MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

SYSTEM-THEORETIC METHODS

for Model Reduction of Linear Dynamical Systems

Peter Benner

October 8, 2019

pyMOR School 2019
Model Order Reduction with Python
October 7–11, 2019

Outline

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$*, where* n *is the dimension of the state space.*

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This is the task of model reduction (also: dimension reduction, order reduction).

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
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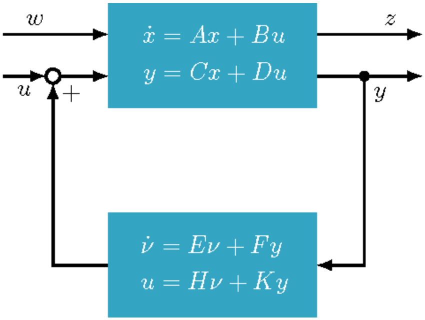
Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



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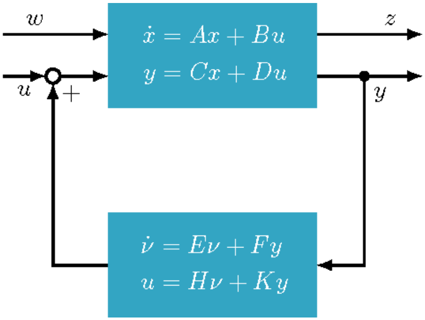

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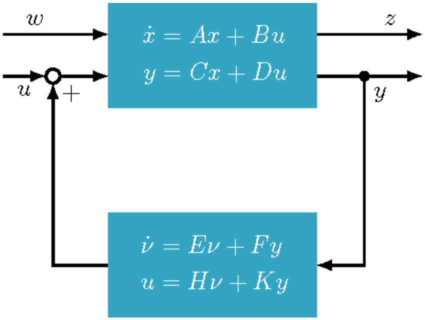

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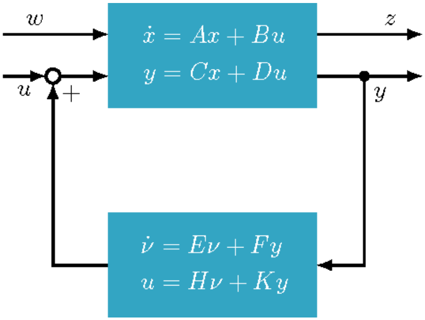

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Standard MOR techniques in systems and control: balanced truncation and related methods.

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.

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- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
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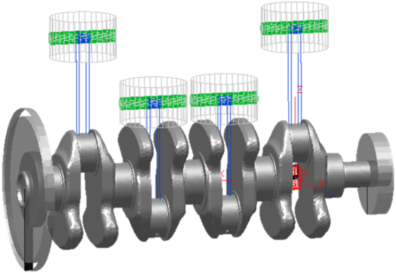
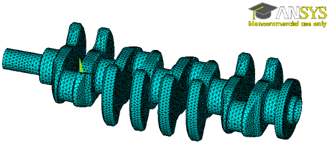
Standard MOR techniques in circuit simulation: Krylov subspace / Padé approximation / rational interpolation methods.



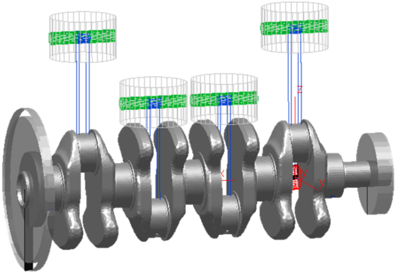
$\rightsquigarrow$



- Resolving complex 3D geometries $\Rightarrow$ millions of degrees of freedom.
- Analysis of elastic deformations requires many simulation runs for varying external forces.

Application Areas

Structural Mechanics / Finite Element Modeling



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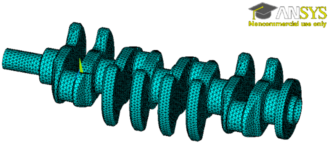

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- Analysis of elastic deformations requires many simulation runs for varying external forces.

Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) ⇝ Craig-Bampton method — not discussed in this course!

An Inspiration: Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
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Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum\nolimits_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
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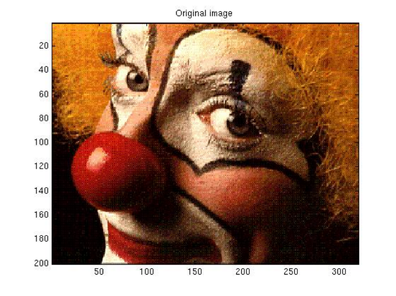
Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r$, $\sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel
$\rightsquigarrow \quad \approx 256$ kb

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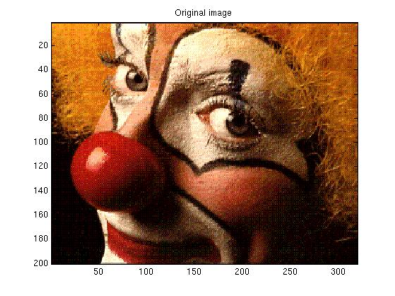

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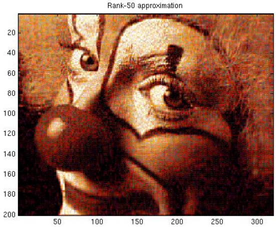
- rank $r = 50$, ≈ 104 kb



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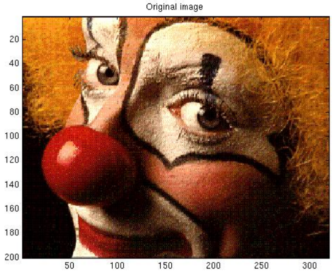

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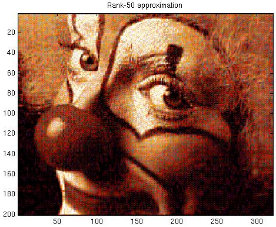
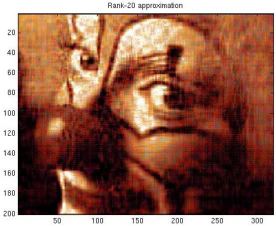
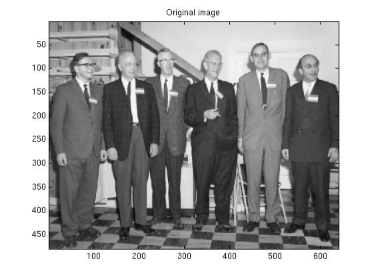


- rank $r = 20$, ≈ 42 kb



Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens, George Forsythe, Alston Householder, Peter Henrici, Fritz L. Bauer.



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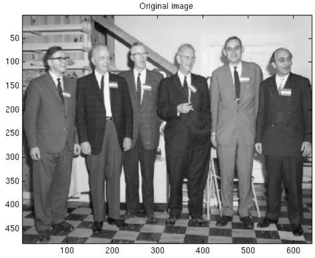
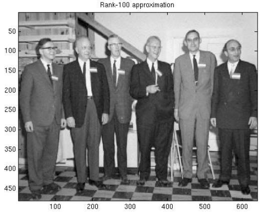
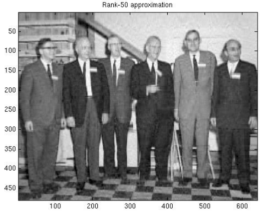


640×480 pixel, ≈ 1229 kb

- rank $r = 100$, ≈ 448 kb



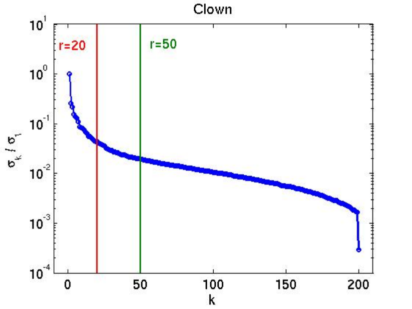
- rank $r = 50$, ≈ 224 kb

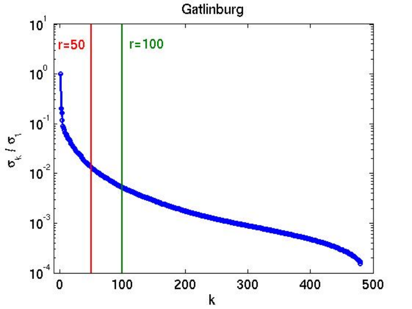


Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices





Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) &= g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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Original System

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Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Model Reduction for Dynamical Systems

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear, Time-Invariant (LTI) Systems

$$
\begin{aligned}
\dot{x} &= f(t,x,u) &= Ax + Bu, \qquad & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\
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Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

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Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t\in\mathbb{R}.$$

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- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

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Alternative to State-Space Operator: Hankel operator

Instead of

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use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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$$\begin{aligned} \dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, & C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

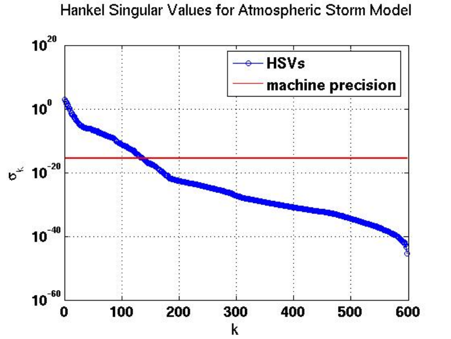
$\mathcal{H}$ compact

$\Downarrow$

$\mathcal{H}$ has discrete SVD

$\Downarrow$

Hankel singular values



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Alternative to State-Space Operator: Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\, d\tau \quad \text{for all } t > 0.$$

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But: computationally infeasible for large-scale systems.

Linear, Time-Invariant (LTI) Systems

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Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

Application of Laplace transform

$$\mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t)\, dt \quad (\Rightarrow \ \dot{x}(t) \mapsto sx(s))$$

with $s \in \mathbb{C}$ leads to linear system of equations:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s).$$

Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

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Laplace Transform / Frequency Domain

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=: G(s)} \Big) u(s) = G(s)u(s).$$

G is the transfer function of Σ, $G : \mathcal{L}_2^m \to \mathcal{L}_2^p \quad (\mathcal{L}_2 := \mathcal{L}(L_2(-\infty, \infty)))$.

Approximation Problem

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, & A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \\ y &= Cx + Du, & C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, & \hat{A} \in \mathbb{R}^{r \times r}, \quad \hat{B} \in \mathbb{R}^{r \times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, & \hat{C} \in \mathbb{R}^{p \times r}, \quad \hat{D} \in \mathbb{R}^{p \times m}. \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \|u\| \leq \text{tolerance} \cdot \|u\|.$$

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$\Longrightarrow$ Approximation problem: $\min_{\text{order}(\hat{G})\leq r} \|G - \hat{G}\|$.

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

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Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of A, denoted by $\Lambda(A)$, satisfies $\Lambda(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ (e.g., $\mathcal{U}_{ad} \in \{C^k[0,T], L_2(0,T), PC[0,T]\}$, possibly with constraints $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$) and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
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Note: for LTI systems $\dot{x} = Ax + Bu$, both concepts are equivalent!

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.

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The controllability set w.r.t. x^1 is defined as $\mathcal{C} := \bigcup_{t_1 > 0} \mathcal{C}(t_1)$ where

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In short: an LTI system is controllable $\iff \mathcal{C} = \mathbb{R}^n$.

Now: characterize controllability.

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Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

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Ansatz: $u(t) = B^T e^{-A^T t}c \Longrightarrow$

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Hence, an LTI system is controllable iff this linear system is solvable for $c \in \mathbb{R}^n$, i.e., iff $P(0,t_1)$ is invertible. (Note: $P(0,t_1) = P(0,t_1)^T \geq 0$ by definition!)

Now: characterize controllability.

Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The finite time Gramian $P(0, t_1)$ is spd $\forall\ t_1 > 0$.*

c) *The controllability matrix*

$$K(A, B) := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n \times n \cdot m}$$

has full rank n. (Note: $\operatorname{range}(K(A, B)) = \mathcal{C}(t_1)\ \forall\ t_1 > 0$!)

d) *If z is a left eigenvector of A, then $z^*B \neq 0$.*

e) *(Hautus test) $\operatorname{rank}([\lambda I - A, B]) = n\ \forall\, \lambda \in \mathbb{C}$.*

The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

$$P := \int_0^\infty e^{As} B B^T e^{A^T s} ds,$$

using congruence of $P(0, t_1)$ to $\int\limits_0^{t_1} e^{As} B B^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

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For a stable LTI system defined by $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The controllability Gramian P is positive definite.*

New question: suppose we have

$$y(t) = \tilde{y}(t)$$

corresponding to two trajectories $x, \tilde{x}$ obtained by the same input function $u(t)$. Can we conclude that $x(0) = \tilde{x}(0)$, or even stronger, that $x(t) = \tilde{x}(t)$ for $t \leq 0, t \geq 0$ (past/future)?

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Definition (Observability)

An LTI system is reconstructable (observable) if for solution trajectories $x(t), \tilde{x}(t)$ obtained with the same input function u, we have

$$\begin{aligned} y(t) = &\ \tilde{y}(t) \quad \forall t \leq 0 \quad (\forall t \geq 0) \\ \Longrightarrow \quad x(t) = &\ \tilde{x}(t) \quad \forall t \leq 0 \quad (\forall t \geq 0). \end{aligned}$$

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

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For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$*, T.F.A.E.:*

a) *The LTI system is reconstructable.*

b) *The LTI system is observable.*

c) *The observability matrix*
$$\mathcal{O}(A, C) = \left[C^T, A^T C^T, (A^2)^T C, \ldots, (A^{n-1})^T C^T\right]^T \in \mathbb{R}^{np\times n} \text{ has } \operatorname{rank} n.$$

d) *If* $Ax = \lambda x$*, then* $C^T x \neq 0$.

e) *(Hautus test)* $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n.$

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An LTI system is reconstructable if and only if the dual system $\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

Theorem

A stable LTI system is observable if and only if the observability Gramian

$$Q := \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

is symmetric positive definite.

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Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, T.F.A.E.:

a) *The LTI system is stabilizable.*
b) $\exists$ *feedback operator/matrix $F \in \mathbb{R}^{m\times n}$ with $\Lambda(A + BF) \subset \mathbb{C}^-$.*
c) *If $p^*A = \tilde{\lambda}p^*$ and $\mathrm{Re}(\lambda) \geq 0$, then $p^*B \neq 0$.*
d) $\mathrm{rank}([A - \lambda I, B]) = n \quad \forall \lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) \geq 0$.
e) *$\Lambda(A_3) \subset \mathbb{C}^-$ in the (controllability) Kalman decomposition of (A, B),*

$$V^T A V = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, V^T B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

$\exists$ dual concept of stabilizability, analogous to duality of controllability and observability.

Definition (Detectability)

An LTI system is detectable if for any solution $x(t)$ of $\dot{x} = Ax$ with $Cx(t) \equiv 0$ we have $\lim_{t\to\infty} x(t) = 0$.
(We can not observe all of x, but the unobservable part is stable.)

∃ dual concept of stabilizability, analogous to duality of controllability and observability.

Theorem

For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$, T.F.A.E.:

a) *The LTI system is detectable.*

b) (A^T, C^T) *is stabilizable.*

c) $Ax = \lambda x, \mathrm{Re}(\lambda) \geq 0 \Rightarrow C^T x \neq 0.$

d) $\mathrm{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ *for all* $\lambda, \mathrm{Re}(\lambda) \geq 0.$

e) *In the observability Kalman decomposition of* (A^T, C^T),

$$W^T A W = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, CW = [C_1 \ 0],$$

we have $\Lambda(A_3) \subset \mathbb{C}^-$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x & \to \quad Tx, \\ (A, B, C, D) & \to \quad (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad \text{with transfer function} \\ y(t) &= Cx(t) + Du(t), \quad G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j\times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1\times m}$, $C_2 \in \mathbb{R}^{q\times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

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Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

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Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

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Theorem

A realization (A, B, C, D) of a linear system is minimal $\Longleftrightarrow$
(A, B) is controllable and (A, C) is observable.

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

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$\sigma_1, \ldots, \sigma_n$ are the Hankel singular values of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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$$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0.$$

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Proof. Exercise!

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Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

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Proof. In balanced coordinates, the HSVs are $\Lambda\left(PQ\right)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

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The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda\left(\hat{P}\hat{Q}\right) = \Lambda\left(PQ\right) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

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Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(\jmath\omega) u(\jmath\omega)\, d\omega.$$

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$$\int_{-\infty}^{\infty} y^*(\jmath\omega) y(\jmath\omega)\, d\omega \quad = \quad \int_{-\infty}^{\infty} u^*(\jmath\omega)\, G^*(\jmath\omega)\, G(\jmath\omega)\, u(\jmath\omega)\, d\omega$$

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&= M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^* u(\jmath\omega)\, d\omega \;<\; \infty.
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$$\Longrightarrow y \in L_2^p(-\infty, \infty) \cong \mathcal{L}_2^p.$$

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Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\,(G(\jmath\omega))\,.$$

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Hardy space $\mathcal{H}_\infty$

Function space of analytic and bounded (in $\mathbb{C}^+$) matrix-/scalar-valued functions.
The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\mathrm{Re}\,(s)>0} \sigma_{\max}\,(F(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\,(F(\jmath\omega))\,.$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_\infty^{p\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

Consider the transfer function

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}u\|_2 \leq \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned} \|F\|_2 &:= \left(\sup_{\mathrm{Re}\,(\sigma)>0} \int_{-\infty}^{\infty} \|F(\sigma + \jmath\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

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$$\|F\|_2 \;=\; \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2\, d\omega\right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \leq \|G - \hat{G}\|_2\|u_0\|.$$

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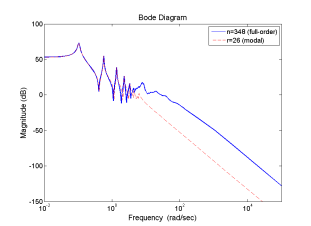
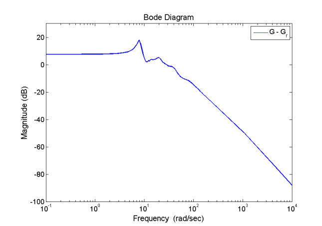
$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory)

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot:
  - for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}(\text{value})$;
  - for MIMO systems, $p \times m$ array of of plots G_{ij}.





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- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

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  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

("system does not generate energy").

Projector

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

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- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

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Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\operatorname{range}(V) = \mathcal{V}, \quad \operatorname{range}(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

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Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T\left(VW^T\dot{x} - AVW^Tx - Bu\right)$$

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Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

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$P(s)$ is a projector onto $\mathcal{V}$:

$\mathrm{range}\,(P(s)) \subset \mathrm{range}\,(V)$, all matrices have full rank $\Rightarrow$ "$=$", and

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$P(s)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

hence $G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*)$, i.e., $\hat{G}$ interpolates G in s_*!

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$$\text{Analogously,} \quad = C(sI_n - A)^{-1}\left(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\right)B.$$

$Q(s)^*$ is a projector onto $\mathcal{W}$ $\Longrightarrow$ Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-T} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0$,

hence $G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*)$, i.e., $\hat{G}$ interpolates G in s_*!

Theorem [Grimme 1997, Villemagne/Skelton 1987]

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \mathrm{range}(V)$, or
- $(s_* I_n - A)^{-T} C^T \in \mathrm{range}(W)$,

then at $s = s_*$, we obtain the (rational) interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

Note: extension to Hermite interpolation conditions later!

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1}b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two-sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$.

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$
\begin{aligned}
0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
\Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u.
\end{aligned}
$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$
\begin{aligned}
\dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\
y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u.
\end{aligned}
$$

Basic principle:

- Recall: an LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T \;=\; 0, \qquad A^TQ + QA + C^TC \;=\; 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- Lyapunov eqns.: $AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$
- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

Proof: Recall Hankel operator

$$y(t) = \mathcal{H}u(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau)\, d\tau = Ce^{At} z.$$

Hankel singular values = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y(t) = \; = B^T e^{-A^T t} \int_0^{\infty} e^{A^T \tau} C^T y(\tau)\, d\tau.$$

Hence,

$$\mathcal{H}^*\mathcal{H}u(t) \;=\; B^T e^{-A^T t} Qz \doteq \sigma^2 u(t).$$

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$\Longleftrightarrow \qquad PQz = \sigma^2 z. \quad \square$

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- Recall: an LTI system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$:

in transformed coordinates, the Gramians satisfy

$$\begin{aligned} (TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \end{aligned}$$

$$\Rightarrow \; (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda(PQ) = \Lambda((TPT^T)(T^{-T}QT^{-1}))$.

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HSVs determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

In balanced coordinates . . . energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int\limits_0^\infty y(t)^T y(t)\, dt}{\int\limits_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

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$\Longrightarrow VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

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- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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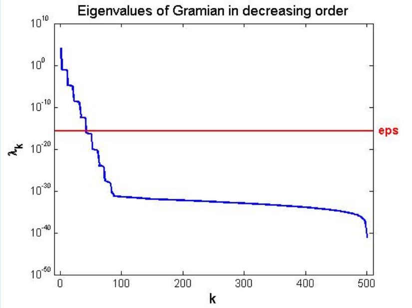
Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n\times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



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Use low-rank techniques ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Implementation using sparse Lyapunov solver ($\rightarrow$ ADI+sparse LU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - + MATLAB toolbox LyaPack (Penzl 1999),
  - + Software library M.E.S.S.[a] in C/MATLAB [B./Saak/Köhler/uvm.],
  - + pyMOR.

[a]Matrix Equation Sparse Solvers

Recall Peaceman-Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$.

ADI iteration idea: decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

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\begin{aligned}
(H + pI)v &= r \\
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can be solved easily/efficiently.

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ADI Iteration

If H, V spd $\Rightarrow \exists p_k,\ k = 1, 2, \ldots,$ such that

$$\begin{array}{rcl} u_0 &=& 0 \\ (H + p_k I)u_{k-\frac{1}{2}} &=& (p_k I - V)u_{k-1} + s \\ (V + p_k I)u_k &=& (p_k I - H)u_{k-\frac{1}{2}} + s \end{array}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

The Lyapunov operator

$$\mathcal{L}: \quad P \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX, \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation **[Wachspress 1988]**

$$\begin{aligned} X_0 &= 0, \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I), \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I). \end{aligned}$$

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n\times m}$ with $m \ll n$.

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For $k = 1, \ldots, k_{\max}$

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Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_k Z_k^T &= -2p_k (A + p_k I)^{-1} BB^T (A + p_k I)^{-T} \\ &\quad + (A + p_k I)^{-1}(A - p_k I) Z_{k-1} Z_{k-1}^T (A - p_k I)^T (A + p_k I)^{-T} \end{aligned}$$

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$\ldots \rightsquigarrow$ low-rank Cholesky factor ADI [Penzl 1997/2000, Li/White 1999/2002, B./Li/Penzl 1999/2008, Gugercin/Sorensen/Antoulas 2003]

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$ [Penzl '00]

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Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$$Z_{k_{\max}} = [z_{k_{\max}},\ P_{k_{\max}-1}z_{k_{\max}},\ P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}),\ \ldots,\ P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})],$$

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I - (p_i + p_{i+1})(A + p_i I)^{-1}\right].$$

[Li/White '02]

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[Li/White '02]

$\rightsquigarrow$ Need to solve only one (sparse) linear system with m right-hand sides per iteration!

Algorithm **[Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]**

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Z_k \leftarrow \left[\begin{array}{cc} Z_{k-1} & V_k \end{array}\right]$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{\% column compression, optional}$$

Algorithm [Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1} B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}} \left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

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At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

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Note: Implementation in real arithmetic is possible: combine two steps [B./Li/Penzl 1999/2008] or employ the relations of consecutive complex factors [B./Kürschner/Saak 2011].

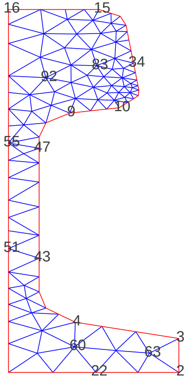
Current implementations (like in pyMOR) employ low-rank property of residual, update residual in each step, and compute new shifts on the fly!

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, & \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), & \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, & \xi \in \Gamma_7.
\end{aligned}
$$

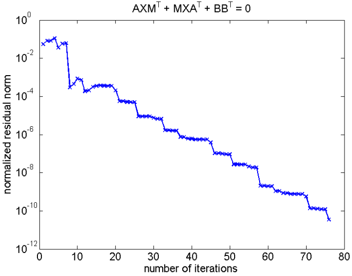
$\Longrightarrow m = 7, p = 6.$

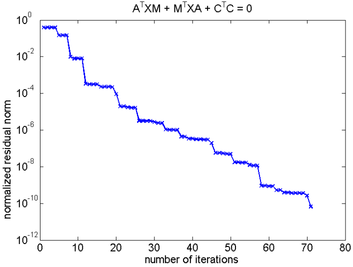
- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841.$



Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,
$$APM^T + MPA^T + BB^T = 0, \quad A^TQM + M^TQA + C^TC = 0,$$
for $79,841$.
- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- M.E.S.S. requires no factorization of mass matrix.
- Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB.





Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\mathrm{range}\,(Z)$, $Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \dots, A^{r-1}B\}$$

[Saad 1990, Jaimoukha/Kasenally 1994, Jbilou 2002–08].

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- Extended (and rational) Krylov method (EKSM, RKSM) [Simoncini 2007, Druskin/Knizhnerman/Simoncini 2011],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

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Examples:

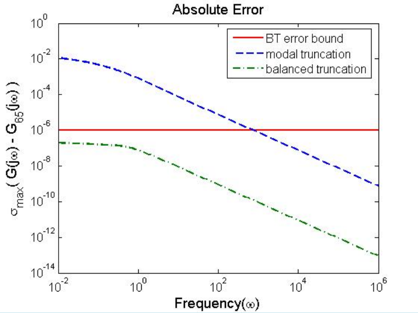
- ADI subspace [B./R.-C. Li/Truhar 2008]:

$$\mathcal{Z} = \mathrm{colspan}\left[\ V_1, \quad \ldots, \quad V_r\ \right].$$

Note:

1. ADI subspace is rational Krylov subspace [J.-R. Li/White 2002].
2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou 2008].

$n = 1357$, Absolute Error



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

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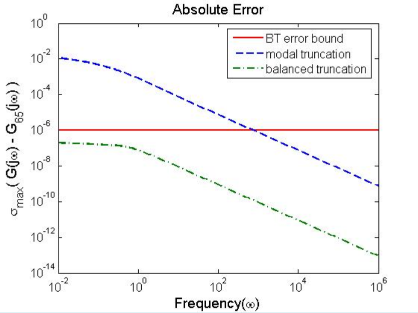

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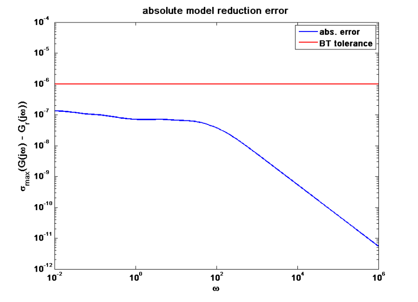
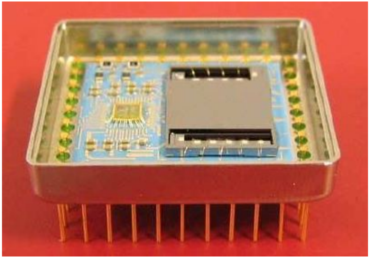
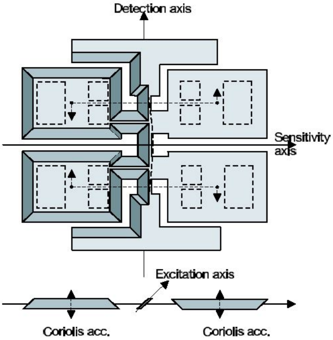
$n = 79841$, **Absolute Error**



- BT model computed using M.E.S.S. in MATLAB,
- dualcore, computation time: <10 min.



- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: http://modelreduction.org/index.php/Modified_Gyroscope

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using SpaRed, $r = 30$.

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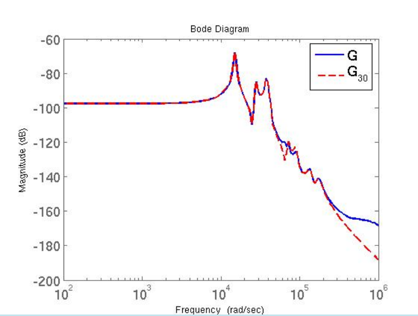
Frequency Repsonse Analysis



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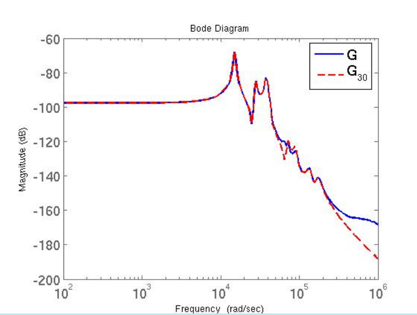
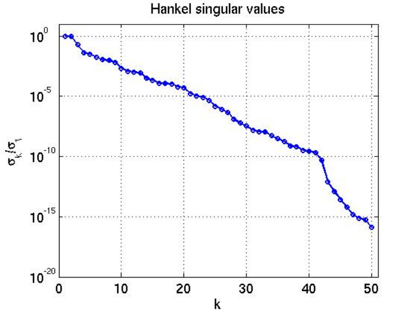


Hankel Singular Values



Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts 1976, Moore 1981]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman 1983]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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Balanced Stochastic Truncation (BST) [Desai/Pal 1984, Green 1988]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Model Reduction

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT)

[Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1}B^T Q + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

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Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

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\text{LQGBT:} \quad & \|G - G_r\|_\infty && \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST:} \quad & \|G - G_r\|_\infty && \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,
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- Computations can be modularized $\rightsquigarrow$ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.

Padé Approximation

Idea:

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- For $s_0 \not\in \Lambda(A)$

$$\begin{aligned}
G(s) &= C((s_0 I_n - A) + (s - s_0)I_n)^{-1}B \\
&= C\left(I - (s - s_0)(s_0 I_n - A)^{-1}\right)^{-1}(s_0 I_n - A)^{-1}B \\
&= m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots
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  - For $s_0 = 0$: $m_j := C(A^{-1})^j B =$ moments.
  - For $s_0 = \infty$: $m_j := CA^{j-1}B =$ Markov parameters.

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- As reduced-order model use rth Padé approximant $\hat{G}$ to G:
$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$
i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r - 1$

  $\rightsquigarrow$ moment matching if $s_0 < \infty$,

  $\rightsquigarrow$ partial realization if $s_0 = \infty$.

Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}\tilde{B}) =: \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0 I_n - A)^{-1},\ \tilde{B} = (s_0 I_n - A)^{-1} B$) along

$$\mathcal{W} = \mathrm{span}(C^T, \tilde{A}^* C^T, \ldots, (\tilde{A}^*)^{r-1} C^T) =: \mathcal{K}(\tilde{A}^*, C^T, r).$$

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Remark: Arnoldi (PRIMA) yields only $G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^r)$.

Padé Approximation

Padé-via-Lanczos Method (PVL)

Difficulties:

- Computable error estimates/bounds for $\|y - \hat{y}\|_2$ often very pessimistic or expensive to evaluate; recent advances using dual-weighted residual-type error estimators [Feng/Antoulas/B. 2017].

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- Good approximation quality only locally.
- Preservation of physical properties only in special cases; usually requires post processing which (partially) destroys moment matching properties.

Computation of reduced-order model by projection

Given an LTI system $\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sI_n - A)^{-1}B,$ a reduced-order model is obtained using projection approach with $V, W \in \mathbb{R}^{n\times r}$ and $W^T V = I_r$ by computing

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

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Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

$$\begin{aligned} \operatorname{span}\left\{(s_1 I_n - A)^{-1} B, \ldots, (s_k I_n - A)^{-1} B\right\} &\subset \operatorname{Ran}(V), \\ \operatorname{span}\left\{(s_1 I_n - A)^{-T} C^T, \ldots, (s_k I_n - A)^{-T} C^T\right\} &\subset \operatorname{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

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Remarks:

using Galerkin/one-sided projection yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j) \neq \frac{d}{ds}\hat{G}(s_j).$$

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If

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Remarks:

$k = 1$, standard Krylov subspace(s) of dimension $K \rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1) = \frac{d^i}{ds^i}\hat{G}(s_1), \quad i = 0, \ldots, K-1(+K).$$

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

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Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme 1997],
- Iterative Rational Krylov Algorithm (IRKA) [Antoulas/Beattie/Gugercin 2007].

Best $\mathcal{H}_2$-norm approximation problem

$$\text{Find} \quad \arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \| G - \hat{G} \|_2 .$$

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$$\text{Find} \quad \arg\min_{\hat{G} \in \mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$$

$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For SISO systems

$$G(-\mu_i) = \hat{G}(-\mu_i),$$
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where μ_i are the poles of the reduced transfer function $\hat{G}$.

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For MIMO systems

$$\begin{aligned} G(-\mu_i)\tilde{B}_i &= \hat{G}(-\mu_i)\tilde{B}_i, && \text{for } i = 1, \ldots, r, \\ \tilde{C}_i^T G(-\mu_i) &= \tilde{C}_i^T \hat{G}(-\mu_i), && \text{for } i = 1, \ldots, r, \\ \tilde{C}_i^T G'(-\mu_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\mu_i)\tilde{B}_i, && \text{for } i = 1, \ldots, r, \end{aligned}$$

where $T^{-1}\hat{A}T = \operatorname{diag}\{\mu_1, \ldots, \mu_r\}$ = spectral decomposition and

$$\tilde{B} = \hat{B}^T T^{-T}, \quad \tilde{C} = \hat{C}T.$$

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

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Then

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for $i = 1, \ldots, r$ as desired.
$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. 2006, Bunse-Gerstner et al. 2007, Van Dooren et al. 2008]

$\mathcal{H}_2$-Optimal Model Reduction
The basic IRKA Algorithm

Algorithm 1 IRKA

Input: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.

Output: A^{opt}, B^{opt}, C^{opt}

$\{\mu_1, \ldots, \mu_r\} = \Lambda(\hat{A})$

while $\left(\max_{j=1,\ldots,r} \left\{ \left|\mu_j - \mu_j^{\text{old}}\right| / \left|\mu_j\right| \right\} > \delta\right)$ **do**

$\quad \operatorname{diag}\{\mu_1, \ldots, \mu_r\} := T^{-1}\hat{A}T, \;\; \tilde{B} = \hat{B}^* T^{-*}, \;\; \tilde{C} = \hat{C}T.$

$\quad V = \left[(-\mu_1 I - A)^{-1} B\tilde{B}_1, \ldots, (-\mu_r I - A)^{-1} B\tilde{B}_r\right]$

$\quad W = \left[(-\mu_1 I - A^T)^{-1} C^T\tilde{C}_1, \ldots, (-\mu_r I - A^T)^{-1} C^T\tilde{C}_r\right]$

$\quad V = \operatorname{orth}(V), \; W = \operatorname{orth}(W)$

$\quad \hat{A} = (W^*V)^{-1} W^*AV, \; \hat{B} = (W^*V)^{-1} W^*B, \; \hat{C} = CV$

end while

$A^{opt} = \hat{A}, \; B^{opt} = \hat{B}, \; C^{opt} = \hat{C}$

IRKA can fairly easily be generalized to structured linear systems with transfer functions of the form

$$G(s) = \mathcal{C}(s)\mathcal{A}(s)^{-1}\mathcal{B}(s), \quad \text{where} \quad \mathcal{A}(s) = \sum_{j=0}^{\ell_\alpha} \alpha_j(s) A_j,$$

$$\mathcal{C}(s) = \sum_{j=0}^{\ell_\gamma} \gamma_j(s) C_j, \quad \mathcal{B}(s) = \sum_{j=0}^{\ell_\beta} \beta_j(s) B_j,$$

if the poles can be computed efficiently for the reduced-order model, or other selection criteria for the shifts are available.

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5) **Integro-differential Volterra systems**, **input delays**, **fractional systems** . . .

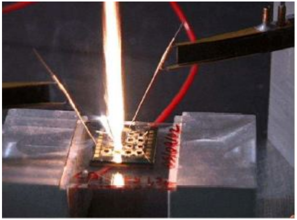
Algorithm 2 TF-IRKA (SISO, B, C constant)

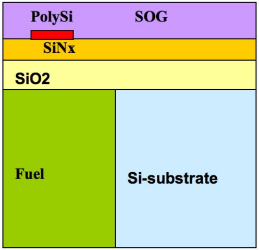
Input: $\mathcal{A}(s)$, B, C, initial shifts $\{\mu_1, \ldots, \mu_r\}$ with $\mu_i \neq \mu_j$ for $i \neq j$, $\delta > 0$.
Output: $\mathcal{A}^{opt}(s)$, B^{opt}, C^{opt}

1: $\mu_j^{\text{old}} = 0$, $j = 1, \ldots, r$.
2: **while** $(\max_{j=1,\ldots,r} \left\{ |\mu_j - \mu_j^{\text{old}}| \,/\, |\mu_j| \right\} > \delta)$ **do**
3: $\quad V = \left[\mathcal{A}(-\mu_1)^{-1}B, \ldots, \mathcal{A}(-\mu_r)^{-1}B\right]$
4: $\quad W = \left[\mathcal{A}(-\mu_1)^{-T}C^T, \ldots, \mathcal{A}(-\mu_r)^{-T}C^T\right]$
5: $\quad V = \text{orth}(V)$, $W = \text{orth}(W)$
6: $\quad \hat{A}_j = (W^*V)^{-1} W^*A_jV$ $(j = 0, \ldots, \ell)$, $\hat{B} = (W^*V)^{-1} W^*B$, $\hat{C} = CV$
7: $\quad$ Compute new shifts as poles of $\hat{C}\hat{\mathcal{K}}(s)^{-1}\hat{B}$.
8: $\quad \{\mu_1, \ldots, \mu_r\} = \Lambda(\hat{A})$
9: **end while**
10: $A_j^{opt} = \hat{A}_j$ $(j = 0, \ldots, \ell)$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

Outline

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretization of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.





Source: http://modelreduction.org/index.php/Micropyros_Thruster

Numerical Comparison of MOR Approaches

Microthruster

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements $\rightsquigarrow$ $n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed. modal truncation using ARPACK, and Z. Bai's PVL implementation.

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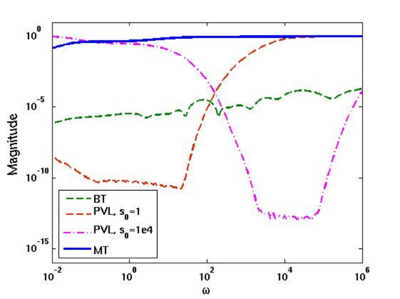
Relative error $n = 4,257$



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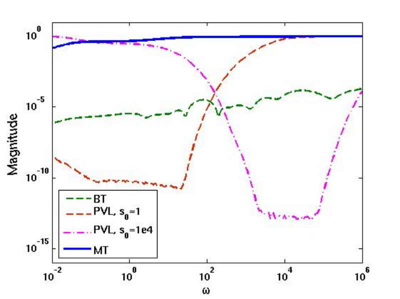
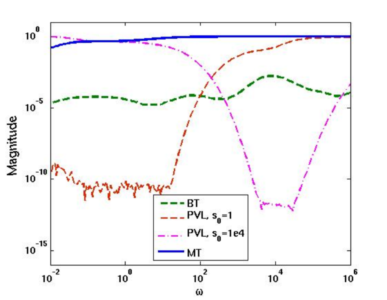


Relative error $n = 11,445$



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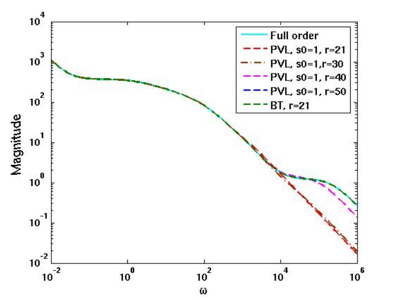
Frequency Response BT/PVL



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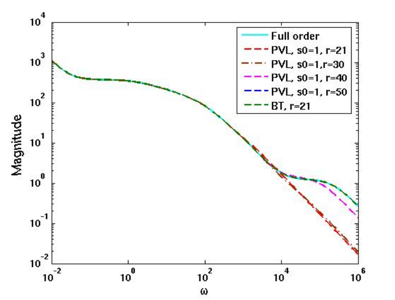
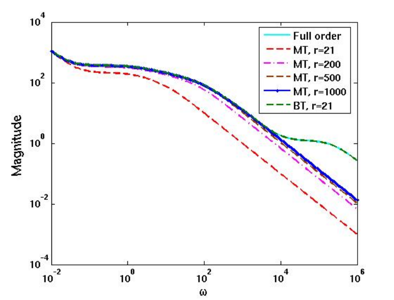


Frequency Response BT/MT



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- Order of reduced model: $r = 28$.

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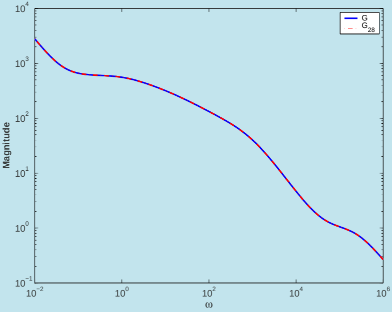
Frequency Repsonse Analysis



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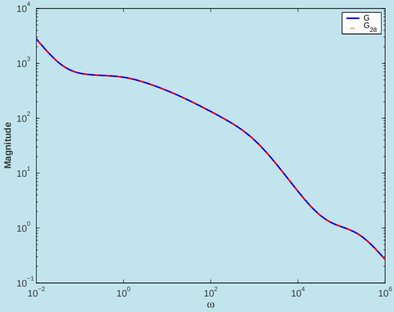
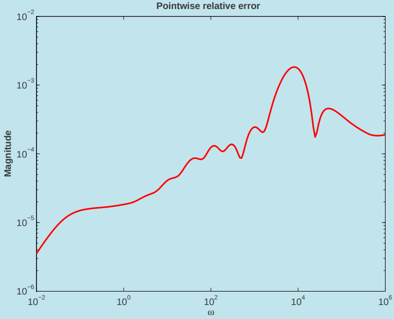


Relative Error



Outline

Current Research Topics

- Special methods for second-order (mechanical) and delay systems.
- Extensions to bilinear, quadratic-bilinear, polynomial, and stochastic systems.
- Rational interpolation methods for nonlinear systems.
- Other MOR techniques like proper orthogonal decomposition (POD) or the reduced basis method (RBM).
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Parametric model reduction:

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

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