Basics of Systems and Control Theory for pyMOR

Jens Saak

October 6, 2021

pyMOR School 2021


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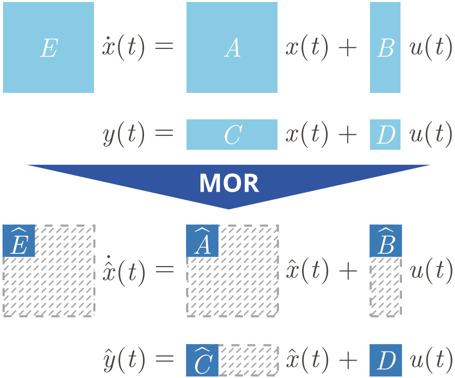

$E\,\dot{x}(t) = A\,x(t) + B\,u(t)$
$y(t) = C\,x(t) + D\,u(t)$
MOR
$\widehat{E}\,\dot{\hat{x}}(t) = \widehat{A}\,\hat{x}(t) + \widehat{B}\,u(t)$
$\hat{y}(t) = \widehat{C}\,\hat{x}(t) + D\,u(t)$

- Only continuous-time systems

  Discrete-time is treated in [Ant05]

- No differential-algebraic systems

  For DAE aspects see [Voi19, GSW13, MS05, Sty04]

- No non-linearities

- No parameter dependencies

First-order State-space Systems

(pyMOR: `LTIModel`)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \end{aligned} \qquad (\Sigma)$$

Here

- $\mathbf{x}(t) \in \mathbb{R}^n$ is called the state,
- $\mathbf{u}(t) \in \mathbb{R}^m$ is called the input,
- $\mathbf{y}(t) \in \mathbb{R}^p$ is called the output

of the LTI system. Correspondingly, we have

$$\mathbf{A} \in \mathbb{R}^{n \times n}, \qquad \mathbf{B} \in \mathbb{R}^{n \times m}, \qquad \mathbf{C} \in \mathbb{R}^{p \times n} \quad \text{and} \quad \mathbf{D} \in \mathbb{R}^{p \times m}.$$

We assume $t \in [0, \infty)$, $\mathbf{x}(0) = 0$.

First-order State-space Systems

(pyMOR: `LTIModel`)

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$$\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{n \times n}, \qquad \mathbf{B} \in \mathbb{R}^{n \times m}, \qquad \mathbf{C} \in \mathbb{R}^{p \times n}.$$

We assume $t \in [0, \infty)$, $\mathbf{x}(0) = 0$ and $\mathbf{E}$ invertible.

Second-order State-space Systems (pyMOR: `SecondOrderModel`)

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{E}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_{\mathsf{v}}\dot{\mathbf{x}}(t) + \mathbf{C}_{\mathsf{p}}\mathbf{x}(t). \end{aligned}$$

Here

- $\mathbf{x}(t) \in \mathbb{R}^n$ is called the position,
- $\dot{\mathbf{x}}(t) \in \mathbb{R}^n$ is called the velocity,
- $\mathbf{u}(t) \in \mathbb{R}^m$ is called the input,
- $\mathbf{y}(t) \in \mathbb{R}^p$ is called the output

of the LTI system. Correspondingly, we have

$$\mathbf{M},\ \mathbf{E},\ \mathbf{K} \in \mathbb{R}^{n \times n}, \qquad \mathbf{B} \in \mathbb{R}^{n \times m}, \qquad \mathbf{C}_{\mathsf{v}},\ \mathbf{C}_{\mathsf{p}} \in \mathbb{R}^{p \times n}.$$

Heat Equation [MORWiki thermal block] I

For $t \in (0, T)$, $\xi \in \Omega$ and initial values

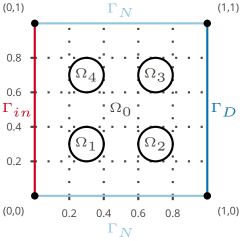
$$\theta(0, \xi) = 0, \text{ for } \xi \in \Omega,$$

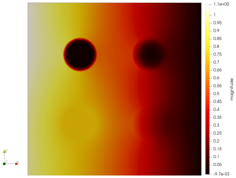
consider

$$\partial_t \theta(t, \xi) + \nabla \cdot (-\sigma(\xi) \nabla \theta(t, \xi)) = 0,$$

with boundary conditions

$$\begin{aligned} \sigma(\xi)\nabla\theta(t,\xi) \cdot n(\xi) &= u(t) & t \in (0,T), \xi \in \Gamma_{in}, \\ \sigma(\xi)\nabla\theta(t,\xi) \cdot n(\xi) &= 0 & t \in (0,T), \xi \in \Gamma_{N}, \\ \theta(t,\xi) &= 0 & t \in (0,T), \xi \in \Gamma_{D}. \end{aligned}$$





Heat Equation [MORWiki thermal block] II

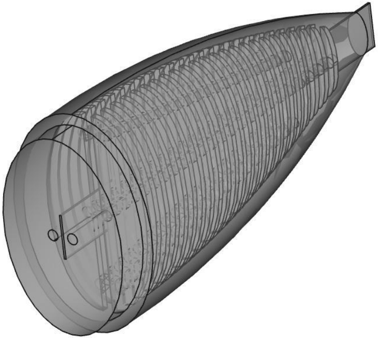
Finite element semi-discretization in space

- pairwise inner products of ansatz functions $\rightsquigarrow$ $\mathbf{E}$
- discretized spatial operator + Dirichlet boundary condition $\rightsquigarrow$ $\mathbf{A}$
- discretized non-zero Neumann boundary condition $\rightsquigarrow$ $\mathbf{B}$
- average temperatures on the inclusions $\rightsquigarrow$ $\mathbf{C}$

- $n = 7\,488$
- $m = 1$
- $p = 4$

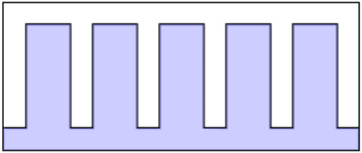
An Artificial Fishtail [MORWiki Artificial Fishtail] I

Construction:

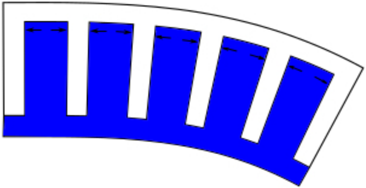


Fluid Elastomer Actuation:

no pressure



under pressure



An Artificial Fishtail [MORWiki Artificial Fishtail] II

Variables:

- displacement $\vec{s}(t, \vec{z})$
- strain $\underline{\vec{\varepsilon}}(\vec{s}(t, \vec{z}))$
- stress $\underline{\vec{\sigma}}(\vec{s}(t, \vec{z}))$

Material parameters:

- density ρ
- Lamé parameters λ, μ

Basic principle:

$$\underline{\vec{\varepsilon}}(\vec{s}(t, \vec{z})) = \frac{1}{2}\left(\nabla \vec{s}(t, \vec{z}) + \nabla^{\mathsf{T}} \vec{s}(t, \vec{z})\right) \qquad \text{(kinematic equation)}$$

$$\underline{\vec{\sigma}}(\vec{s}(t, \vec{z})) = \lambda \operatorname{tr}((\underline{\vec{\varepsilon}}(\vec{s}(t, \vec{z})))\, \underline{I} + 2\mu \underline{\vec{\varepsilon}}(\vec{s}(t, \vec{z}))) \qquad \text{(material equation)}$$

$$\rho \frac{\partial^2 \vec{s}(t, \vec{z})}{\partial t^2} = \nabla \cdot \underline{\vec{\sigma}}(\vec{s}(t, \vec{z})) + \vec{f}(t, \vec{z}) \qquad \text{(equation of motion)}$$

+ initial and boundary conditions

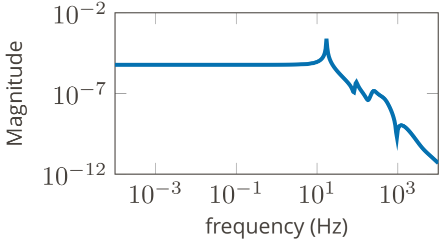
An Artificial Fishtail [MORWiki Artificial Fishtail] III

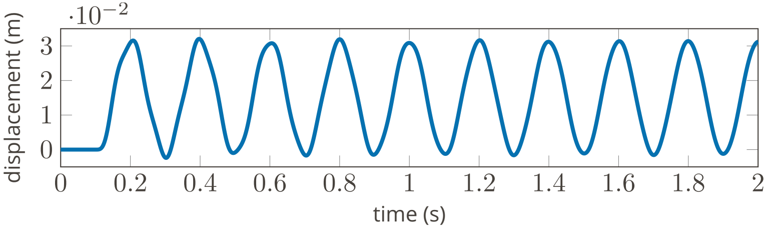
FEM semi-discretization:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{E}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_{\mathsf{p}}\mathbf{x}(t), \end{aligned}$$

with

- $\mathbf{M}, \mathbf{E}, \mathbf{K} > 0$, $\mathbf{C}_{\mathsf{v}} = 0$,
- $n = 779\,232$, $m = 1$, $p = 3$.





Definition

Let $f : [0, \infty) \to \mathbb{R}^n$ be exponentially bounded with bounding exponent α. Then

$$\mathcal{L}\left\{f\right\}(s) := \int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau$$

for $\mathrm{Re}(s) > \alpha$ is called the **Laplace transform** of f. The process of forming the Laplace transform is called **Laplace transformation**.

It can be shown that the integral converges uniformly in a domain with $\mathrm{Re}(s) \geq \beta$ for all $\beta > \alpha$.

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Allows us to map time signals to frequency signals.

Theorem

Let $f,\ g,\ h : [0, \infty) \to \mathbb{R}^n$ be given. Then the following two statements hold true:

a) *The Laplace transformation is linear, i. e., if f and g are exponentially bounded, then $h := \gamma f + \delta g$ is also exponentially bounded and*

$$\mathcal{L}\left\{h\right\} = \gamma\mathcal{L}\left\{f\right\} + \delta\mathcal{L}\left\{g\right\}$$

holds for all $\gamma,\ \delta \in \mathbb{C}$.

b) *If $f \in \mathcal{PC}^1([0, \infty), \mathbb{R}^n)$ and $\dot{f}$ is exponentially bounded, then f is exponentially bounded and*

$$\mathcal{L}\{\dot{f}\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

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- $X(s) := \mathcal{L}\{\mathbf{x}\}(s)$, $U(s) := \mathcal{L}\{\mathbf{u}\}(s)$, and $Y(s) := \mathcal{L}\{\mathbf{y}\}(s)$
- $\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \rightsquigarrow \mathbf{A}X(s) + \mathbf{B}U(s)$
- $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \rightsquigarrow Y(s) = \mathbf{C}X(s)$
- $sX(s) := \mathcal{L}\{\dot{x}\}(s)$ since $\mathbf{x}(0) = 0$

Rational Matrix Function Representation

In summary we have:

- $s\mathbf{E}X(s) = \mathbf{A}X(s) + \mathbf{B}U(s)$
- $Y(s) = \mathbf{C}X(s)$

Thus the mapping from inputs to outputs in frequency domain can be expressed as

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}.$$

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Analogously, for second-order systems we get

$$\mathbf{H}(s) = (s\mathbf{C}_{\mathsf{v}} + \mathbf{C}_{\mathsf{p}}) \left(s^2\mathbf{M} + s\mathbf{E} + \mathbf{K}\right)^{-1}\mathbf{B}.$$

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$\mathbf{H}$ is analytic in $\mathbb{C} \setminus \Lambda(\mathbf{E}, \mathbf{A})$, or $\mathbb{C} \setminus \Lambda(\mathbf{M}, \mathbf{E}, \mathbf{K})$, respectively

Important Representations of $\hat{H}(s)$

(Laurent) series expansion

$$\mathbf{H}(s) = \sum_{k=0}^{\infty} (s - s_0)^k M_k(s_0) \qquad \mathbf{H}(s) = \sum_{k=0}^{\infty} s^{-k} M_k(\infty)$$

The matrices $M_k(s_0)$ are called **moments** of $\mathbf{H}$. At infinity they are also referred to as **Markov parameters**.

Pole Residue Form

Let (λ_i, w_i, v_i) be the eigentriplets of the pair $(\mathbf{A}, \mathbf{E})$ with no degenerate eigenspaces. Then we have

$$\mathbf{H}(s) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i},$$

where $R_i = (\mathbf{C}v_i)(w_i^{\mathsf{H}}\mathbf{B})$, assuming $w_i^{\mathsf{H}}v_i = 1$.

The representation of $\mathbf{H}$ using $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is not unique.

In fact for any invertible matrix $\mathbf{T} \in \mathbb{R}^{n \times n}$, we have

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\\ &= \mathbf{C}\mathbf{T}^{-1}\mathbf{T}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{T}^{-1}\mathbf{T}\mathbf{B}\\ &= \mathbf{C}\mathbf{T}^{-1}\left(s\mathbf{T}\mathbf{E}\mathbf{T}^{-1} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)^{-1}\mathbf{T}\mathbf{B}\end{aligned}$$

and thus a system given, by $(\mathbf{T}\mathbf{E}\mathbf{T}^{-1}, \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \mathbf{T}\mathbf{B}, \mathbf{C}\mathbf{T}^{-1})$ realizes the exact same input/output behavior.

Definition

- All sets of matrices leading to the same function $\mathbf{H}$ are called its **realizations**.
- The matrix $\mathbf{T}$ above is called **state-space transformation**.

Important Realizations

- Minimal Realizations

  Can we realize $\mathbf{H}$ with less equations?

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- Balanced Realizations

  see here

  Can we find state coordinates that allow us to decide what is important?

McMillan Degree and Minimal Realization

Example

Realizations can even be of different dimensions. Take for example:

$$\mathbf{E} = \mathbf{I} \text{ the identity}, \mathbf{A} = \begin{bmatrix} -11 & 0 \\ 0 & -5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Truncating the second state component does not change $\mathbf{H}$.

Definition

There exists a minimum number of equations necessary to describe $\mathbf{H}$. The state dimension n of this minimal set of equations is called **McMillan degree** of the system. A realization of $\mathbf{H}$ with this dimension is called **minimal realization**.

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t) &= 0, \\ \mathbf{y}(t) - \mathbf{C}\mathbf{x}(t) - \mathbf{D}\mathbf{u}(t) &= 0. \end{aligned}$$

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned}\mathbf{E}\mathbf{V}\dot{\hat{\mathbf{x}}}(t) - \mathbf{A}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{B}\mathbf{u}(t) &= e_{\text{res}}(t),\\ \mathbf{y}(t) - \mathbf{C}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t) &= e_{\text{output}}(t).\end{aligned}$$

Step I: Use truncated state transformation

Replace

$$\mathbf{x}(t) \approx \mathbf{V}\hat{\mathbf{x}}(t)$$

with $\mathbf{V} \in \mathbb{R}^{n\times r}$ and $\hat{\mathbf{x}}(t) \in \mathbb{R}^r$.

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned} \mathbf{V}^\mathsf{T}\mathbf{E}\mathbf{V}\dot{\hat{\mathbf{x}}}(t) - \mathbf{V}^\mathsf{T}\mathbf{A}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{V}^\mathsf{T}\mathbf{B}\mathbf{u}(t) &= 0,\\ \mathbf{y}(t) - \mathbf{C}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t) &= e_{\text{output}}(t). \end{aligned}$$

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Step II: Mitigate transformation error

Suppress truncation residual through left projection.

- one-sided method: use $\mathbf{V}$ again.

Truncated Realizations via Ritz/Petrov-Galerkin Projection

$$\begin{aligned} \mathbf{W}^{\mathsf{T}}\mathbf{E}\mathbf{V}\dot{\hat{\mathbf{x}}}(t) - \mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{W}^{\mathsf{T}}\mathbf{B}\mathbf{u}(t) &= 0, \\ \mathbf{y}(t) - \mathbf{C}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t) &= e_{\text{output}}(t). \end{aligned}$$

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Step II: Mitigate transformation error

Suppress truncation residual through left projection.

- one-sided method: use $\mathbf{V}$ again.
- two-sided method: find $\mathbf{W} \in \mathbb{R}^{n \times r}$.

$\hat{\mathbf{A}} \approx$
$\mathbf{W}^\top$
$\mathbf{A}$
$\mathbf{V}$

Reduced order model (ROM) (pyMOR: `LTIPGReductor`)

Define $\hat{\mathbf{E}} = \mathbf{W}^{\mathsf{T}}\mathbf{E}\mathbf{V}$, $\hat{\mathbf{A}} = \mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V} \in \mathbb{R}^{r\times r}$, $\hat{\mathbf{B}} = \mathbf{W}^{\mathsf{T}}\mathbf{B} \in \mathbb{R}^{r\times m}$ and $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V} \in \mathbb{R}^{p\times r}$. Then

$$\begin{aligned}\hat{\mathbf{E}}\dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t),\\ \hat{\mathbf{y}}(t) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \tag{ROM}$$

approximates the dynamics of the full-order model (Σ) with output error

$$\mathbf{y}(t) - \hat{\mathbf{y}}(t) = e_{\text{output}}(t).$$

- We call the corresponding transfer function $\hat{\mathbf{H}}$.
- Model order reduction (MOR) $\rightsquigarrow$ Find $\mathbf{W}, \mathbf{V} \in \mathbb{R}^{n\times r}$ such that $e_{\text{output}}(t)$ is small in a suitable sense.
- We will see energy-based and interpolation-based methods today and tomorrow.

We have

$$Y(s) = \mathbf{H}(s)U(s)$$

and

$$\hat{Y}(s) = \hat{\mathbf{H}}(s)U(s).$$

Question

What are suitable norms such that

$$\|y - \hat{y}\| \leq \left\|\mathbf{H} - \hat{\mathbf{H}}\right\| \|u\|?$$

The Banach Space $\mathcal{H}_\infty^{p\times m}$

$$\mathcal{H}_\infty^{p\times m} := \left\{ G : \mathbb{C}^+ \to \mathbb{C}^{p\times m} \ : \ G \text{ is analytic in } \mathbb{C}^+ \text{ and } \sup_{s\in\mathbb{C}^+} \|G(s)\|_2 < \infty \right\}.$$

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$\mathcal{H}_\infty^{p\times m}$ is a Banach space equipped with the $\mathcal{H}_\infty$**-norm**

$$\left\|G\right\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \left\|G(\mathrm{i}\omega)\right\|_2 .$$

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$\mathcal{H}_\infty^{p\times m}$ is a Banach space equipped with the **$\mathcal{H}_\infty$-norm**

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Can show: $$\|\mathbf{y} - \hat{\mathbf{y}}\|_{\mathcal{L}_2} \le \left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_{\mathcal{H}_\infty} \|\mathbf{u}\|_{\mathcal{L}_2}.$$

This bound can even be shown to be sharp.

The Hilbert Space $\mathcal{H}_2^{p\times m}$

$$\mathcal{H}_2^{p\times m} := \left\{ G : \mathbb{C}^+ \to \mathbb{C}^{p\times m} \;:\; G \text{ is analytic in } \mathbb{C}^+ \text{ and } \sup_{\xi>0} \int_{-\infty}^{\infty} \|G(\xi + \mathrm{i}\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega < \infty \right\}.$$

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$\mathcal{H}_2^{p\times m}$ is a Hilbert space with the inner product

$$\langle F, G\rangle_{\mathcal{H}_2} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left(F(\mathrm{i}\omega)^{\mathsf{H}} G(\mathrm{i}\omega) \right) \mathrm{d}\omega$$

and induced norm

$$\|G\|_{\mathcal{H}_2} := \langle G, G\rangle_{\mathcal{H}_2}^{1/2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(\mathrm{i}\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega \right)^{1/2}.$$

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and induced norm

$$\|G\|_{\mathcal{H}_2} := \langle G, G\rangle_{\mathcal{H}_2}^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(\mathrm{i}\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega \right)^{1/2}.$$

Can show: $\|\mathbf{y} - \hat{\mathbf{y}}\|_{\mathcal{L}_\infty} \leq \left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_{\mathcal{H}_2} \|\mathbf{u}\|_{\mathcal{L}_2}.$

System Gramians and $\mathcal{H}_2$-trace-formula

A system (Σ) with $\Lambda(\mathbf{E},\mathbf{A}) \subset \mathbb{C}^-$ is called **asymptotically stable**. Then, all state trajectories decay exponentially as $t \to \infty$ and

a) the infinite controllability and observability **Gramians** exist:

$$\mathbf{P} = \int_0^\infty \mathrm{e}^{\mathbf{E}^{-1}\mathbf{A}t}\mathbf{E}^{-1}\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{E}^{-\mathsf{T}}\mathrm{e}^{\mathbf{A}^\mathsf{T}\mathbf{E}^{-\mathsf{T}}t}\mathrm{d}t$$
$$\mathbf{E}^\mathsf{T}\mathbf{Q}\mathbf{E} = \int_0^\infty \mathrm{e}^{\mathbf{A}^\mathsf{T}\mathbf{E}^{-\mathsf{T}}t}\mathbf{C}^\mathsf{T}\mathbf{C}\mathrm{e}^{\mathbf{E}^{-1}\mathbf{A}t}\mathrm{d}t.$$

b) $\mathbf{P}$, $\mathbf{Q}$ solve the two **Lyapunov equations**

$$\mathbf{A}\mathbf{P}\mathbf{E}^\mathsf{T} + \mathbf{E}\mathbf{P}\mathbf{A}^\mathsf{T} = -\mathbf{B}\mathbf{B}^\mathsf{T}, \qquad \mathbf{A}^\mathsf{T}\mathbf{Q}\mathbf{E} + \mathbf{E}^\mathsf{T}\mathbf{Q}\mathbf{A} = -\mathbf{C}^\mathsf{T}\mathbf{C}$$

c) the $\mathcal{H}_2$-norm can be expressed as

$$\|\mathbf{H}\|_{\mathcal{H}_2}^2 = \mathrm{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T}\right) = \mathrm{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B}\right).$$

Bode Plots

The Bode plot for $\mathbf{H}$ consists of a **magnitude plot** and a **phase plot**.

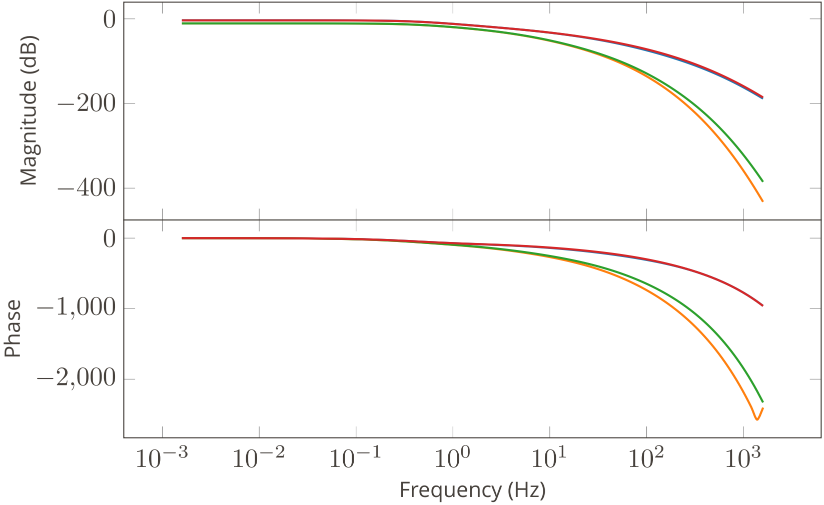
Bode magnitude plot

- component-wise graph of the function $|\mathbf{H}(\mathrm{i}\omega)|$ for frequencies $\omega \in [\omega_{\mathrm{min}}, \omega_{\mathrm{max}}] \subset \mathbb{R}$.
- ω-axis is logarithmic.
- magnitude is given in decibels, i.e., $|\mathbf{H}(\mathrm{i}.)|$ is plotted as $20\log_{10}(|\mathbf{H}(\mathrm{i}.)|)$.

Bode phase plot

- component-wise graph of the function $\arg \mathbf{H}(\mathrm{i}\omega)$ for frequencies $\omega \in [\omega_{\mathrm{min}}, \omega_{\mathrm{max}}] \subset \mathbb{R}$.
- ω-axis is logarithmic.
- phase is given in degrees on a linear scale.

Bode Plot for the Thermal Block Example



(Sigma) Magnitude Plots

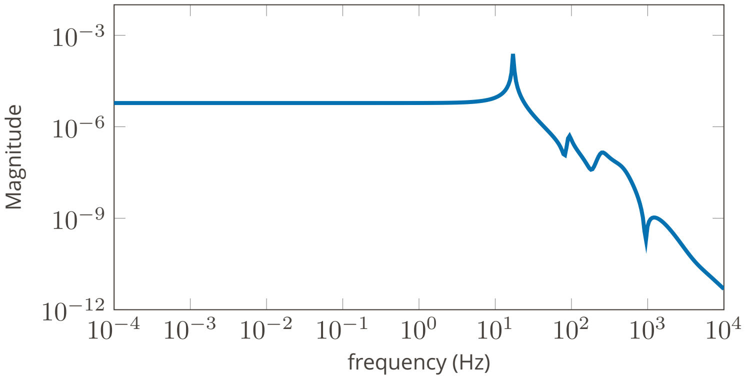
Sigma magnitude plot

- 2-norm-wise graph of the function $\mathbf{H}(\mathrm{i}\omega)$ for frequencies $\omega \in [\omega_{\mathrm{min}}, \omega_{\mathrm{max}}] \subset \mathbb{R}$.
- ω-axis is logarithmic.

The name is due to the fact that for a given matrix $\mathbf{M}$ the norm $\|\mathbf{M}\|_2$ is given by its largest singular value.

The real sigma magnitude plot depicts all singular values as functions of ω.

Sigma Magnitude Plot for the Artificial Fishtail



Modal Coordinates

Assume that the pair (E, A), respectively the triple (M, E, K), is simultaneously diagonalizable in $\mathbb{C}^{n\times n}$.

Classic Modal Truncation

- Compute diagonal realization from an eigendecomposition.
- State-space transformation matrices contain eigenvectors (modes).
- Use $W = V$.
- Populate V with modes corresponding to eigenvalues closest to $\mathrm{i}\mathbb{R}$.
- Add a few domain specific or “anxiety” modes.

Problem

- Does not take inputs and outputs into account!
- How many “anxiety” modes are necessary?

Dominant Poles Approximation

(pyMOR: `MTReductor`)

Recall the pole residue form of the transfer function

$$\mathbf{H}(s) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i},$$

where $R_i = (\mathbf{C}v_i)(w_i^{\mathsf{H}}\mathbf{B})$, assuming $w_i^{\mathsf{H}}v_i = 1$.

Sort and select modes by the magnitude of the $\|R_i\| \,/\, \mathrm{Re}(\lambda_i)$. Then

Error bound

$$\left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_{\infty} \leq \sum_{i=r+1}^{n} \frac{\|R_i\|}{|\mathrm{Re}(\lambda_i)|}$$

Computation is feasible via *subspace accelerated MIMO dominant pole algorithm* (SAMDP).

Balanced Truncation aka. Lyapunov Balancing

Idea:

- The system (Σ), in realization $(\mathbf{E} = \mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C})$, is called balanced, if the solutions $\mathbf{P},\ \mathbf{Q}$ of the Lyapunov equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{B}^{\mathsf{T}} \ = \ 0, \qquad \mathbf{A}^{\mathsf{T}}\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^{\mathsf{T}}\mathbf{C} \ = \ 0,$$

satisfy: $\mathbf{P} = \mathbf{Q} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0.$

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- A balanced realization is computed via state space transformation

$$\begin{aligned}\mathcal{T} : (\mathbf{I}, \mathbf{A}, \mathbf{B}, \mathbf{C}) &\mapsto (\mathbf{I}, \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \mathbf{T}\mathbf{B}, \mathbf{C}\mathbf{T}^{-1}) \\ &= \left(\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \right).\end{aligned}$$

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- Truncation $\rightsquigarrow$ reduced order model: $(\mathbf{I}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}) = (\mathbf{I}, \mathbf{A}_{11}, \mathbf{B}_1, \mathbf{C}_1)$.

Implementation: The Square Root Method

The SR Method (pyMOR: `BTReductor`)

1. Compute (Cholesky) factors of the solutions to the Lyapunov equation,

$$\mathbf{P} = \mathbf{S}^{\mathsf{T}}\mathbf{S}, \quad \mathbf{Q} = \mathbf{R}^{\mathsf{T}}\mathbf{R}.$$

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$$\mathbf{S}\mathbf{R}^{\mathsf{T}} = [\,\mathbf{U}_1,\, \mathbf{U}_2\,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^{\mathsf{T}} \\ \mathbf{V}_2^{\mathsf{T}} \end{bmatrix}.$$

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3. Define

$$\mathbf{W} := \mathbf{R}^{\mathsf{T}}\mathbf{V}_1\Sigma_1^{-1/2}, \qquad \mathbf{V} := \mathbf{S}^{\mathsf{T}}\mathbf{U}_1\Sigma_1^{-1/2}.$$

4. Then the reduced order model is $(\mathbf{W}^{\mathsf{T}}\mathbf{A}\mathbf{V}, \mathbf{W}^{\mathsf{T}}\mathbf{B}, \mathbf{C}\mathbf{V})$.

Properties

- Lyapunov balancing **preserves asymptotic stability**.
- We have the **a priori error bound**: $\left\| \mathbf{H} - \hat{\mathbf{H}} \right\|_{\mathcal{H}_\infty} \leq 2 \sum\limits_{k=r+1}^{n} \sigma_k$

Variants (pyMOR: `BRBTReductor`, `LQGBTReductor`)

Other versions for special classes of systems or applications exist, such as

- **positive-real balancing**, (passivity-preserving)
- **bounded-real balancing**, (contractivity-preserving)
- **linear-quadratic Gaussian balancing**. (stability preserving) (aims at low-order output feedback controllers)

The given ones all compute $\mathbf{P}$, $\mathbf{Q}$ as solutions of **algebraic Riccati equations** of the form:

$$0 = \tilde{A}\mathbf{P}\tilde{E}^\mathsf{T} + \tilde{E}\mathbf{P}\tilde{A}^\mathsf{T} + \tilde{B}\tilde{B}^\mathsf{T} \pm \tilde{E}\mathbf{P}\tilde{C}^\mathsf{T}\tilde{C}\mathbf{P}\tilde{E}^\mathsf{T}$$
$$0 = \tilde{A}^\mathsf{T}\mathbf{Q}\tilde{E} + \tilde{E}^\mathsf{T}\mathbf{Q}\tilde{A} + \tilde{C}^\mathsf{T}\tilde{C} \pm \tilde{E}^\mathsf{T}\mathbf{Q}\tilde{B}\tilde{B}^\mathsf{T}\mathbf{Q}\tilde{E}.$$

Tools I

Lemma (Neumann series)

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ *with spectral radius* $\rho(\mathbf{A}) < 1$ *be given. Then* $\mathbf{I} - \mathbf{A}$ *is invertible and it holds that*

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k.$$

Will be important to identify the actual shape of Markov parameters and system moments.

Tools II

Definition ((polynomial) Krylov subpace)

Given an invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^n$ the k**-dimensional (polynomial) Krylov subspace** is defined as

$$\mathcal{K}_k(\mathbf{A},\mathbf{b}) := \operatorname{span}\left\{\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \ldots, \mathbf{A}^{k-1}\mathbf{b}\right\}.$$

Definition (rational Krylov subpace)

Given an invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ a vector $\mathbf{b} \in \mathbb{R}^n$ and a vector of shifts $s \in \mathbb{R}^k$ the k**-dimensional rational Krylov subspace** is defined as

$$\mathcal{K}_k(\mathbf{A},\mathbf{b},s) := \operatorname{span}\left\{(s_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}, (s_2\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}, \ldots, (s_k\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}\right\}.$$

Orthonormal bases of these spaces should be computed via the **Arnoldi iteration**.

Padé-type approximations

Goal

Match the coefficients $\mathbf{M}_k(s_0)$ or $\mathbf{M}_k(\infty)$ in

$$\mathbf{H}(s) = \sum_{k=0}^{\infty} (s - s_0)^k M_k(s_0) \qquad \mathbf{H}(s) = \sum_{k=0}^{\infty} s^{-k} M_k(\infty)$$

Motivation **(assume: $m = p = 1$, s large enough)**

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \tfrac{1}{s}\mathbf{C}\underbrace{\left(\mathbf{I} - \tfrac{1}{s}\mathbf{E}^{-1}\mathbf{A}\right)^{-1}}_{=\sum_{k=0}^{\infty}\frac{1}{s^k}(\mathbf{E}^{-1}\mathbf{A})^k}\mathbf{E}^{-1}\mathbf{B}$$

$$= \sum_{k=1}^{\infty} \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{k-1}\mathbf{E}^{-1}\mathbf{B}\frac{1}{s^k}.$$

Padé-type approximations

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$$= \sum_{k=1}^{\infty}\mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{k-1}\mathbf{E}^{-1}\mathbf{B}\tfrac{1}{s^k}.$$

Therefore, we have

$$M_k(\infty) = \begin{cases} 0, & \text{if } k = 0, \\ \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{k-1}\mathbf{E}^{-1}\mathbf{B}, & \text{if } k \geq 1. \end{cases} \rightsquigarrow \text{ use } \mathbf{V} = \mathcal{K}_r(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{B})$$

Padé-type approximations

Approximation at ∞

$$\mathbf{V} = \mathcal{K}_r(\mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{B}), \qquad \mathbf{W} = \mathbf{V} \text{ or } \mathbf{W} = \mathcal{K}_r(\mathbf{A}^\mathsf{T}\mathbf{E}^{-\mathsf{T}}, \mathbf{C}^\mathsf{T})$$

Approximation at $s_0 = 0$

$$\mathbf{V} = \mathcal{K}_r(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B}), \qquad \mathbf{W} = \mathbf{V} \text{ or } \mathbf{W} = \mathcal{K}_r(\mathbf{E}^\mathsf{T}\mathbf{A}^{-\mathsf{T}}, \mathbf{C}^\mathsf{T})$$

Approximation at $s_0 \in (0, \infty)$

$$\mathbf{V} = \mathcal{K}_r((s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}, (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}), \qquad \mathbf{W} = \mathbf{V}$$

or

$$\mathbf{W} = \mathcal{K}_r(\mathbf{E}^\mathsf{T}(s_0\mathbf{E}^\mathsf{T} - \mathbf{A}^\mathsf{T})^{-1}, \mathbf{C}^\mathsf{T})$$

Multi-point Moment Matching, Interpolation and IRKA/TSIA

Approximation at $s_1, \ldots, s_r$

$$\mathbf{V} = \mathcal{K}_r(s, \mathbf{E}^{-1}\mathbf{A}, \mathbf{E}^{-1}\mathbf{B}), \qquad \mathbf{W} = \mathbf{V} \text{ or } \mathbf{W} = \mathcal{K}_r(s, \mathbf{A}^\mathsf{T}\mathbf{E}^{-\mathsf{T}}, \mathbf{C}^\mathsf{T}).$$

- $\mathbf{W} = \mathbf{V}$ as above matches first r moments of (Σ).
- $\mathbf{W} \neq \mathbf{V}$ as above matches first $2r$ moments of (Σ).
- $\mathbf{W} \neq \mathbf{V}$ as above actually achieves Hermite interpolation of $\mathbf{H}$, see, e.g., [ABG20].

How do we choose $s_1, \ldots, s_r$?

$\mathcal{H}_2$-optimal MOR

Find $\mathbf{s} = [s_1, \ldots, s_r]^\mathsf{T}$, such that $\left\|\mathbf{H} - \hat{\mathbf{H}}\right\|_{\mathcal{H}_2}$ is minimized.

IRKA iterative improvement of $\mathbf{s}$ using $\Lambda(\hat{\mathbf{E}}_j, \hat{\mathbf{A}}_j)$.

(pyMOR: `IRKAReductor`)

TSIA run a fixed point iteration on the first order necessary conditions.

(pyMOR: `TSIAReductor`)

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QUESTIONS?