Gradient for multivariate normal CDF

For a set of $d > 1 \in \mathbb{Z}_+$ random variables $\boldsymbol{X} := \{X_1, \ldots, X_d\}$, let $\mathbf{X} \sim MVN(\boldsymbol{\mu}, \Sigma)$, with $\boldsymbol{\mu} \in \mathbb{R}^d, \Sigma \in \mathbb{R}_+^{d \times d}$ for dimension $d > 1 \in \mathbb{Z}_+$. If the CDF evaluated at a vector $\boldsymbol{x} := \{x_i \in \mathbb{R} : i \in \{1, \ldots, x_d\}\}$ is $\boldsymbol{\Phi}(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) = \Pr\{X_1 < x_1, X_2 < x_2, \ldots, X_d < x_d\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}x_i} \boldsymbol{\Phi}(\boldsymbol{x}) = \phi(x_i; \mu_i, \sigma_i) \boldsymbol{\Phi}\left(\boldsymbol{x}_{-i}; \boldsymbol{\mu}_{-i|i}, \Sigma_{-i|i}\right),$$

where μ_i is the ith element of $\boldsymbol{\mu}$, and σ_i^2 is the ith diagonal component of Σ, and $\boldsymbol{\mu}_{-i|i}$, $\Sigma_{-i|i}$ are the mean and covariance conditional on the ith component, respectively. The conditional mean and variance are given by

$$\boldsymbol{\mu}_{-i|i} = \boldsymbol{\mu}_{-i} + \Sigma_{-i,i} \Sigma_{i,i}^{-1} (x_i - \mu_i), \quad \Sigma_{-i|i} = \Sigma_{-i,-i} - \Sigma_{-i,i} \Sigma_{i,i}^{-1} \Sigma_{i,-i},$$

where $\Sigma_{-i,i}$ represents the off diagonal components. Note that in the case of i being one-dimensional, that $\Sigma_{i,i}^{-1} = 1/\sigma_i^2$.