$$y_t=\alpha+\beta x_t+\varepsilon_t,\quad \varepsilon_t\sim N(0,\sigma^2),\quad E(y_t)=\alpha+\beta x_t,\quad D(y_t)=\sigma^2$$

$$\hat y_t=a+bx_t:\quad S=\sum\hat\varepsilon_t^2=\sum(y_t-\hat y_t)^2=\sum(y_t-a-bx_t)^2=\sum(y_t^2-2ay_t-2bx_ty_t+a^2+2abx_t+b^2x_t^2)$$

$$\bar y=\frac1T\sum y_t\qquad \frac{\partial S}{\partial a}=\sum_t(-2y_t+2a+2bx_t)=0;\quad \frac{\partial S}{\partial b}=\sum_t(-2x_ty_t+2ax_t+2bx_t^2)=0$$

$$\bar x=\frac1T\sum x_t\qquad \boxed{a=\bar y-b\bar x};\quad \sum_t(-x_ty_t+\bar y\bar x-b\bar x x_t+bx_t^2)=0\Rightarrow \boxed{b=\frac{\sum[(x_t-\bar x)y_t]}{\sum[(x_u-\bar x)x_u]}}$$

$$b=\sum k_ty_t,\ k_t=\frac{x_t-\bar x}{\sum[(x_t-\bar x)x_t]},\quad \sum k_t=0,\ \sum k_tx_t=1$$

$$\hat y_t=\bar y+b(x_t-\bar x)=\sum_u y_u\left[\frac1T+\frac{(x_u-\bar x)(x_t-\bar x)}{\sum_v(x_v-\bar x)x_v}\right]$$

$$E(b)=\sum k_tE(y_t)=\sum k_t(\underbrace{\alpha}_{0}+\underbrace{\beta x_t}_{\beta})=\beta\quad;\quad E(a)=E(\bar y)-E(b)\bar x=[E(\bar y)=\alpha+\beta\bar x]=\alpha$$

$$D(b)=\sum k_t^2D(y_t)+\sum_t\sum_{u\ne t}k_tk_u\underbrace{\operatorname{cov}(y_t,y_u)}_{0}=\sigma^2\sum k_t^2=\sigma^2\sum\frac{(x_t-\bar x)^2}{(\sum(x_t-\bar x)^2)^2}=\frac{\sigma^2}{\sum[(x_t-\bar x)x_t]}$$

$$D(\bar y)=\frac{\sigma^2}{T}$$

$$D(a)=D(\bar y)+D(b)\bar x^2-2\bar x\underbrace{\operatorname{cov}(\bar y,b)}_{0}=\sigma^2\left(\frac1T+\frac{\bar x^2}{\sum(x_t-\bar x)^2}\right);\quad \operatorname{cov}(a,b)=\operatorname{cov}(\bar y,b)-\bar xD(b)=-\bar xD(b)$$

$$\hat y_t=a+bx_t;\quad E(\hat y_t)=\alpha+\beta x_t=E(y_t);\quad D(\hat y_t)=D(a)+D(b)x_t^2+2x_t\operatorname{cov}(a,b)=\sigma^2\left(\frac1T+\frac{(\bar x-x_t)^2}{\sum(\bar x-x_t)^2}\right)$$

$$S=\sum\hat\varepsilon_t^2=\sum(y_t-\hat y_t)^2=\sum[(y_t-E(y_t))-(\hat y_t-E(\hat y_t))]^2;\ E(S)=\overbrace{\sum D(y_t)}^{T\sigma^2}+\overbrace{\sum D(\hat y_t)}^{2\sigma^2}-2\sum\operatorname{cov}(y_t,\hat y_t)=$$

$$\operatorname{cov}(y_t,\hat y_t)=\operatorname{cov}\left(y_t,\sum_u y_u\left[\frac1T+\frac{(x_u-\bar x)(x_t-\bar x)}{\sum_v(x_v-\bar x)x_v}\right]\right)=\sigma^2\left[\frac1T+\frac{(x_t-\bar x)^2}{\sum_v(x_v-\bar x)^2}\right]$$

$$y_i=\alpha+\beta(x_i-\bar x)+\varepsilon$$

$$=\sigma^2\left[T+2-2-2\sum\frac{(x_t-\bar x)^2}{\sum_v(x_v-\bar x)^2}\right]=\sigma^2(T-2)$$

A — ортонорм. матрица, первый столбец $\begin{pmatrix}1/\sqrt n\\ \vdots\\ 1/\sqrt n\end{pmatrix}$, второй $\begin{pmatrix}\frac{x_i-\bar x}{\sqrt{\sum(x_i-\bar x)^2}}\end{pmatrix}$

$$\vec z=A\vec y;\ z_1=\sqrt n\,\bar y=\sqrt n\,a\sim N(\sqrt n\,\alpha,\sigma^2)$$

$$z_2=\sqrt{\sum(x-\bar x)^2}\,b\sim N\left(\sqrt{\sum(x-\bar x)^2}\,\beta,\sigma^2\right)$$

$$\sum z_i^2=\sum y_i^2$$

$$s^2=\frac{\sum\hat\varepsilon_t^2}{T-2}$$ — несмещ. оценка σ^2

$$R=1-\frac{S}{\sum(y_t-\bar y)^2}=1-\frac{\sum(y_t-\hat y_t)^2}{\sum(y_t-\bar y)^2}$$ — качество модели

$$y_t=\beta x_t+\varepsilon_t,\ \varepsilon_t\sim N(0,\sigma^2).\ E(y_t)=\beta x_t,\ D(y_t)=\sigma^2;\ \hat y_t=bx_t;\ E(\hat y_t)=\beta x_t\quad D(\hat y_t)=\frac{\sigma^2x_t^2}{\sum x_t^2}$$

$$S=\sum(y_t-bx_t)^2;\ \frac{\partial S}{\partial b}=-2\sum(y_t-bx_t)x_t=0\Rightarrow b=\frac{\sum x_ty_t}{\sum x_t^2},\ E(b)=\beta;\ D(b)=\frac{\sigma^2}{\sum x_t^2};\ D(\bar y)=\frac{\sigma^2}{T}$$

$$E(S)=\underbrace{\sum D(y_t)}_{T\sigma^2}+\underbrace{\sum D(\hat y_t)}_{\sigma^2}-2\underbrace{\sum\operatorname{cov}\left(y_t,\frac{\sum x_uy_u}{\sum x_u^2}x_t\right)}_{\sigma^2}=\sigma^2(T+1-2)=\sigma^2(T-1)\Rightarrow s^2=\frac{\sum\hat\varepsilon_t^2}{T-1}$$

$$R=1-\frac{\sum(y_t-\frac{\bar y}{\bar x}x_t)^2}{\sum(y_t-\bar y)^2}$$

$$y_t = \alpha + \beta x_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2), \quad y_t \sim N(\alpha + \beta x_t, \sigma^2)$$

$$\mathcal{L}(x_t, y_t; \alpha, \beta, \sigma^2) = \prod_t \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left[-\frac{1}{2\sigma^2}(y_t - \alpha - \beta x_t)^2\right]$$

$$\ln \mathcal{L} = -\frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum(y_t - \alpha - \beta x_t)^2 - \frac{T}{2}\ln 2\pi$$

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = -\frac{1}{\sigma^2}\sum(y_t - \alpha - \beta x_t) = 0; \quad \frac{\partial \ln \mathcal{L}}{\partial \beta} = -\frac{1}{\sigma^2}\sum(y_t - \alpha - \beta x_t)x_t$$

$$\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} = -\frac{T}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(y_t - \alpha - \beta x_t)^2 = 0$$

$$\tilde{a} = \bar{y} - \tilde{b}\bar{x}; \quad \tilde{b} = \frac{\sum[(x_t - \bar{x})y_t]}{\sum[(x_t - \bar{x})x_t]}; \quad \tilde{s}^2 = \frac{\sum(y_t - \tilde{a} - \tilde{b}x_t)^2}{T}$$

$$y_t = \varepsilon_t, \quad \varepsilon_t \sim \underset{\frac{\lambda^t}{t!}e^{-\lambda}}{\overset{\shortparallel}{P}}(\lambda). \quad \mathcal{L} = \prod_t \left(\frac{\lambda^{y_t}}{y_t!}e^{-\lambda}\right) = \frac{\lambda^{\bar{y}T}}{\prod y_t!}e^{-\lambda T}; \quad \ln \mathcal{L} = \bar{y}T\ln\lambda - \ln\prod y_t! - \lambda T$$

$$\frac{\partial \ln \mathcal{L}}{\partial \lambda} = 0 = \frac{\bar{y}T}{\lambda} - T \Rightarrow \lambda = \bar{y}$$

$$\underline{y} \sim N(\underline{\mu}, \sigma_{ij}):$$

$$\underline{P}(y(x_1'), \dots y(x_k') \mid y(x_1) \dots y(x_n)) \sim N(\mu, \Sigma) : \text{Rank}(\mu) = n, \ \Sigma \text{ shape}: n \times k$$

$$\Sigma_{ij} = k(x_i, x_j')$$

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t-распред: $\dfrac{X_{N+1}}{\left(\frac{1}{N}\sum X_i^2\right)^{1/2}}$ $X_i \sim N(0,1)$.

χ^2-распред: $\sum x_i^2$, $x_i \sim N(0,1)$; $P(x_1^2 \le x) = P(x_1 \le \sqrt{x}) + P(x_1 \ge \sqrt{x}) =$

$$= 2\int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} dx_1 = \left[y = \frac{x_1}{\sqrt{2}};\ x_1 = \sqrt{2}y\right] = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\frac{x}{2}}} e^{-y^2} dy = \operatorname{erf}\left(\sqrt{\frac{x}{2}}\right)$$

$$p(x) = \frac{dP}{dx} = \frac{\cancel{2}}{\sqrt{\pi}} e^{-\frac{x}{2}} \frac{1}{\cancel{2}\sqrt{2}\sqrt{x}} = \frac{1}{\sqrt{2\pi}}\frac{e^{-x/2}}{\sqrt{x}}$$

$y = z^2$
$z^2 = \sum x_i^2$

$z = y^{1/2}$; $dz = \frac{1}{2}dy\, y^{-1/2}$
$z^{N-1}dz = y^{\frac{N-1}{2}-\frac{1}{2}}\frac{dy}{2} = \frac{y^{\frac{N}{2}-1}dy}{2}$

$$P\left(\sum x_i^2 \le x\right) = \int dx_1 \ldots \int dx_N \frac{1}{(2\pi)^{N/2}} e^{-\sum x_i^2/2} = \left[y = \sum x_i^2;\ dx_1\ldots dx_N = \frac{\pi^{N/2} N}{\Gamma\left(\frac{N}{2}+1\right)}\frac{1}{2} y^{\frac{N}{2}-1} dy,\ y_{max} = x\right] =$$

$$= \frac{N/2}{2^{N/2}\Gamma\left(\frac{N}{2}+1\right)}\int_0^x dy\, y^{\frac{N}{2}-1} e^{-y/2};\quad p(x) = \frac{dP}{dx} = \frac{x^{\frac{N}{2}-1}e^{-x/2}}{2^{N/2}\Gamma\left(\frac{N}{2}\right)}$$

$$y: \chi^2;\ P\left(\sqrt{\tfrac{y}{N}} \le x\right) = \int_0^{Nx^2} \frac{y^{\frac{N}{2}-1}e^{-y/2}}{2^{N/2}\Gamma\left(\frac{N}{2}\right)} dy = \left[\begin{array}{l} z = y^{1/2}\frac{1}{2N} \\ y = Nz;\ dy = 2z\,dz\,N \\ y_{max} = Nx^2 = z^2_{max} N \end{array}\right] = \int_0^x 2\frac{N^{\frac{N}{2}} z^N e^{-\frac{Nz^2}{2}}}{2^{N/2}\Gamma\left(\frac{N}{2}\right)} dz;\ p(x) = 2\left(\frac{N}{2}\right)^{\frac{N}{2}}\frac{x^N e^{-\frac{Nx^2}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

$$n \sim N(0,1);\ s \sim \sqrt{\frac{\chi^2}{N}}:\ P\left(\frac{n}{s} \le x\right) = \int_0^\infty ds \int_{-\infty}^{sx} dn\, 2\left(\frac{N}{2}\right)^{\frac{N}{2}} s^N e^{-\frac{Ns^2}{2}}\frac{1}{\Gamma\left(\frac{N}{2}\right)}\frac{1}{\sqrt{2\pi}} e^{-n^2/2} = \left[\frac{n}{s} = y,\ dn = s\,dy\right]$$

$$= \int_0^\infty ds\, \cancel{2}\left(\frac{N}{2}\right)^{\frac{N}{2}} s^N e^{-\frac{Ns^2}{2}}\frac{1}{\Gamma\left(\frac{N}{2}\right)}\frac{1}{\cancel{2}}\left(1 + \int_0^x \sqrt{\frac{2}{\pi}} e^{-s^2y^2/2} dy\right)$$

$$p(x) = \frac{dP}{dx} = \int_0^\infty ds \left(\frac{N}{2}\right)^{\frac{N}{2}} s^N \frac{1}{\Gamma\left(\frac{N}{2}\right)} e^{-\frac{s^2(N+x^2)}{2}}\sqrt{\frac{2}{\pi}} = \left(\frac{\cancel{N}}{\cancel{2}}\right)^{\frac{N}{2}}\frac{1}{\Gamma\left(\frac{N}{2}\right)}\sqrt{\frac{\cancel{2}}{\pi}}\, 2^{\frac{N}{2}-\frac{1}{2}}\frac{1}{\cancel{x}(N+x^2)^{\frac{N}{2}+\frac{1}{2}}}\Gamma\left(\frac{N}{2}+\frac{1}{2}\right) =$$

$$= \frac{\Gamma\left(\frac{N+1}{2}\right)}{\sqrt{\pi N}\left(1+\frac{x^2}{N}\right)^{\frac{N+1}{2}}\Gamma\left(\frac{N}{2}\right)}$$

$$T(y) = \frac{\frac{\bar{y}-\mu_0}{\sigma}}{\sqrt{\frac{\sum(y-y_t)^2}{(n-1)\sigma^2 n}}} = \frac{\bar{y}-\mu_0}{\sqrt{\frac{\sum(y-y_t)^2}{(n-1)n}}}$$

Двухвыборочный: $t = \dfrac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$

Линейная регр: $a \sim N\left(\alpha, \frac{\sigma^2}{T}\right)$; $b \sim N\left(\beta, \frac{\sigma^2}{\sum(x-x_t)^2}\right)$; $\frac{S}{\sigma^2} \sim \chi^2_{T-2}$

Д.И.: $\dfrac{a-a_0}{\sqrt{\frac{S}{(n-2)n}}} \sim t_{n-2}\left(\frac{\gamma}{2}\right)$ $\quad \dfrac{b-b_0}{\sqrt{\frac{S}{(n-2)\sum(x-x_t)^2}}} \sim t_{n-2}$. $\quad \dfrac{\alpha + \ell\beta}{\sqrt{\frac{S}{n-2}}\sqrt{\frac{1}{n}+\frac{\ell^2}{\sum(x-x_t)^2}}}$

Критерий Колмогорова

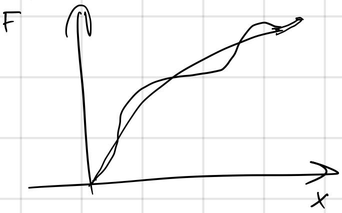
Эмпирическая ф-ия распред: $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{[-\infty,x]}(X_i)$, $I_{[-\infty,x]}(X_i) = \begin{cases} 1, & X_i \le x \\ 0, & X_i > x \end{cases}$

Колмогоровск. стат. $D_n = \max_x |F_n(x) - F(x)|$, где $F(x)$ — аналит. ф-ия распред.

Распред. Колм.: $K = \max_{t\in[0;1]} |B(t)|$; $D_n\sqrt{n} \xrightarrow{n\to\infty} K$; $F_K = \begin{cases} \sum_{k=-\infty}^{+\infty} (-1)^k e^{-2k^2x^2}, & x>0 \\ 0, & x \le 0 \end{cases}$

броуновский мост

При $\alpha \to 1$, $n \to \infty$: $C_\alpha = \sqrt{-\frac{1}{2}\ln\alpha}$



Критерий Колмогорова — Смирнова

$\sqrt{\frac{nm}{n+m}}\, D_{n,m} \sim K$, $D_{n,m} \equiv \max|F_{1,n} - F_{2,m}|$

Тесты k-мерного $N(\mu, \Sigma)$

$S = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})'$; $Y_i \equiv \frac{X_i - \bar{X}}{S^{1/2}}$. Скалярные произведения $D_{ij} = Y_i' Y_j$ — инвариант

Тест коэф-та ассиметрии Мардиа

$$\gamma = E\left(\frac{x-\mu}{\sigma}\right)^3;\quad b = \frac{1}{n}\sum (Y_i' Y_k)^3 \sim \alpha_1 \chi^2_d + \alpha_2 \chi^2_{\frac{d(d-1)(d+4)}{6}},\quad \alpha_1 = \alpha_2 = 6$$

Тест коэф-экспесса Мардиа

$$\gamma = E\left(\frac{x-\mu}{\sigma}\right)^4;\quad b = \frac{1}{n} E(y_i)^4;\quad \sqrt{n}\, b \sim N(d(d+2), 8d(d+2))$$

Тест углов:

$$U_n = \left\| \frac{1}{n}\sum U_i \right\|^2 = \frac{1}{n^2}\sum_{i,j}\frac{Y_i' Y_j}{(Y_i' Y_j)^{1/2}(Y_i' Y_k)^{1/2}}$$

$$nU_n \sim \frac{1}{d}\left(1 - \frac{2}{d}\left(\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}\right)^2\right)\chi^2_d$$

Тест расстояний:

$\sqrt{n}$