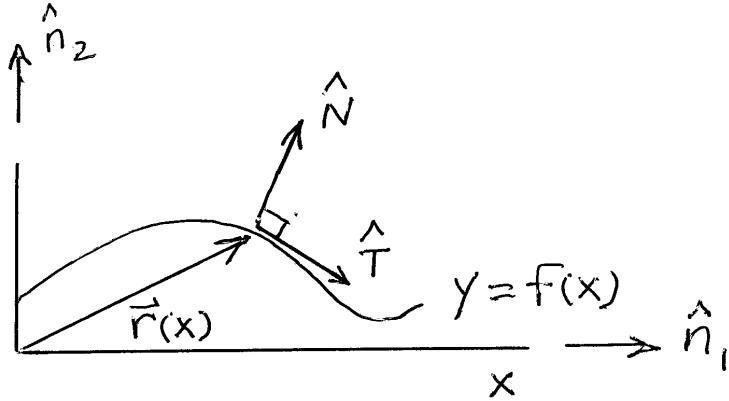


Figure 2: Unit Tangent and Unit Normal to Curve

when  $f(x)$  is equal to a constant. Substitution from Eqs. (5) and (9) into (4), and subsequent dot multiplication with  $\hat{\mathbf{e}}_1$ , yields

$$\frac{e_1}{a^2} = \frac{\lambda(S_q - f'C_q)}{\sqrt{1 + (f')^2}} = \frac{\lambda z_1}{\sqrt{1 + (f')^2}} \quad (10)$$

whereas dot multiplication with  $\hat{\mathbf{e}}_2$  produces

$$\frac{e_2}{b^2} = \frac{\lambda(C_q + f'S_q)}{\sqrt{1 + (f')^2}} = \frac{\lambda z_2}{\sqrt{1 + (f')^2}} \quad (11)$$

where two auxilliary variables  $z_1$  and  $z_2$  have been introduced and defined, respectively, as

$$z_1 \triangleq S_q - f'C_q \quad z_2 \triangleq C_q + f'S_q \quad (12)$$

There will be occasion in what follows to make use of the time derivatives

$$\dot{z}_1 = C_q \dot{q} - f'' \dot{x} C_q + f' S_q \dot{q} = z_2 \dot{q} - f'' \dot{x} C_q \quad (13)$$

$$\dot{z}_2 = -S_q \dot{q} + f'' \dot{x} S_q + f' C_q \dot{q} = f'' \dot{x} S_q - z_1 \dot{q} \quad (14)$$

Substitution from Eqs. (10) and (11) into (3) yields

$$\frac{(a\lambda z_1)^2}{1 + (f')^2} + \frac{(b\lambda z_2)^2}{1 + (f')^2} - 1 = 0 \quad (15)$$

which can be solved for  $\lambda$

$$\lambda = \pm \frac{\sqrt{1 + (f')^2}}{[(az_1)^2 + (bz_2)^2]^{\frac{1}{2}}} \quad (16)$$

The correct choice of sign on the right-hand side is negative, based on the following observations. In the case when the curve is a horizontal line, the outward normal to the ellipse [on the left-hand side of Eq. (4)] at  $\bar{E}$  should have the same direction as  $-\hat{\mathbf{n}}_2$ ; however,  $\hat{\mathbf{N}}$  on the right-hand side of Eq. (4) reduces to  $\hat{\mathbf{n}}_2$ . Therefore, the coordinates of  $\bar{E}$  are given by

$$e_1 = -\frac{a^2 z_1}{[(az_1)^2 + (bz_2)^2]^{\frac{1}{2}}} \quad (17)$$

$$e_2 = -\frac{b^2 z_2}{[(az_1)^2 + (bz_2)^2]^{\frac{1}{2}}} \quad (18)$$

Thus, given the slope of the curve,  $f'$ , and any value of the rotation angle  $q$ , the position vector from  $E^*$  to the contact point  $\bar{E}$  is given by

$$\mathbf{r}^{E^* \bar{E}} = e_1 \hat{\mathbf{e}}_1 + e_2 \hat{\mathbf{e}}_2 \quad (19)$$

## Velocity of the Contact Point on the Ellipse

The velocity  ${}^N \mathbf{v}^{E^*}$  of  $E^*$  in  $N$  can be obtained by proceeding as in Problem 3.3 of Ref. [2] in connection with a circular disk in contact with a horizontal surface. The position vector from a point  $O$  fixed in  $N$  to  $E^*$ , given by

$$\mathbf{r}^{OE^*} = x \hat{\mathbf{n}}_1 + f(x) \hat{\mathbf{n}}_2 + \mathbf{r}^{\bar{E} E^*} \quad (20)$$

is differentiated with respect to time in  $N$ ,

$${}^N \mathbf{v}^{E^*} = \frac{N d \mathbf{r}^{OE^*}}{dt} = \dot{x} \hat{\mathbf{n}}_1 + f' \dot{x} \hat{\mathbf{n}}_2 + \frac{E d \mathbf{r}^{\bar{E} E^*}}{dt} + {}^N \boldsymbol{\omega}^E \times \mathbf{r}^{\bar{E} E^*} \quad (21)$$

where  ${}^N\boldsymbol{\omega}^E$  is the angular velocity of  $E$  in  $N$ . However, as in Problem 3.6 of Ref. [2], the velocity in  $N$  of the point of contact is

$$\begin{aligned} {}^N\mathbf{v}^{\bar{E}} &= {}^N\mathbf{v}^{E^*} + {}^N\boldsymbol{\omega}^E \times \mathbf{r}^{E^*\bar{E}} = \dot{x}\hat{\mathbf{n}}_1 + f'\dot{x}\hat{\mathbf{n}}_2 + \frac{{}^E d\mathbf{r}^{\bar{E}E^*}}{dt} + {}^N\boldsymbol{\omega}^E \times (\mathbf{r}^{\bar{E}E^*} + \mathbf{r}^{E^*\bar{E}}) \\ &= \dot{x}\hat{\mathbf{n}}_1 + f'\dot{x}\hat{\mathbf{n}}_2 + \frac{{}^E d\mathbf{r}^{\bar{E}E^*}}{dt} \end{aligned} \quad (22)$$

where the term in parentheses vanishes because  $\mathbf{r}^{\bar{E}E^*} = -\mathbf{r}^{E^*\bar{E}}$ . In Problem 3.8 of Ref. [2], the condition of rolling (which is the absence of slipping) is imposed on a circular disk. In the present case, the velocities in any reference frame of the two points in contact,  $\bar{C}$  and  $\bar{E}$ , must be equal. As  $\bar{C}$  is fixed in  $N$ ,  ${}^N\mathbf{v}^{\bar{C}} = \mathbf{0}$ ; therefore,  ${}^N\mathbf{v}^{\bar{E}} = \mathbf{0}$  when  $E$  rolls on  $C$ . It turns out the objective of the present exercise is conveniently obtained from dot multiplying the foregoing equation with  $\hat{\mathbf{e}}_1$ :

$${}^N\mathbf{v}^{\bar{E}} \cdot \hat{\mathbf{e}}_1 = \left( \dot{x}\hat{\mathbf{n}}_1 + f'\dot{x}\hat{\mathbf{n}}_2 + \frac{{}^E d\mathbf{r}^{\bar{E}E^*}}{dt} \right) \cdot \hat{\mathbf{e}}_1 = \mathbf{0} \cdot \hat{\mathbf{e}}_1 = 0 \quad (23)$$

Now,

$$\frac{{}^E d\mathbf{r}^{\bar{E}E^*}}{dt} \cdot \hat{\mathbf{e}}_1 = -(\dot{e}_1 \hat{\mathbf{e}}_1 + \dot{e}_2 \hat{\mathbf{e}}_2) \cdot \hat{\mathbf{e}}_1 = -\dot{e}_1 = \frac{d}{dt} \left\{ \frac{a^2 z_1}{[(az_1)^2 + (bz_2)^2]^{\frac{1}{2}}} \right\} \quad (24)$$

or

$$\begin{aligned} -\dot{e}_1 &= \frac{a^2 \dot{z}_1}{[(az_1)^2 + (bz_2)^2]^{\frac{1}{2}}} + \frac{a^2 z_1 (-\frac{1}{2})(2a^2 z_1 \dot{z}_1 + 2b^2 z_2 \dot{z}_2)}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} \\ &= \frac{a^2 \dot{z}_1 [(az_1)^2 + (bz_2)^2] - a^2 z_1 (a^2 z_1 \dot{z}_1 + b^2 z_2 \dot{z}_2)}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} \\ &= \frac{(ab z_2)^2 \dot{z}_1 - (ab)^2 z_1 z_2 \dot{z}_2}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} \\ &= \frac{(ab z_2)^2 (z_2 \dot{q} - f'' \dot{x} C_q) - (ab)^2 z_1 z_2 (f'' \dot{x} S_q - z_1 \dot{q})}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} \\ &= \frac{(ab)^2 z_2 (z_1^2 + z_2^2) \dot{q} - (ab)^2 z_2 (z_1 S_q + z_2 C_q) f'' \dot{x}}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} \end{aligned} \quad (25)$$

In view of Eqs. (12), one finds that

$$z_1^2 + z_2^2 = (S_q - f' C_q)^2 + (C_q + f' S_q)^2 = 1 + (f')^2 \quad (26)$$

and

$$z_1 S_q + z_2 C_q = S_q^2 - f' S_q C_q + C_q^2 + f' S_q C_q = 1 \quad (27)$$

so that Eq. (25) can be restated as

$$-\dot{e}_1 = \frac{(ab)^2 z_2 [1 + (f')^2] \dot{q} - (ab)^2 z_2 f'' \dot{x}}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} \quad (28)$$

Returning to Eq. (23),

$$\dot{x}C_q + f'\dot{x}S_q - \dot{e}_1 = 0 \quad (29)$$

or, following substitution from Eqs. (12) and (28),

$$z_2\dot{x} + \frac{(ab)^2 z_2 [1 + (f')^2] \dot{q} - (ab)^2 z_2 f'' \dot{x}}{[(az_1)^2 + (bz_2)^2]^{\frac{3}{2}}} = 0 \quad (30)$$

which can be rearranged first as

$$\left\{ [(az_1)^2 + (bz_2)^2]^{\frac{3}{2}} - (ab)^2 f'' \right\} \dot{x} = -(ab)^2 [1 + (f')^2] \dot{q} \quad (31)$$

and finally as

$$\dot{x} = -\frac{(ab)^2 [1 + (f')^2] \dot{q}}{[a^2(S_q - f'C_q)^2 + b^2(C_q + f'S_q)^2]^{\frac{3}{2}} - (ab)^2 f''} \quad (32)$$

## References

- [1] Mitiguy, P. C., and Banerjee, A. K., "Efficient Simulation of Motions Involving Coulomb Friction," *Journal of Guidance, Control, and Dynamics*, Vol. 22, No. 1, 1999.
- [2] Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications*, McGraw-Hill, New York, 1985.