Integer Programming (IP)

2024 Fall

Doo Ho Lee

1. Introduction to Integer Programming (IP)

- **IP = Integer(정수) + Programming(계획법) : Integer Programming**
  - Mathematical programming where some or all decision variables should be integers.
  - ILP (Integer Linear programming): Among IPs, constraints and an objective function are all linear.

(1) Variations of IP problems
- Pure IP problem: All decision variables should be integers.
- Mixed IP problem: Some decision variables are integers and others are real numbers.
- 0-1 (Binary) IP problem: All decision variables should be 0 or 1.

(2) Importance of IP
- For real decision problems, the integer form of the optimal solution is required.
  ex) How many ~~?, Should we do ~~?, …
- Decision problems can be solved more easily by modeling them as IP problems.

(3) Solution methods of IP
- Enumeration
- LP relaxation
- Cutting plane
- Branch and bound

2. Solution methods of IP – (1) Enumeration

[EX 6-1] Logistics company Y is planning to establish new facilities, including wholesale shops (WSs) and distribution centers (DCs) with a budget of 38 million dollars. Establishment costs are 5 million dollars for a WS and 10 million dollars for a DC, respectively. Monthly expected revenues are 0.4 million dollars for a WS and 0.6 million dollars for a DC, respectively. At least one DC should be established and the total number of facilities does not exceed 5. Y wants to maximize their total monthly revenue.

- **Decision variables**

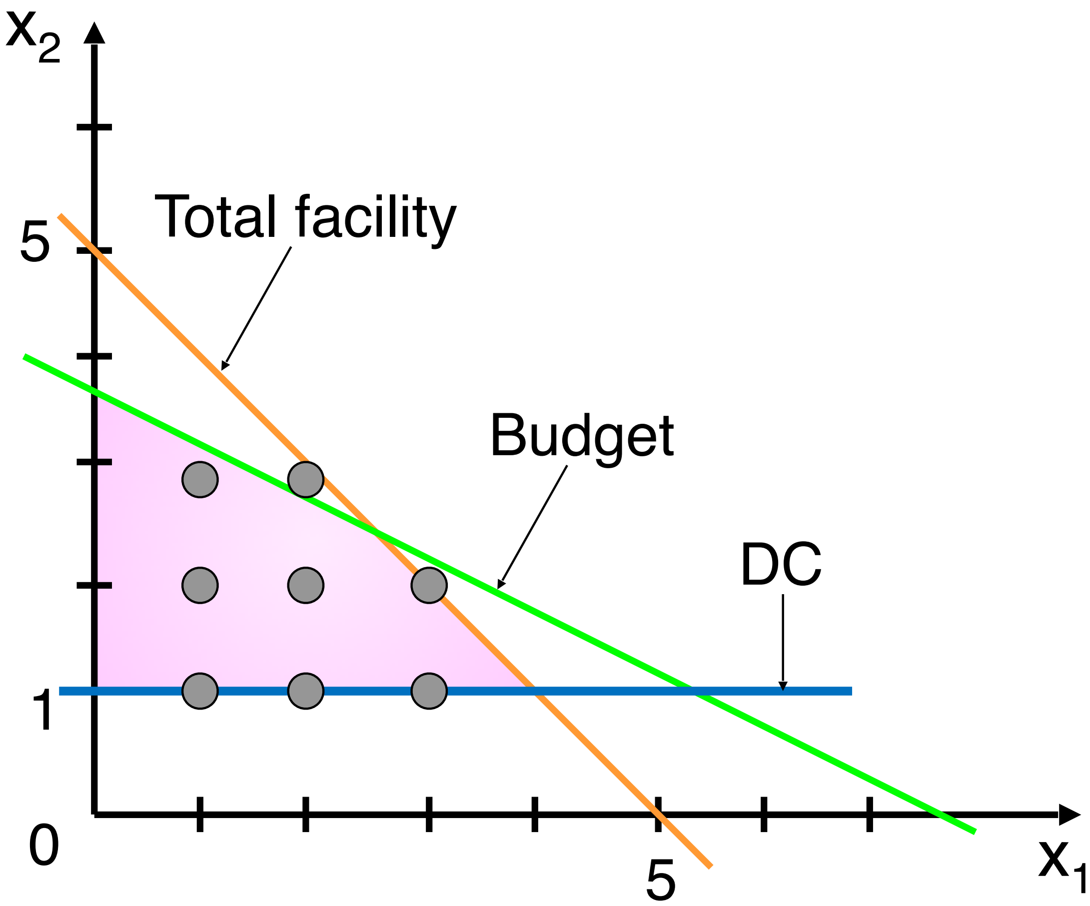
  X_1 = the number of wholesale shops to be established

  X_2 = the number of distribution centers to be established

  *** Note that X_1 and X_2 are non-negative integers.**

$$
\begin{aligned}
\max\ Z &= 0.4X_1 + 0.6X_2 && \text{(monthly revenue)} \\
\text{s.t.}\quad & 5X_1 + 10X_2 \le 38 && \text{(budget)} \\
& X_1 + X_2 \le 5 && \text{(total facility)} \\
& X_2 \ge 1 && \text{(distribution center)} \\
& X_1, X_2 \in I_+ \cup 0 && \text{(non-negative integer)}
\end{aligned}
$$

- **Enumeration (열거법)**
  - Find all feasible solutions and pick one of them which maximize or minimize the objective function.



C1	**C2**			
0.4	0.6			
X1	**X2**	**Obj**		**Obj***
1	1	1		2.4
1	2	1.6		
1	3	2.2		
2	1	1.4		
2	2	2		
3	1	1.8		
3	2	2.4		

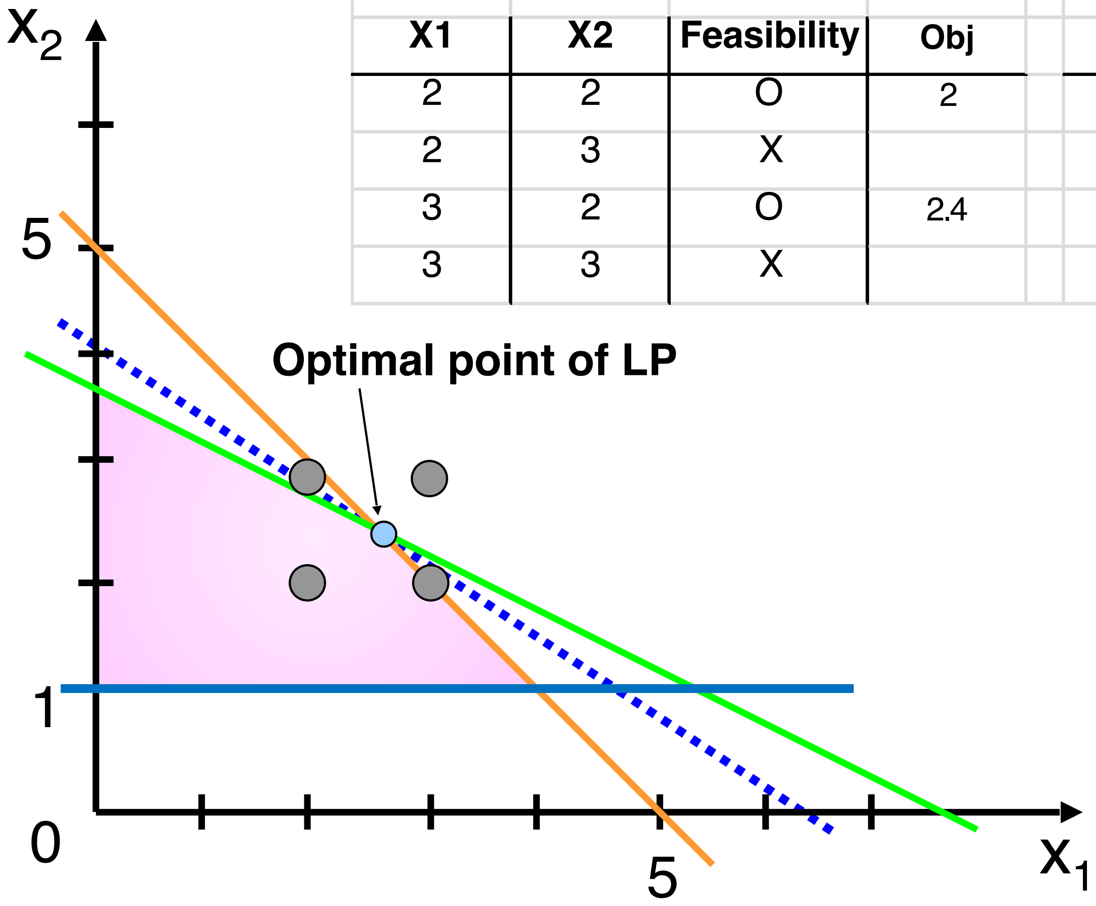
- Not proper in case of the infinite number of feasible solutions.

2. Solution methods of IP – (2) LP relaxation

- **LP relaxation (LP 완화법)**
  - Remove integer constraints find the optimal solution of the relaxed LP problem.
  - Search integer points around the LP solution point.

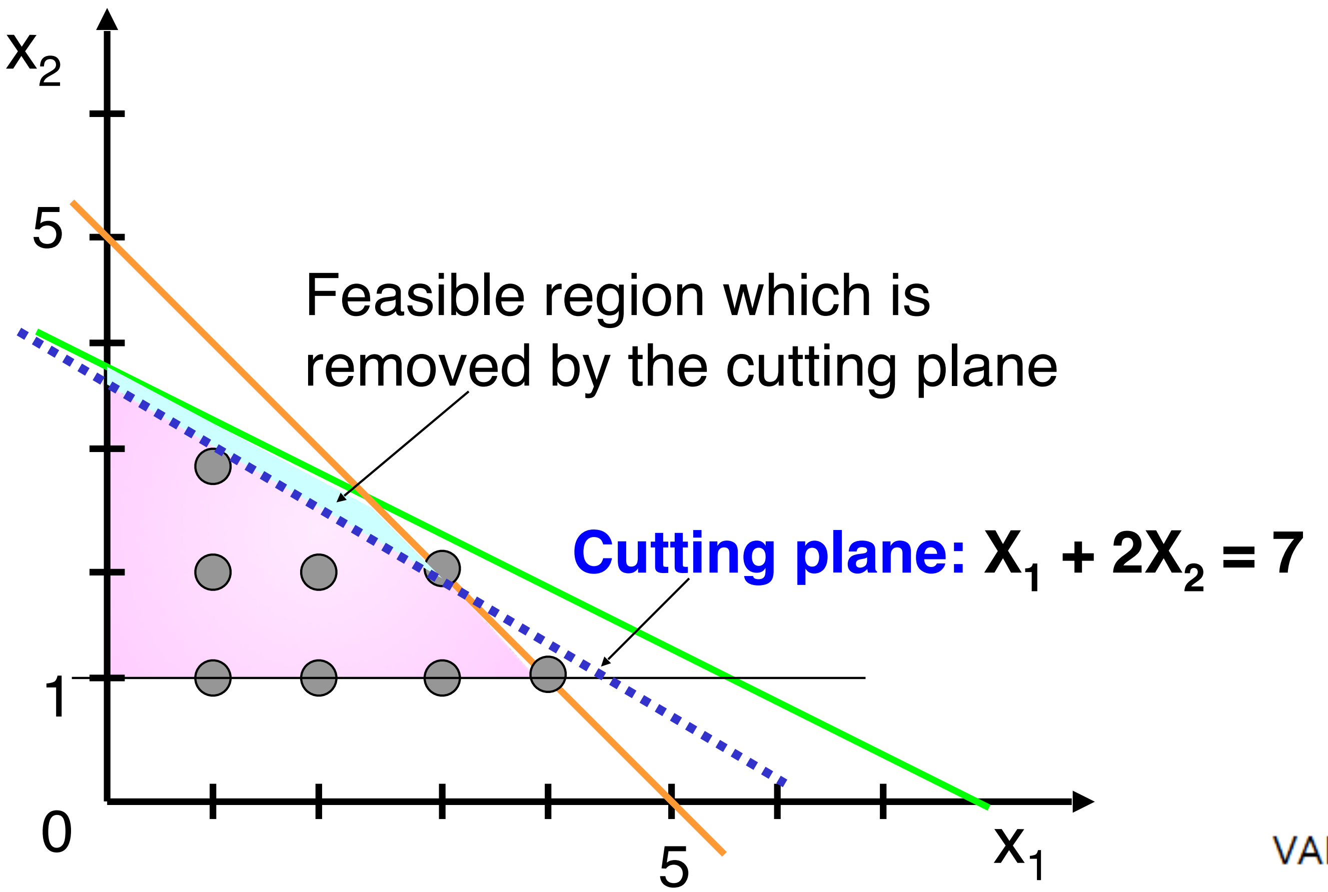
C1	C2				
0.4	0.6				
X1	**X2**	**Feasibility**	**Obj**		**Obj***
2	2	O	2		2.4
2	3	X			
3	2	O	2.4		
3	3	X			

```
&lt;untitled&gt;
max 0.4X1 + 0.6X2
st
5X1 + 10X2 <= 38
X1 + X2 <= 5
X2 >= 1
X1 >= 0
end
OBJECTIVE FUNCTION VALUE

1)      2.520000

VARIABLE      VALUE        REDUCED COST
X1            2.400000     0.000000
X2            2.600000     0.000000
```



- Not proper in case of a big problem (curse of dimensionality).

- **Cutting plane (평면분할법)**
  - Add new constraints and remove a part of the feasible region which does not include integer point.
  - Repeat above procedure until we find the optimal integer point.




```
<untitled>
max 0.4X1 + 0.6X2
st
5X1 + 10X2 <= 38
X1 + X2 <= 5
X2 >= 1
X1 >= 0
X1 + 2X2 <= 7
end
```

Add

```
OBJECTIVE FUNCTION VALUE

1)      2.400000

VARIABLE      VALUE        REDUCED COST
   X1        3.000000        0.000000
   X2        2.000000        0.000000
```

- Inefficient in view of computation time.

2. Solution methods of IP – (4) Branch and bound

- **Branch and bound (B&B, 분지한계법)**
  - By enumerating the set of solutions, validate the optimality.
  - By excepting impossible sets repeatedly, narrow the feasible region and find the optimal point.
  - Often called as the partial enumeration method.
  - Superior to previous three methods in view of computing efficiency.

- **Procedure of branch and bound**

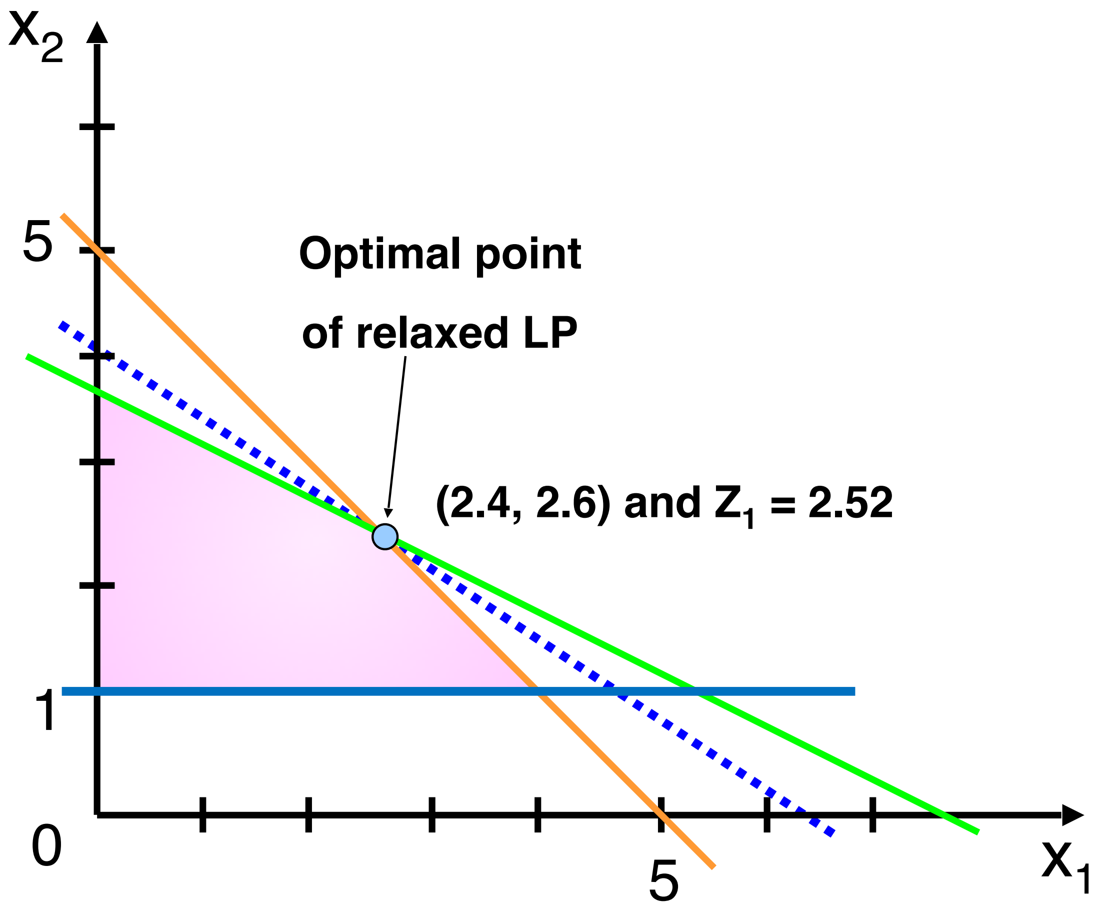
  (1) Branching
  - Split the feasible region into two exclusive regions and make two LP partial problems.

  (2) Bounding
  - Solve two LP partial problems.
  - Based on the optimal solutions of two LP partial problems, define the upper bound and the lower bound of the optimal solution of the original IP problem.

  (3) Forthcoming
  - For the IP problem where the upper bound and lower bound are decided, validate the optimality and except impossible solutions.

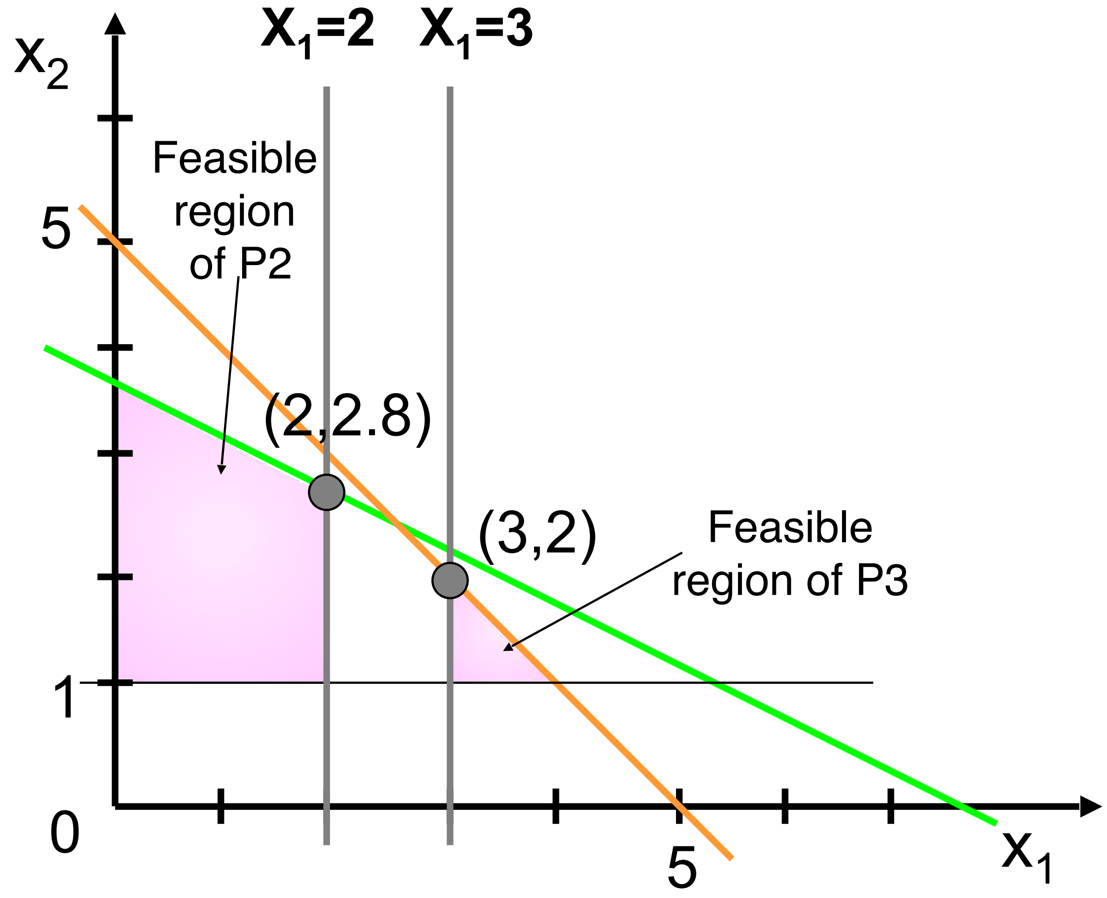
Why don't you guess my number?

- **Graphical method of relaxed LP (Problem 1, P1)**
  - $(X_1, X_2)_1 = (2.4, 2.6)$, $Z_1 = 2.52$
  - **Set the upper bound of the objective function of EX 6-1 to $Z_1 = 2.52$**



2. Solution methods of IP – (4) Branch and bound

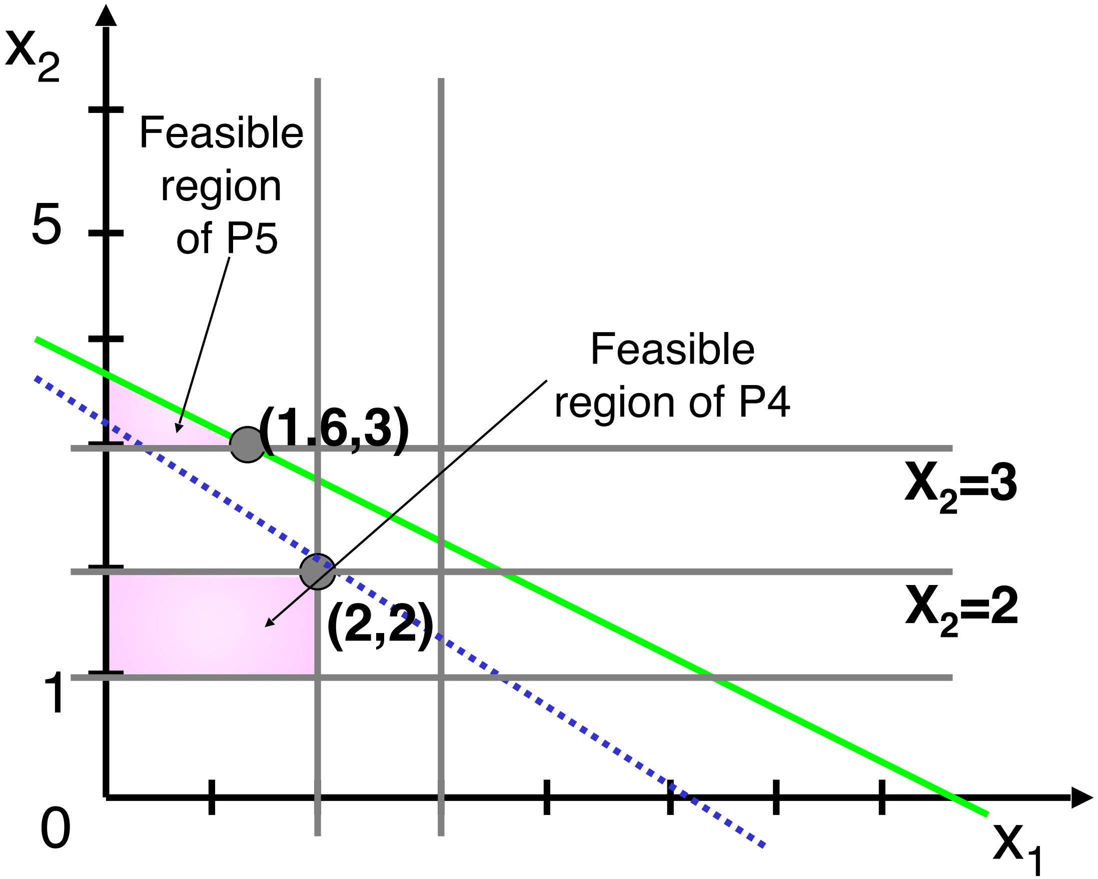
- **Branching**
  - The current solution X_1 = 2.4 and X_2 = 2.6 are not both integers. ← Need to branch
  - Generally, branch the solution element nearer to the integer.
  - In our case, we first branch X_1 and make two problems, namely P2 and P3.
  - In P2, we add constraint **$X_1 \leq 2$** in P1.
  - In P3, we add constraint **$X_1 \geq 3$** in P1.



- **Bounding**
  - Optimal solution of P2
    $(X_1, X_2)_2$ = (2, 2.8), Z_2 = 2.48
  - Optimal solution of P3
    $(X_1, X_2)_3$ = (3, 2), Z_3 = 2.4
  - The optimal solution of P3 is integer and it is the candidate of the optimal solution of the original IP problem.
  - **The lower bound of the objective function is set to 2.4. → 2.4 ≤ optimal value ≤ 2.52**

2. Solution methods of IP – (4) Branch and bound

- **Forthcoming**
  - Branch X_2 in P2 and make two problems, namely P4 and P5.
  - In P4, we add constraint $\mathbf{X_2 \leq 2}$ in P2.
  - In P5, we add constraint $\mathbf{X_2 \geq 3}$ in P2.



- Optimal solution of P4

  $\mathbf{(X_1, X_2)_4 = (2, 2), Z_4 = 2}$

  Satisfy integer but eliminate it due to

  $\mathbf{Z_4 < \text{lower bound} = 2.4}$

- Optimal solution of P5

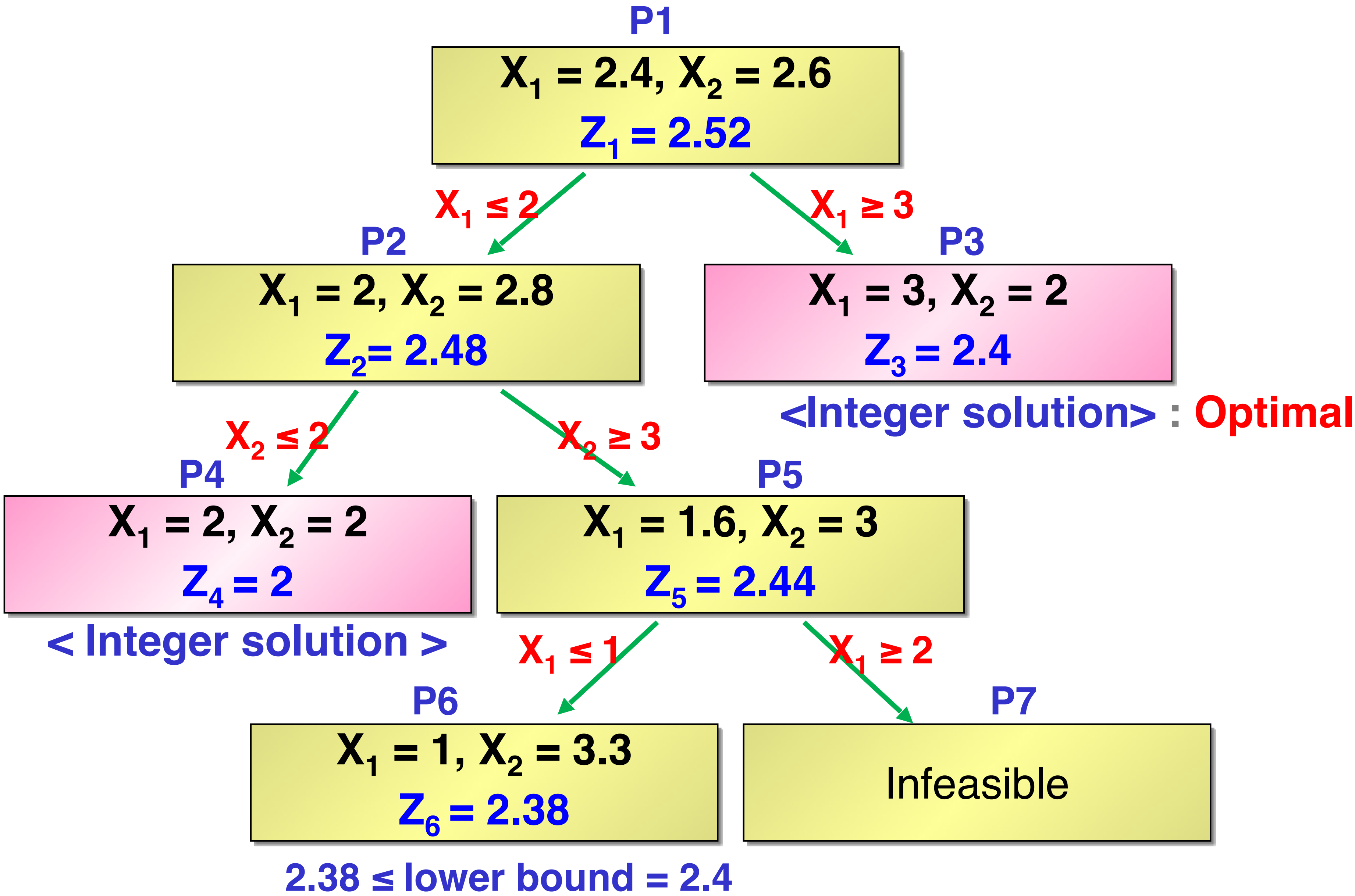
  $\mathbf{(X_1, X_2)_5 = (1.6, 3), Z_5 = 2.44}$

  Does not satisfy integer and branch

  Branch X_1 in P5 and make two problems,

  namely P6 and P7.

- **Branch and bound tree**



3. Solution methods of IP – LINDO

- **Solving IP problem using LINDO**
  - Actually, LINDO provides IP solution by utilizing the branch and bound method.
  - At the end of the LINDO code, just define the variable type.
  - Variable type
    - √ GIN: general integer
    - √ INT: binary (0-1) integer

<untitled>

```
max 0.4X1 + 0.6X2
st
5X1 + 10X2 <= 38
X1 + X2 <= 5
X2 >= 1
X1 >= 0
end
gin X1
gin X2
```

Add

```
LP OPTIMUM FOUND AT STEP       2
OBJECTIVE VALUE =    2.51999998

NEW INTEGER SOLUTION OF     2.40000010      AT BRANCH       0 PIVOT
BOUND ON OPTIMUM:   2.400000
ENUMERATION COMPLETE. BRANCHES=      0 PIVOTS=         3

LAST INTEGER SOLUTION IS THE BEST FOUND
RE-INSTALLING BEST SOLUTION...

       OBJECTIVE FUNCTION VALUE

       1)       2.400000

 VARIABLE        VALUE          REDUCED COST
       X1         3.000000        -0.400000
       X2         2.000000        -0.600000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)         3.000000         0.000000
       3)         0.000000         0.000000
       4)         1.000000         0.000000
       5)         3.000000         0.000000

NO. ITERATIONS=         3
BRANCHES=     0 DETERM.=   1.000E    0
```

Homework

한국슈퍼연쇄점은 최근 사세확장을 위하여 신규점포와 중간저장설비를 신설할 계획으로 155억 원의 자금을 확보하였다. 신규 점포는 신설비용이 **개당 10억 원이 소요**되며 이를 통해 **한 달에 3,000만 원의 수익**을 올릴 수 있고, 저장설비는 **개당 30억 원의 자금이 소요**되며 이를 통해 **한 달에 7,200만 원의 수익**을 올릴 수 있다. 경영진에서는 최소한 한 개의 신규저장설비를 설치할 예정이며 신규점포는 5개를 넘지 않는 범위에서 설치하려 한다. 한국슈퍼는 확보된 자금으로 최대의 수익을 얻을 수 있는 신설 계획을 수립하고자 한다.

4. Applications of IP – (1) Knapsack problem (배낭문제)

- **Knapsack problem**
  - A typical and the most famous binary integer problem (이진정수계획문제).
  - Locate items into the given knapsack which has a limited capacity.
  - Employing integer 0 and 1, select the item in case of 1 and discard the item in case of 0.

EX 6-2) Limitation of a given knapsack: 12kg

Item	Weight (kg)	Utility (value)
1. Tent	5.0	75
2. Sleeping bag	2.0	20
3. Burner	1.2	40
4. Food	4.0	50
5. Plate	2.5	30
6. Fishing rod	4.5	25
7. First-aid medicine	0.5	15

- **Define decision variables**
  - If we decide to select i^{th} item, $X_i = 1$. Otherwise, $X_i = 0$.
- **Define objective function**
  - We want to maximize the total utilities of the selected items.
- **Define constraints**
  - Total weight of the selected items is less than or equal to 12 kg.

- **Formulation**

$$\max \quad 75X_1 + 20X_2 + 40X_3 + 50X_4 + 30X_5 + 25X_6 + 15X_7$$

$$\text{st} \quad 5X_1 + 2X_2 + 1.2X_3 + 4X_4 + 2.5X_5 + 4.5X_6 + 0.5X_7 \leq 12$$

$$X_i \in \{0, 1\}, \ \forall\, i$$

```
<untitled>
max 75X1 + 20X2 + 40X3 + 50X4 + 30X5
    + 25X6 + 15X7
st
5X1+2X2+1.2X3+4X4+2.5X5+4.5X6+0.5X7 <= 12
end
int X1
int X2
int X3
int X4
int X5
int X6
int X7
```

```
        OBJECTIVE FUNCTION VALUE

        1)      180.0000

  VARIABLE        VALUE
        X1         1.000000
        X2         0.000000
        X3         1.000000
        X4         1.000000
        X5         0.000000
        X6         0.000000
        X7         1.000000
```

4. Applications of IP – (2) Investment planning

[EX 6-3] For next 3 years, Samjung electronics is planning to invest their money to four possible projects, including the development of new products, the factory automation, the factory expansion and the establishment of the foreign offices. For each project, costs, expected revenues and investment limits are given in the below table. Which projects should be chosen to maximize the total revenue?

Project	Costs			Revenue
	1st year	2nd year	3rd year	
1. New products	10	10	10	40
2. Automation	1	20	5	30
3. Expansion	5	30	20	50
4. Foreign offices	10	15	15	45
Investment limits	20	60	40	

- **Define decision variables**

  - If we decide to choose i^{th} project, $\mathbf{X_i = 1}$. Otherwise, $\mathbf{X_i = 0}$.

- **Define objective function**

  - We want to maximize the total revenue of the chosen projects.

- **Define constraints**

  - In each year, total costs of the chosen projects does not exceed the given investment limit.

4. Applications of IP – (2) Investment planning

- **Formulation**

$$\max \ 40X_1 + 30X_2 + 50X_3 + 45X_4$$

$$\text{st} \quad 10X_1 + X_2 + 5X_3 + 10X_4 \le 20 \text{ (1st year)}$$

$$10X_1 + 20X_2 + 30X_3 + 15X_4 \le 60 \text{ (2nd year)}$$

$$10X_1 + 5X_2 + 20X_3 + 15X_4 \le 40 \text{ (3rd year)}$$

$$X_i \in \{0, 1\} \ , \ \forall \ i$$

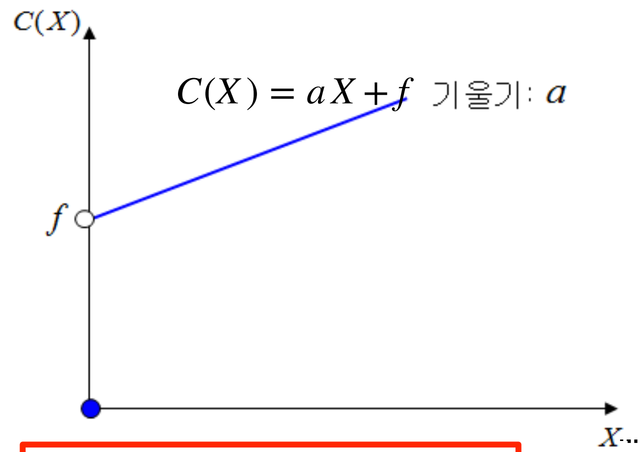
```
<untitled>
max  40X1 + 30X2 + 50X3 + 45X4
st
10X1 + X2 + 5X3 + 10X4 <= 20
10X1 + 20X2 + 30X3 + 15X4 <= 60
10X1 + 5X2 + 20X3 + 15X4 <= 40
end
int X1
int X2
int X3
int X4
```

```
      OBJECTIVE FUNCTION VALUE

      1)        120.0000

VARIABLE        VALUE
      X1         1.000000
      X2         1.000000
      X3         1.000000
      X4         0.000000
```

4. Applications of IP – (3) Fixed cost problem

- **Production cost consists of**

- Variable cost (변동비): occurs in proportional to the production volume
- Fixed cost (고정비): occurs regardless of production volume



- **Cost function, $C(X)$**

- Production volume = X (unit)
- Variable cost = a (dollar/unit)
- Fixed cost = f (dollar)

$$C(X) = \begin{cases} f + aX, & X > 0 \text{ 일} \\ 0, & X = 0 \text{ 일} \end{cases}$$

- **Integer programming modeling**

- Exploiting the binary decision variable, Y
- $$Y = \begin{cases} 1, & \text{if } X > 0, \leftarrow \text{Decide to produce} \\ 0, & \text{if } X = 0. \leftarrow \text{Decide not to produce} \end{cases}$$
- To guarantee above relation, we need to add the following constraint.

$$X \le MY$$

,where M is any big, big and big number which is larger than X.

- If $X > 0$, then $Y = 1$. And If $X = 0$, then $Y = 0$ or 1, but $Y = 0$ at the optimal point

4. Applications of IP – (3) Fixed cost problem

[EX 6-4]

K industries Inc. produce their products by using three types of machines. To operate each machine, the constant fixed cost occurs. The fixed cost, the variable cost and the production capacity are given in the below table. K industries Inc. want to produce at least 20,000 units of their product with a minimal production costs.

Machine	Fixed cost (dollars)	Variable cost (dollars)	Production Capa. (uints)
1	300	2	8,000
2	100	10	13,000
3	200	5	14,000

- **Define decision variables**

$$Y_j = \begin{cases} 1, & \text{if machie j will be operated,} \\ 0, & \text{if machine j won't be operated,} \end{cases} \quad j = 1, 2, 3$$

$$X_j = \text{production volume at machine j.}$$

4. Applications of IP – (3) Fixed cost problem

- **Define objective function**
  - We want to minimize the total production costs. 총생산비 = 고정비 + 변동비

$$\min\ 300Y_1 + 100Y_2 + 200Y_3 + 2X_1 + 10X_2 + 5X_3$$

- **Define constraints**
  - Production volume: $X_1 + X_2 + X_3 \geq 20{,}000$
  - Capa. of Machine 1: $X_1 \leq 8{,}000Y_1$
  - Capa. of Machine 2: $X_2 \leq 13{,}000Y_2$
  - Capa. of Machine 3: $X_3 \leq 14{,}000Y_3$

```
<untitled>
min  300Y1 + 100Y2 + 200Y3 + 2X1 + 10X2 + 5X3
st
X1 + X2 + X3 >= 20000
X1 - 8000Y1 <= 0
X2 - 13000Y2 <= 0
X3 - 14000Y3 <= 0
end
int Y1
int Y2
int Y3
gin X1
gin X2
gin X3
```

```
 OBJECTIVE FUNCTION VALUE

 1)       76500.00

VARIABLE        VALUE
      Y1         1.000000
      Y2         0.000000
      Y3         1.000000
      X1      8000.000000
      X2         0.000000
      X3     12000.000000
```

4. Applications of IP – (4) Set cover problem (집합덮음문제)

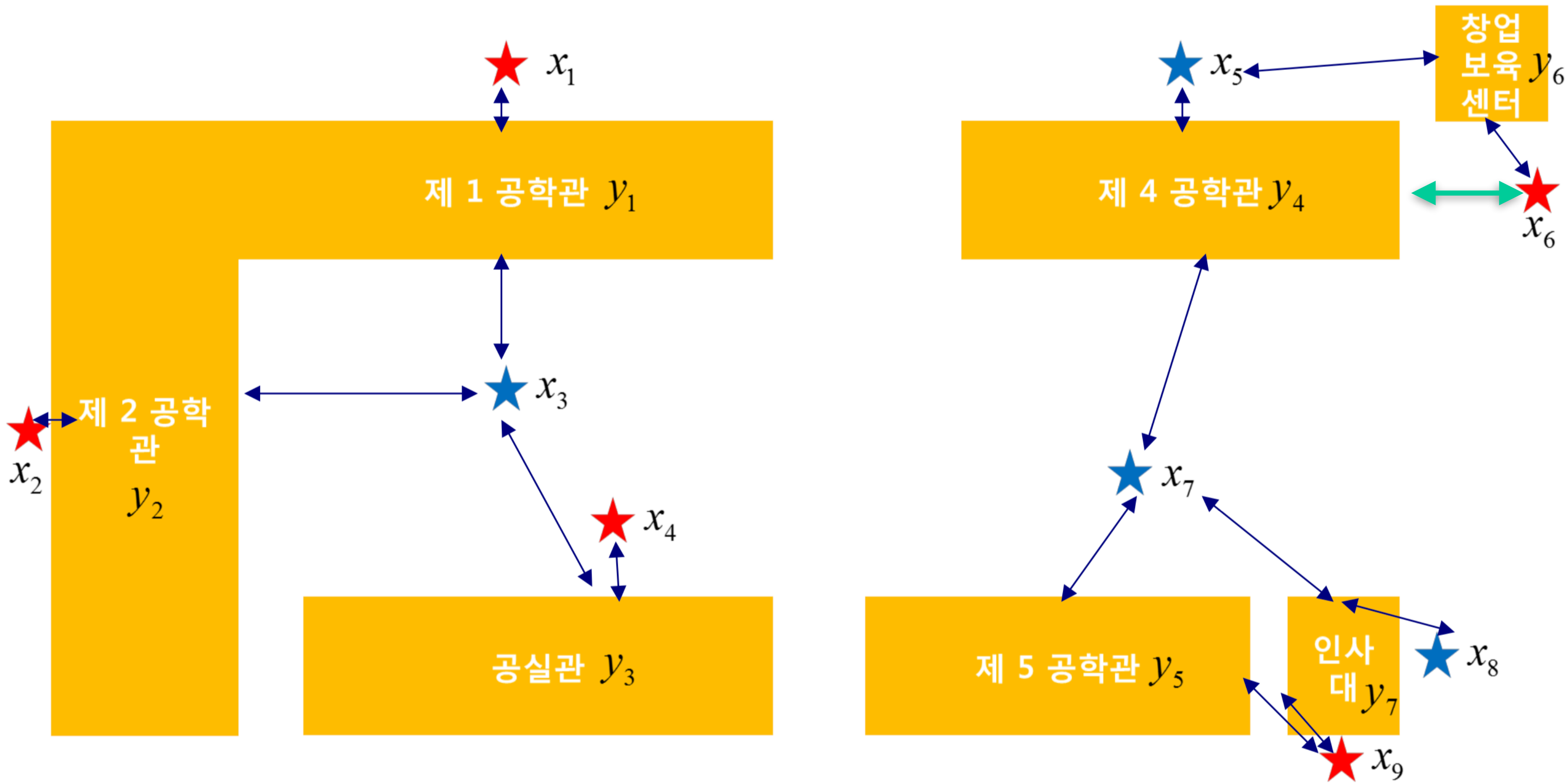
- **강원대학교 내 흡연장소 설치 최소화**
  - 현재의 흡연장소 설치 상태가 최적인지 판단

$$x_i = \begin{cases} 1, & \text{흡연장소 } i\text{를 설치함} \\ 0, & \text{흡연장소 } i\text{를 설치하지 않음} \end{cases} \qquad i = 1,\cdots,9$$

x1 + x3 >= 1 (1공학관)

x2 + x3 >= 1 (2공학관)

x5 + x6 + x7 >= 1 (4공학관)



4. Applications of IP – (4) Set cover problem

- **강원대학교 내 흡연장소 설치 최소화**

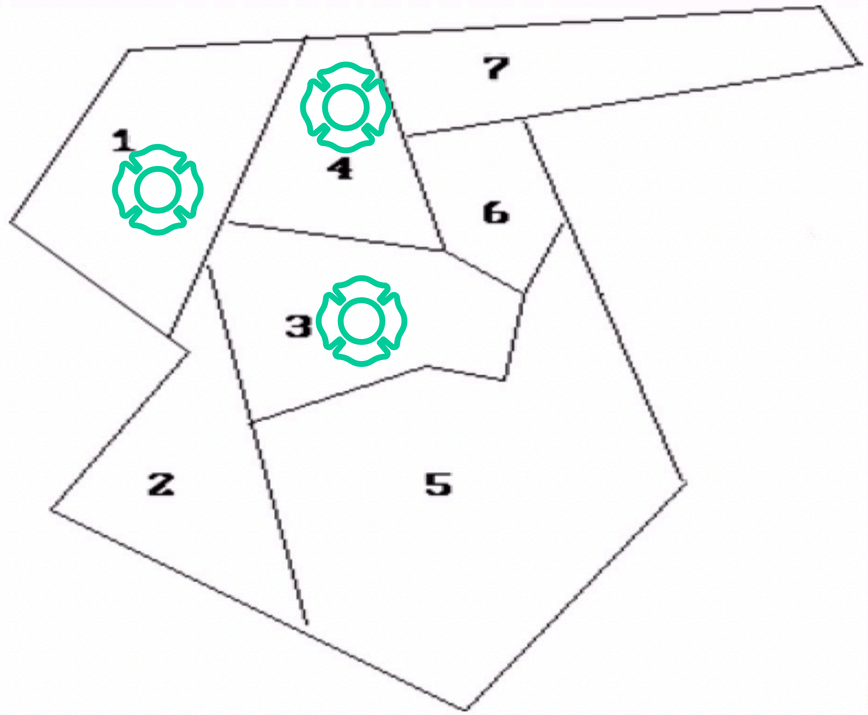
  - 현재의 흡연장소 설치 상태가 최적인지 판단

```
min x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 (흡연장소의 설치갯수)
st
x1 + x3 >= 1 (1공학관)
x2 + x3 >= 1 (2공학관)
x3 + x4 >= 1 (공실관)
x5 + x6 + x7 >= 1 (4공학관)
x7 + x9 >= 1 (5공학관)
x5 + x6 >= 1 (창업보육센터)
x7 + x8 + x9 >= 1 (인사대)
end
int x1
int x2
int x3
int x4
int x5
int x6
int x7
int x8
int x9
```

4. Applications of IP – (5) Set cover problem

- 소방서 설치 문제

6. (27 marks) The administrative map of a city is as follows. The city can be divided into seven regions as shown in the figure below, and new fire stations will be built in these regions. If a fire station is installed in region i, fire trucks are accessible to region i and its adjacent regions. The fire station service should cover the whole city. The mayor of this city wants to install fire stations with the smallest number.



4. Applications of IP – (5) Set cover problem

- **소방서 설치 최소화**

min x1 + x2 + x3 + x4 + x5 + x6 + x7 (소방서의 설치갯수)
st
x1 + x2 + x3 + x4 >= 1 (1번 지역 화재 발생 시)
? (2번 지역 화재 발생 시)
? (3번 지역 화재 발생 시)
…
? (7번 지역 화재 발생 시)
x1 + x4 = 1
end
int x1
int x2
int x3
int x4
int x5
int x6
int x7

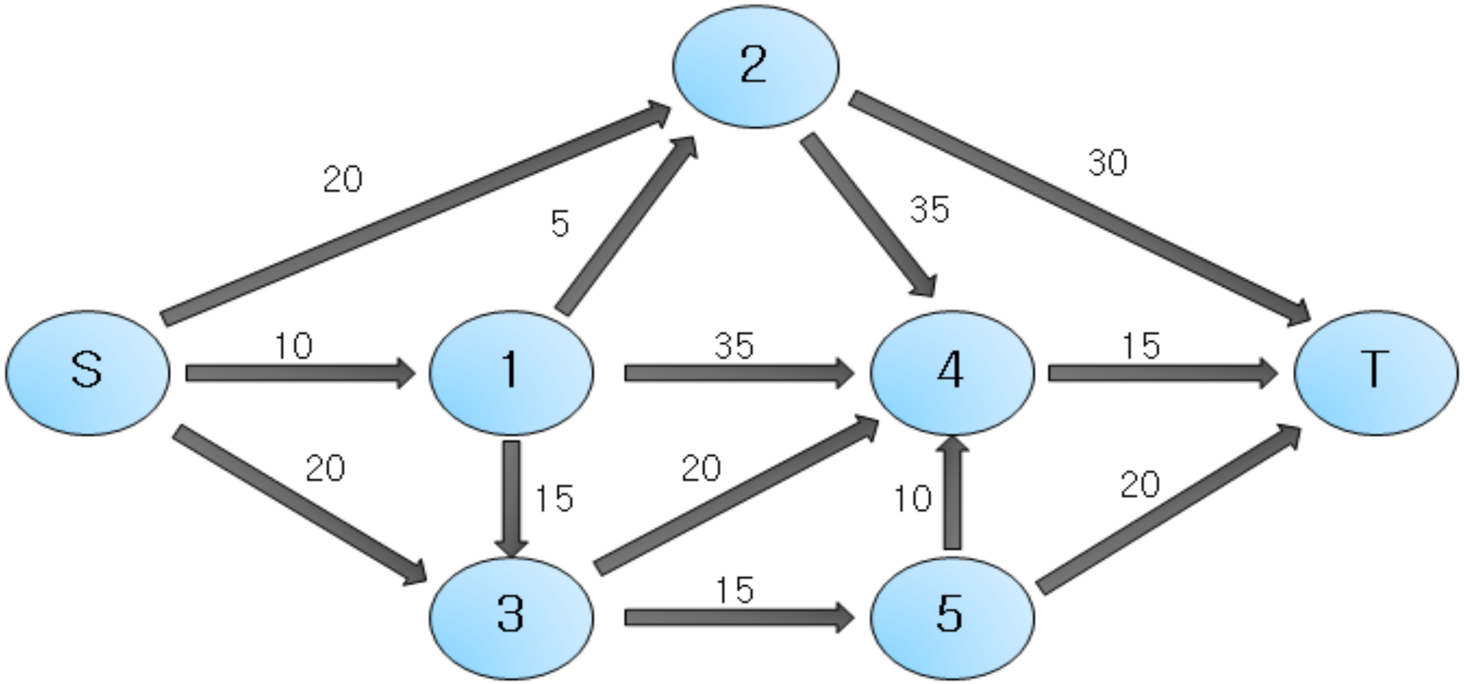
1. 지역 1과 4 중에 한개의 소방서는 무조건 설치되어야 한다. 이를 하나의 식으로 표현하라.

4. Applications of IP – (6) Shortest path

- **Shortest path problem (최단경로문제)**
  - To find the shortest or the minimal cost path between two nodes in a given network.
  - Traffic (cars, ships, air-planes) routing, Circuit board design, and navigation system

[EX 7-5] Shortest path problem

Find the shortest path from node S to node T in the below network. Distance information is given on each arc.



- **Decision variables**

X_{ij} = 1 if arc (i, j) is included in the shortest path, otherwise, 0

4. Applications of IP – (6) Shortest path

- **Objective function**

  - To minimize the total moving distance (time) from node S to node T.

$$\min \ 10X_{S1} + 20X_{S2} + 20X_{S3} + 5X_{12} + 15X_{13} + 35X_{14}$$
$$+ 35X_{24} + 30X_{2T} + 20X_{34} + 15X_{35} + 15X_{4T} + 10X_{54} + 20X_{5T}$$

- **Constraints**

  - Only one arc should be selected from node S.

$$X_{S1} + X_{S2} + X_{S3} = 1$$

  - The number of arcs entering node T should be one.

$$X_{2T} + X_{4T} + X_{5T} = 1$$

  - At each node excepting node S and node T, the number of entering arcs should be equal to the number of exiting arcs.

$$X_{S1} = X_{12} + X_{13} + X_{14} \quad \text{(node 1)}$$
$$X_{S2} + X_{12} = X_{24} + X_{2T} \quad \text{(node 2)}$$
$$X_{S3} + X_{13} = X_{34} + X_{35} \quad \text{(node 3)}$$
$$X_{14} + X_{24} + X_{34} + X_{54} = X_{4T} \quad \text{(node 4)}$$
$$X_{35} = X_{54} + X_{5T} \quad \text{(node 5)}$$
$$X_{ij} \in \{0, 1\}, \ \forall i, \ \forall j$$

4. Applications of IP – (6) Shortest path

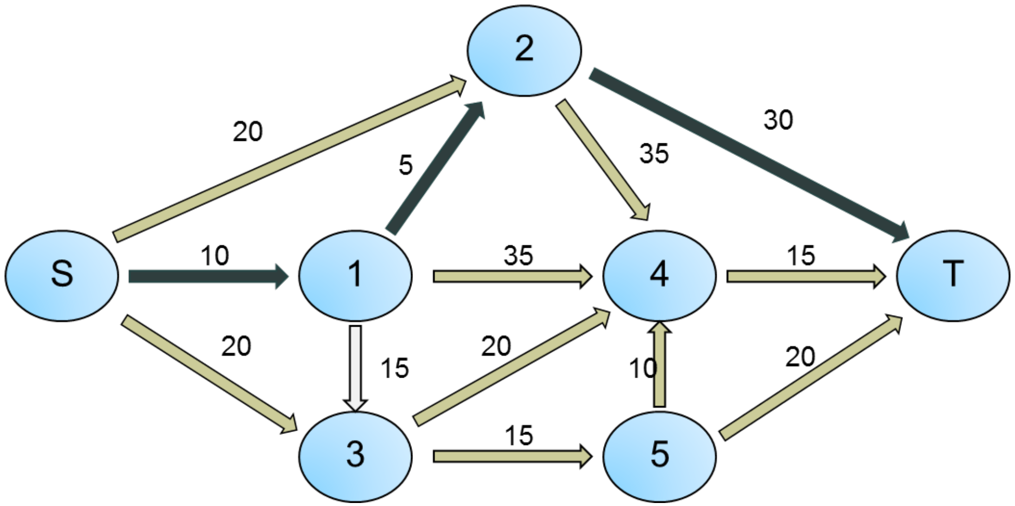
- **LINDO result**

<untitled>

```
min  10XS1 + 20XS2 + 20XS3
   + 5X12 + 15X13 + 35X14
   + 35X24 + 30X2T
   + 20X34 + 15X35
   + 15X4T
   + 10X54 + 20X5T
st
XS1 + XS2 + XS3 = 1
X2T + X4T + X5T = 1
XS1 - X12 - X13 - X14 = 0
XS2 + X12 - X24 - X2T = 0
XS3 + X13 - X34 - X35 = 0
X14 + X24 + X34 + X54 - X4T = 0
X35 - X54 - X5T = 0
end
```

```
int XS1
int XS2
int XS3
int X12
int X13
int X14
int X24
int X2T
int X34
int X35
int X4T
int X54
int X5T
```

```
        OBJECTIVE FUNCTION VALUE

        1)       45.00000

  VARIABLE        VALUE
       XS1         1.000000
       XS2         0.000000
       XS3         0.000000
       X12         1.000000
       X13         0.000000
       X14         0.000000
       X24         0.000000
       X2T         1.000000
       X34         0.000000
       X35         0.000000
       X4T         0.000000
       X54         0.000000
       X5T         0.000000
```



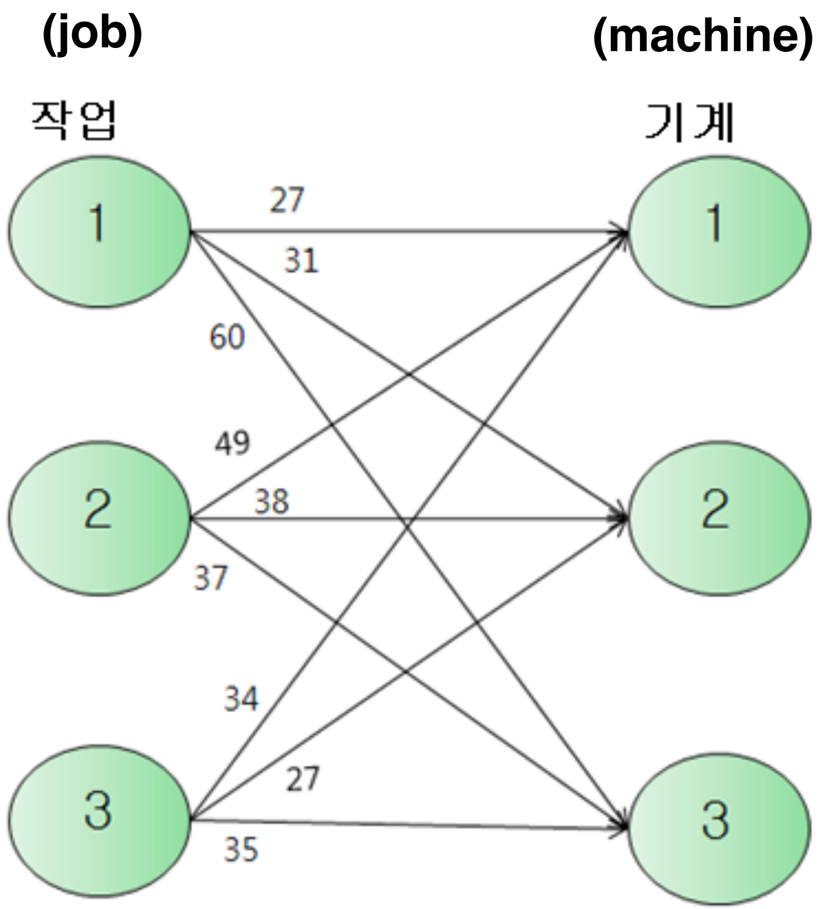
4. Applications of IP – (7) Assignment

- **Assignment model (할당모형)**

[EX 7-4]

Sewon industries Ltd. wants to process three jobs with three machines. A job should be processed at one of the three machines. Information on processing costs is given in the below table. What is the optimal assignment for minimizing the total costs?

Job \ Machine	1	2	3
1	27	31	60
2	49	38	37
3	34	27	35



- **Decision variables**

X_{ij} = 1 if job i is assigned to machine j and, otherwise, 0

4. Applications of IP – (7) Assignment

- **Formulation**
  - Objective: We want minimize the total processing costs

$$\min \; 27X_{11} + 31X_{12} + 60X_{13} + 49X_{21} + 38X_{22} + 37X_{23} + 34X_{31} + 27X_{32} + 35X_{33}$$

  - Constraints: One-to-one correspondence between jobs and machines

$X_{11} + X_{12} + X_{13} = 1$ **(job 1)**

$X_{21} + X_{22} + X_{23} = 1$ **(job 2)**

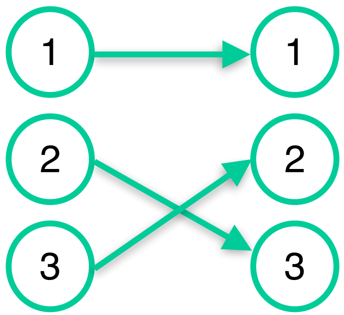
$X_{31} + X_{32} + X_{33} = 1$ **(job 3)**

$X_{11} + X_{21} + X_{31} >= 1$ **(machine 1)**

$X_{12} + X_{22} + X_{32} >= 1$ **(machine 2)**

$X_{13} + X_{23} + X_{33} >= 1$ **(machine 3)**

$X_{ij} \in \{0, 1\}, \forall i, \forall j$




```
OBJECTIVE FUNCTION VALUE

1)        91.00000

VARIABLE         VALUE
     X11          1.000000
     X12          0.000000
     X13          0.000000
     X21          0.000000
     X22          0.000000
     X23          1.000000
     X31          0.000000
     X32          1.000000
     X33          0.000000
```

Homework 1

- **Find the shortest path from 동대입구 to 종로5가.**
  - Establish mathematical model and solve by using any SW.
  - Neglect the transferring time.

Homework 2

농구선수 선발문제

강원대학의 농구 코치는 <대학농구연맹전> 결승전에서 뛸 스타팅 라인업 5명을 고르려고 한다. 지금까지의 각 선수의 전적은 다음 표와 같다.

선수번호	포지션	신장(cm)	게임당 평균 리바운드	게임당 평균 득점	게임당 평균 어시스트
1	가드	182	1	4	5
2	가드	175	3	16	2
3	가드	185	3	4	1
4	포워드	190	4	10	2
5	포워드	196	2	12	3
6	포워드	201	6	8	1
7	센터	203	3	6	5
8	센터	208	9	22	1

코치는 다음과 같은 4개의 목표를 가지고 있다.

1. 출전선수들의 평균신장은 193cm 이상 이어야 한다.
2. 각 선수의 게임당 평균 리바운드의 합은 23 이상이어야 한다.
3. 각 선수의 게임당 평균 득점의 합은 58 이상이어야 한다.
4. 각 선수의 게임당 평균 어시스트의 합은 13 이상이어야 한다.

Homework 2

코치는 다음의 제약조건들을 가지고 있다.

a. 적어도 가드 한 명은 뛰어야 한다.

b. 센터는 한 명만 뛴다.

c. 1번이나 4번이 뛰면 6번은 뛸 수 없다.

(문제 1) 되도록 많은 목표를 만족시키는 스타팅 라인업을 선발하기 위한 수리계획 모델을 작성하시오.

(문제 2) 문제 1의 해답을 제시하시오.

Notice) 농구는 5명이 하는 운동임.