Motivated Reasoning and Information Aggregation*

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Abstract

If agents engage in motivated reasoning, how does that affect the aggregation of information across society? We study the effects of motivated reasoning in two canonical settings—that of the Condorcet jury theorem (CJT), and the sequential social learning model (SLM). We define a notion of motivated reasoning that applies to these and a broader class of other settings. We show in the case of the CJT that information aggregates even when agents are motivated reasoners, and the rate of aggregation can be faster than in the benchmark case where all agents are fully Bayesian. In the case of the SLM, we find that motivated reasoning can improve information aggregation; but if agents place too little weight on truth-seeking, this can lead to worse aggregation relative to the benchmark case of full Bayesian rationality.

Key words: Condorcet voting, sequential social learning

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1 Introduction

We study how the Condorcet jury theorem and the sequential social learning model are affected when agents in these models are motivated reasoners. Our theory of motivated reasoning is straightforward, and builds directly upon the foundational principals of probability theory, and the perspective in Acharya, Blackwell and Sen (2018) that motivated reasoning is the outcome of a dissonance reduction problem.

Given sample and event spaces $(\Omega, \mathcal{F})$, let an agent's *objective belief* be a probability measure P. This could either be a prior belief or an interim posterior belief that is formed by updating some signal that we associate with the agent having learned some event in $\mathcal{F}$. The agent's *motivation* is a conditional probability $P_Z := P(\cdot|Z)$ for some event $Z \in \mathcal{F}$ such that $P(Z) > 0$. We refer to Z as the *motivating event*, with the interpretation being that the agent would like to believe that event Z is true. Finally, the agent's *motivated belief* $\hat{P}$ is a probability law that solves

$$\max_{\hat{P} \ll P} (1-w)\underbrace{[-||P-\hat{P}||^2_{L^2(P)}]}_{\text{truth-seeking}} + w\underbrace{[-||P_Z-\hat{P}||^2_{L^2(P)}]}_{\text{motivation}} \tag{$*$}$$

for some fixed $w \in (0,1)$, where $L^2(P)$ is the Hilbert space of square-integrable functions with respect to P and $||\cdot||_{L^2(P)}$ is the standard L^2 norm in this space. Thus, an agent who is a motivated reasoner chooses their motivated belief to minimize the weighted average of dissonances between their belief and the truth and their belief and what they would like to believe. w is the weight the agent puts on motivation, and $1-w$ is the weight he puts on truth-seeking. In this setting, we have the following result.

Theorem 1. *The unique solution to* $(*)$ *is the probability measure*

$$\hat{P} = wP_Z + (1-w)P.$$

This follows from the standard projection theorem, and allows us to interpret w as a parameter capturing the strength of the agent's motivation.[1] If $w = 0$ the agent is fully Bayesian while if $w = 1$ he is fully motivated. For $w \in (0,1)$ the agent is partially motivated, partially truth-seeking. In applications, the modeler specifies the motivating event Z (and

[1]As P_Z is a conditional probability of P, we have that $P_Z \ll P$ and the Radon-Nikodym derivative dP_Z/dP exists and belongs to $L^2(P)$. Therefore, the solution to problem $(*)$ is the unique probability law $\hat{P}$ such that $d\hat{P}/dP = wdP_Z/dP + (1-w)$. This is $\hat{P} = wP_Z + (1-w)P$, as claimed.

hence the motivation P_Z) as part of the assumptions of the model, just as they would specify the agent's prior beliefs, information, and utility over outcomes.

In our two applications, all agents receive private information about a binary state of the world, and we specify that agents are motivated to believe that the state (imperfectly) indicated by their private signal is the true state. We show that motivation affects welfare defined as the expected sum of agents' utilities. In the Condorcet jury model, where agents simultaneously vote under a majority rule, the group increasingly selects one option as the population grows, and whether that option is correct determines overall welfare. Motivation accelerates the convergence of welfare to its limit by shaping the agents' equilibrium strategies, but without necessarily affecting the rate of convergence. Therefore, increasing the level of motivation has either always positive or always negative marginal effets on the welfare.

In the sequential learning model, where agents observe the choices of their predecessors, a cascade occurs when the absolute difference in the number of signals favoring one option over the other becomes sufficiently large. Motivation affects equilibrium strategies by encouraging agents to follow their own private signals, thereby delaying the onset of the cascade. We formally analyze the exploration-exploitation trade-off and prove that while the effect of motivation is ambiguous, the optimal level of motivation increases with population size. Motivation here directly affects the limiting welfare, but as before, without affecting the rate of convergence.

Our model of motivated reasoning is related to existing models that rely on belief distortions based on KL divergence. An important limitation of KL divergence in many applications is that if two measures P_A and P are mutually singular (for example, P_A is a point mass on an event that the agent would like to believe while P is continuous) then the KL-divergence between them will be infinite, and so is not straightforward how we would write the problem of finding a motivated belief that minimizes the weighted sum of KL-divergences with the objective belief and with the motivation. Our approach of specifying the motivated belief to minimize the weighted sum of L^2 distances does not have this limitation. In addition, our approach provides a simple formula and transparent interpretation of how the gap between the objective belief and the motivated belief depends on a scale parameter w that measures the strength of motivation.

We now study the consequences of motivated reasoning in the applications.

2 The Condorcet Jury Theorem

2.1 The Model

There are two states of the world $\theta \in \{A, B\}$, equally likely. Each agent $n \in \{1, ..., 2N+1\}$ receives a conditionally independent signal $\omega_n \in \{a, b\}$, such that

$$\forall n, \qquad \Pr\left[\omega_n = a \mid \theta = A\right] = q_A \in (\tfrac{1}{2}, 1), \qquad \Pr\left[\omega_n = b \mid \theta = B\right] = q_B \in (\tfrac{1}{2}, 1)$$

Assume without loss of generality that $q_A \geq q_B$, and let Q denote the objective common prior over the state and signal spaces $\{A, B\} \times \{a, b\}^{2N+1}$. After receiving their private signals, all agents update their beliefs and must simultaneously vote for one of two alternatives, also called A and B, and votes are aggregated by majority rule. Let ρ denote the policy that is elected. Each agent n's payoff is

$$u_n = u(\rho, \theta) = \begin{cases} 1 & \text{if } \rho = \theta \\ 0 & \text{if } \rho \neq \theta \end{cases}$$

Thus, policy A gives a payoff of 1 to all agents when the state is A and 0 when the state is B, while policy B gives all a payoff of 1 when the state is B and 0 when it is A.

Let $\hat{P}_s$ denote any agent n's updated posterior belief after seeing signal $\omega_n = s \in \{a, b\}$, which we assume is possibly motivated. In particular, if an agent receives signal a, let his motivating event be the event that A is the true state, while if he receives signal b, let it be the event that B is the true state. Thus the objective updated beliefs for agent n following signals a and b respectively are $P_a = Q(\cdot|\omega_n = a)$ and $P_b = Q(\cdot|\omega_n = b)$, the motivations are $P_{a,A} = Q(\cdot|\omega_n = a, \theta = A)$ and $P_{b,B} = Q(\cdot|\omega_n = b, \theta = B)$, and motivated beliefs are

$$\hat{P}_a = wP_{a,A} + (1-w)P_a$$
$$\hat{P}_b = wP_{b,B} + (1-w)P_b$$

Let N_A denote the number of votes from agents other than n in favor of A. Given $\hat{P}_s$, agent n who has private signal s weakly prefers to vote for alternative A if

$$\hat{P}_s\left[N_A = N,\ \theta = A\right] \geq \hat{P}_s\left[N_A = N,\ \theta = B\right]$$

and weakly prefers to vote for B if the reverse holds. This follows from the standard calculation that no matter how agents update their beliefs—motivated reasoning or not—they care only about how their vote affects outcomes in the event that it is pivotal.[2]

2.2 Equilibrium Analysis

We examine type-symmetric responsive Nash equilibria—hereafter "equilibria"—in which all agents who receive the same signal use the same strategy, and all agents face a positive probability that their vote is pivotal. Given parameter N of the model, which governs the total number of agents, we denote a type-symmetric strategy profile with the pair (σ_N^a, σ_N^b) which are, respectively, the probability that a voter who receives the a signal votes for A and the probability that a voter who receives the b signal votes for B. We define "sincere voting" as the case in which $(\sigma_N^a, \sigma_N^b) = (1, 1)$.

Theorem 2. *Let*

$$N^*(w) = \left\lceil \log \psi(w) / \log \frac{q_A}{q_B} \frac{1-q_A}{1-q_B} \right\rceil \quad \textit{where} \quad \psi(w) = \frac{(1-w)(1-q_B)}{w + (1-w)q_A}$$

For $N < N^*(w)$ the unique equilibrium is sincere voting, while for $N \geq N^*(w)$ it is

$$(\sigma_N^a, \sigma_N^b) = \left(\frac{q_A - \psi(w)^{1/N}(1-q_B)}{(q_A)^2 - \psi(w)^{1/N}(1-q_B)^2}, 1 \right)$$

Proof. For agents with motivated beliefs $\hat{P}_a$ and $\hat{P}_b$, the motivated probabilities of the state being $\theta = \tilde{\theta} \in \{A, B\}$ and the pivotal event occurring are

$$\hat{P}_a(N_A = N,\ \theta = \tilde{\theta}) = \varphi_{\tilde{\theta}} \left[(w + (1-w)q_A)\mathbf{1}_{\{\tilde{\theta}=A\}} + (1-w)(1-q_B)\mathbf{1}_{\{\tilde{\theta}=B\}} \right]$$
$$\hat{P}_b(N_A = N,\ \theta = \tilde{\theta}) = \varphi_{\tilde{\theta}} \left[(w + (1-w)q_B)\mathbf{1}_{\{\tilde{\theta}=B\}} + (1-w)(1-q_A)\mathbf{1}_{\{\tilde{\theta}=A\}} \right]$$

[2] To see this, the agent weakly prefers to vote for A if

$$\hat{P}_s[N_A > N,\ \theta = A] + \hat{P}_s[N_A = N,\ \theta = A] + \hat{P}_s[N_A < N,\ \theta = B]$$
$$\geq \hat{P}_s[N_A > N,\ \theta = A] + \hat{P}_s[N_A = N,\ \theta = B] + \hat{P}_s[N_A < N,\ \theta = B]$$

The left hand side of the inequality is the agent's expected payoff from voting for A while the right side the expected payoff from voting for B. Then cancel common terms on both sides.

where

$$\varphi_A = \binom{2N}{N}[q_A\sigma_N^a + (1-q_A)(1-\sigma_N^b)]^N[q_A(1-\sigma_N^a) + (1-q_A)\sigma_N^b]^N$$
$$\varphi_B = \binom{2N}{N}[q_B\sigma_N^b + (1-q_B)(1-\sigma_N^a)]^N[q_B(1-\sigma_N^b) + (1-q_B)\sigma_N^a]^N$$

are the pivotal probabilities conditional on the states being A and B respectively. Next, note that $(\sigma_N^a, \sigma_N^b) \in \{(0,1),(1,0)\}$ cannot be responsive while $(\sigma_N^a, \sigma_N^b) = (0,0)$ is not an equilibrium. In pure strategies, that leaves sincere voting. In mixed strategies, if $\sigma_N^a \in (0,1)$ then we must have $\sigma_N^b = 1$, as if the a-type is indifferent then the b-type will strictly prefer to vote for B. Similarly, if $\sigma_N^b \in (0,1)$ then $\sigma_N^a = 1$. If $\sigma_N^a \in (0,1)$ then $\hat{P}_a(N_A = N, A) = \hat{P}_a(N_A = N, B)$ which holds iff

$$[q_A\sigma_N^a]^N[q_A(1-\sigma_N^a) + 1 - q_a]^N = [q_B + (1-q_B)(1-\sigma_N^a)]^N[(1-q_B)\sigma_N^a]^N\psi(w)$$

which follows from substituting the expressions for $\hat{P}_a(N_A = N, A)$ and $\hat{P}_a(N_A = N, B)$ that follow from above. This implies that

$$\sigma_N^a = \frac{q_A - \psi(w)^{1/N}(1-q_B)}{(q_A)^2 - \psi(w)^{1/N}(1-q_B)^2} \tag{1}$$

which we can verify directly is strictly less than 1 iff $N \geq N^*(w)$ defined above. Given our assumption that $q_A \geq q_B$, it follows that we cannot have an equilibrium with $\sigma_N^b \in (0,1)$ and $\sigma_N^a = 1$, and that when $N < N^*(w)$ then sincere voting is the unique equilibrium. [to do here: Fill in details and improve proof clarity.] ■

A few observations about the equilibrium. As $w \to 1$ sincere voting is the unique equilibrium for all N, as in this case $N^*(w) \to 1$. In fact, $N^*(w)$ is increasing in w, and so is the mixing probability σ_N^a in (1). To see this, note that

$$\frac{\partial \sigma_N^a}{\partial w} = \frac{d\sigma_N^a}{d\psi^{1/N}}\frac{1}{N}\psi^{\frac{1}{N}-1}\frac{d\psi}{dw} = \frac{q_A(1-q_B)(1-q_A-q_B)}{[(q_A)^2 - \psi^{1/N}(1-q_B)^2]^2}\frac{1}{N}\psi^{\frac{1}{N}-1}\frac{d\psi}{dw} > 0$$

as the first term is negative and $d\psi/dw < 0$ as well. Therefore, motivated reasoning pushes equilibrium behavior towards sincere voting, and therefore the question of whether motivated reasoning improves the rate of information aggregation or not depends on a comparison of aggregation rates under sincere voting to aggregation rates under strategic voting in the full Bayesian rationality benchmark of $w = 0$.

2.3 Information Aggregation

A limit equilibrium $(\sigma_\infty^a, \sigma_\infty^b)$ is the limit of a sequence of type-symmetric responsive equilibria $\{(\sigma_N^a, \sigma_N^b)\}_{N=1}^\infty$ with each (σ_N^a, σ_N^b) being the equilibrium of a game with $2N+1$ voters. This section studies how the equilibrium decision is affected by the motivation and how fast it approaches its limit as the number of voters grows to infinity.

To measure welfare, we take the expected average sum of utilities of the agents conditioning on σ being the elected alternative and θ being the better restaurant:

$$W_N(\theta) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} u(\rho, \theta)\right].$$

Denote Θ for the exact convergence rate. Formally, $\Theta(X) = f(N)$ means that $a < \lim_{N\to\infty} X/f(N) < b$ for some finite constants a and b such that $ab > 0$.

Theorem 3. *Suppose agents adopt the pivotal equilibrium in Theorem 2.*
(a) Motivation improves $W_N(A)$ but worsens $W_N(B)$.
(b) Fix motivation w. $1 - W_N(A) = W_N(B) = \Theta\big(\lambda^{N/2}/\sqrt{N}\big)$ regardless of w for $\lambda = \exp(-(2\bar{q}_\infty - 1)^2/2\bar{q}_\infty(1-\bar{q}_\infty))$.

Proof. (a) $W_N(A)$ is the probability of A being elected. Agents independently cast their votes at the equilibrium. For each agent, the probability of voting for A is $\bar{q}_N \equiv q_A\sigma_N^a + (1-q_A)(1-\sigma_N^b)$. Therefore,

$$\mathrm{P}\big[\text{elect } A\big] = \mathrm{P}\big[\mathrm{Binom}(2N+1,\, \bar{q}_N) > N\big].$$

$\bar{q}_N$ increases in w. The conclusion for $W_N(B)$ follows from $W_N(A) + W_N(B) = 1$.

(b) By the Central Limit Theorem, the above probability is equivalent to

$$\mathrm{P}\big[\mathcal{N}\big((2N+1)\bar{q}_N,\, (2N+1)\bar{q}_N(1-\bar{q}_N)\big) > N + o_p(\sqrt{N})\big].$$

Since $\sigma_N^a \to \sigma_\infty^a$ and $\sigma_N^b \to \sigma_\infty^b$ as $N \to \infty$, we have $\bar{q}_N \to \bar{q}_\infty$ and $\bar{q}_N = \bar{q}_\infty + o(1)$. Divide both sides of the inequality by N and obtain

$$\begin{aligned}\mathrm{P}\big[\text{elect } A\big] &= \mathrm{P}\big[\mathcal{N}\big(2\bar{q}_\infty,\, 2\bar{q}_\infty(1-\bar{q}_\infty)/N\big) + o(1) > 1 + o_p\big(1/\sqrt{N}\big)\big] \\ &= \mathbb{E}\left[\Phi\left(\frac{\big(2\bar{q}_\infty - 1\big)\sqrt{N}}{\sqrt{2\bar{q}_\infty(1-\bar{q}_\infty)}} + R_N\right)\right]\end{aligned}$$

for the standard normal CDF Φ and a residual random variable $R_N = o_p\big(\sqrt{N}\big)$. Φ is bounded within [0, 1], so R_N is ignorable by the dominance convergence theorem. The Mills ratio suggests

$$\frac{x}{x^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} < \Phi(-x) < \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

for $x > 0$, hence $\Phi(-N) = \Theta\big(e^{-N^2/2}/N\big)$. ■

Under full rationality, the probability that the right choice is elected converges to one if the signal favoring that choice is more likely than that for the alternative. Motivation accelerates the convergence by increasing the ex-ante probability of voting for the correct choice as motivation rises. However, interestingly, motivation affects only the convergence rate of the individual strategy, not the overall convergence rate of the equilibrium probability.

3 Sequential Social Learning

3.1 The Model

We study the effects of motivated reasoning on information cascades in the model of Bhikchandani, Hirschleifer, and Welch (1992), hereafter BHW.

Suppose there are two possible restaurants, A and B, and an infinite sequence of agents $n \in \{1, 2, ..., \infty\}$. Each agent has prior belief that the two restaurants are equally likely to be the better restaurant. Each agent n receives a private signal $\omega_n \in \{a, b\}$ about which restaurant is better. Denote by $\theta \in \{A, B\}$ the better restaurant. Then,

$$\forall n, \qquad \Pr\left[\omega_n = a \mid \theta = A\right] = \Pr\left[\omega_n = b \mid \theta = B\right] = p \in (\tfrac{1}{2}, 1)$$

Thus, signal a (imperfectly) indicates that A is better while b (also imperfectly) indicates that B is better. All agents' signals are independently delivered. Agents sequentially decide which restaurant to go to, and each would like to go to the better restaurant. Denote by $\rho_n \in \{A, B\}$ the restaurant choice of agent n. Then, each agent n's payoff is

$$u_n = u(\rho_n, \theta) = \begin{cases} 1 & \text{if } \rho_n = \theta \\ 0 & \text{if } \rho_n \neq \theta \end{cases}$$

Thus, each receives a payoff of 1 if they go to the better one and 0 if they go to the worse one. Each agent observes the restaurant choice of every agent prior in the sequence than them before making their decision.

BHW assume that when indifferent, an agent will go to each of the two restaurants with equal probability. We, however, adopt a different "tie-breaking rule:" if an agent is indifferent between the two restaurants given his updated beliefs, he follows his personal signal and goes to the restaurant indicated by that signal.

Say that we are *in a cascade at n agents* if this agent goes to the same restaurant given his information about the choices of his predecessors, regardless of his own signal. Say that we are *in a cascade at n agents to restaurant A (respectively, B)* if that restaurant happens to be A (respectively, B). Given our tie-breaking rule, any agent can perfectly infer all of the signals of his predecessors from their restaurant choices so long as we are not already in a cascade. For an agent who can infer from his private signal and the behavior of his predecessors that there were n_A total signals in favor of A and $n_B = n_A - k$ in favor of B, including their own signal, the objective updated belief that A is the better restaurant is

$$\begin{aligned} P\big[\theta = A\big] &:= Q\big[\theta = A \mid \#\{\omega_n = a\} = n_A,\ \#\{\omega_n = b\} = n_A - k\big] \\ &= \frac{p^{n_A}(1-p)^{n_B}}{p^{n_A}(1-p)^{n_B} + (1-p)^{n_A}p^{n_B}} = \frac{1}{1 + \left(\frac{1-p}{p}\right)^k} \end{aligned}$$

However, we assume that agents are motivated reasoners. For an agent who receives private signal a, let his motivating event be the event that A is the better restaurant, while if he receives private signal b, let it be the event that B is the better restaurant. Therefore, for an agent n who receives private signal $\omega_n = a$, the motivation is $P_a = Q\big[\,\cdot \mid \omega_n = a,\ \theta = A\big]$. If he were to receive private signal $\omega_n = b$, it would be $P_b = Q\big[\,\cdot \mid \omega_n = b,\ \theta = B\big]$. Therefore, for any agent who receives private signal $s \in \{a, b\}$ and would objectively infer a total of $n_A - 1$ signals that are a and $n_A - k$ signals that are b (based on his own signal and the behavior of his predecessors), the motivated belief that A is the better restaurant is

$$\begin{aligned} \hat{P}_s\big[\theta = A\big] &= wP_s\big[\theta = A\big] + (1-w)P\big[\theta = A\big] \\ &= w\mathbf{1}_{\{s=a\}} + (1-w)\frac{1}{1 + \left(\frac{1-p}{p}\right)^{k - \mathbf{1}_{\{s=a\}} + \mathbf{1}_{\{s=b\}}}}. \end{aligned}$$

3.2 Analysis

Suppose the agent has signal a and is thus motivated to think that A is better. Then for all k such that

$$\hat{P}_a\big[\theta = A\big] < \frac{1}{2}$$

the agent will go against his private signal and choose restaurant B while for all k such that this inequality doesn't hold, then the agent will choose A. Inserting $\hat{P}_a\big[\theta = A\big]$ from above, rearranging, and solving, we find that the critical threshold on k is

$$k \le -k^*(w) := -\left\lceil \frac{\log(1-2w)}{\log(1-p) - \log p} \right\rceil - 2$$

provided $w < 1/2$. When $w = 0$, for example, and agents are fully rational (or when w is small and they are close enough to fully rational), $k^*(w)$ must equal 2. As a result note that $k^*(w) > 0$ when $w < 1/2$. For $w \ge 1/2$, the agent will always follow his private signal and go to restaurant A for all values of k. Therefore, we can set $k^*(w) = \infty$, indicating when the strength of motivation is sufficiently strong, an agent who receives a private a signal will never go to restaurant B for any value of k.

The same calculation for an agent who receives signal b and is thus motivated to think that B is better shows that if $k \geq k^*(w)$ then the agent will go to A while for all $k < k^*(w)$ he will follow his private signal and go to B. Therefore, there is a set of values of k,

$$\mathcal{K}(w) := \{-k^*(w), -k^*(w)+1, ..., k^*(w)-1, k^*(w)\}$$

such that if $k \in \mathcal{K}(w) \backslash \{-k^*(w), k^*(w)\}$ all agents follow their private signals but if k ever reaches $k^*(w)$ at some agent then a cascade to A will have started while if k ever reaches $-k^*(w)$ then a cascade to B will have started.

Theorem 4. *If the strength of motivated reasoning is $w \geq 1/2$ then no cascades ever occur, all agents follow their private signals and welfare is p. If it is $w < 1/2$ then a cascade eventually occurs almost surely, and agents cascade to the correct restaurant with probability*

$$\frac{p^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}}$$

Proof. When $w \geq 1/2$ then $k^*(w) = \infty$ while if $w < 1/2$ then $2 \geq k^*(w) < \infty$. If $k^*(w) = \infty$, then all agents follow their own signals.

Let $\{\omega_1, \omega_2, ..., \omega_n\}$ be the sequence of private signals received by agents up to agent n and let $A_n = \#\{\omega_i \mid \omega_i = a,\ i \leq n\}$ and $B_n = \#\{\omega_i \mid \omega_i = b,\ i \leq n\}$, with $A_0 = B_0 = 0$. So long as $|A_n - B_n| < k^*(w)$ we are *not* in a cascade, and each agent up to agent n will optimally go to the restaurant indicated by their private signal. However, if $|A_n - B_n| \geq k^*(w)$, we are in a cascade. In particular, if $A_n - B_n \geq k^*(w)$ we are in a cascade to A while if $A_n - B_n \leq -k^*(w)$ we are in a cascade to B.

Given this, define the random walk $\{X_n\}$ by $X_n = A_n - B_n$ with $X_n = 0$ and

$$X_{n+1} - X_n = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

Suppose we stop at the first (random) agent n^* at which $X_{n^*} = k^*(w)$ or $X_{n^*} = -k^*(w)$. Let $\mathcal{A} = \{X_{n^*} = k^*(w)\}$ and $\mathcal{B} = \{X_{n^*} = -k^*(w)\}$. $\mathcal{A}$ and $\mathcal{B}$ are clearly disjoint events. As X_n takes values in $\mathcal{K}(w)$ with $\pm k^*(w)$ absorbing, the following lemma establishes that they are also exhaustive, meaning that a cascade to one of the two restaurants will almost surely occur. The proof follows by considering the classical gambler's ruin difference-equation.

Lemma 1. $\Pr(\mathcal{A}) + \Pr(\mathcal{B}) = 1$.

Proof. For any $k \in \mathcal{K}(w)\backslash\{-k^*(w), k^*(w)\}$, let η_k be the probability that starting from $X_i = k$ we will eventually arrive at some $X_{n^*} \in \{-k^*(w), k^*(w)\}$ at some $n^* > k$. We want to show that $\eta_k = 1$ for all $k \in \mathcal{K}(w)\backslash\{-k^*(w), k^*(w)\}$. Since $\pm k^*(w)$ are absorbing set $\eta_{-k^*(w)} = \eta_{k^*(w)} = 1$. For all $k \in \mathcal{K}(w)\backslash\{-k^*(w), k^*(w)\}$, we have $\eta_k = p\eta_{k+1} + (1-p)\eta_{k-1}$, which rearranges to $p(\eta_{k+1} - \eta_k) = (1-p)(\eta_k - \eta_{k-1})$. Thus the increments $\delta_k = \eta_k - \eta_{k-1}$ satisfy $p\delta_{k+1} = (1-p)\delta_k$, whence $\delta_{k+1} = [(1-p)/p]\delta_k$. Then by induction, $\delta_k = [(1-p)/p]^{k+k^*(w)}\delta_{-k^*(w)}$ for all $k \in \mathcal{K}(w)\backslash\{-k^*(w), k^*(w)\}$. Therefore, $\eta_{k^*(w)} - \eta_{-k^*(w)} = \sum_{k=-k^*(w)+1}^{k^*(w)}(\eta_k - \eta_{k-1}) = \sum_{l=1}^{2k^*(w)}[(1-p)/p]^l\delta_{-k^*(w)}$. But since $\eta_{k^*(w)} - \eta_{-k^*(w)} = 1 - 1 = 0$, this implies that $\delta_{-k^*(w)} = 0$, thus $\delta_k = 0$ for all $k \in \mathcal{K}(w)\backslash\{-k^*(w), k^*(w)\}$. Therefore $\eta_k - \eta_{k-1} = 0$ meaning that η_k is constant in k throughout $\mathcal{K}$. This implies $\eta_k = 1$ for all $k \in \mathcal{K}(w)$. ■

Now returning to the proof of the theorem, consider the process

$$Z_n = \left(\frac{1-p}{p}\right)^{X_n}.$$

Then let $\mathcal{F}_n$ be the filtration of the process up to agent n. Then we have

$$\begin{aligned}
\mathbb{E}[Z_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_{n+1}} \mid \mathcal{F}_n\right] \\
&= \left(\frac{1-p}{p}\right)^{X_n}\left[p\left(\frac{1-p}{p}\right)^{+1} + (1-p)\left(\frac{1-p}{p}\right)^{-1}\right] \\
&= \left(\frac{1-p}{p}\right)^{X_n} \\
&= Z_n
\end{aligned}$$

so $(Z_n, \mathcal{F}_n)$ is a martingale.

Because the stopping time

$$n^* = \inf\{n \geq 0 \ : \ X_n = +k^*(w) \text{ or } X_n = -k^*(w)\}$$

is almost surely finite, and $(Z_n, \mathcal{F}_n)$ is a martingale, the optional stopping theorem implies that

$$\mathbb{E}[Z_{n^*}] = \mathbb{E}[Z_0] = \left(\frac{1-p}{p}\right)^0 = 1$$

Since at event $\mathcal{A}$ we have $Z_{n^*} = [(1-p)/p]^{k^*(w)}$ while on $\mathcal{B}$ we have $Z_{n^*} = [(1-p)/p]^{-k^*(w)}$, we have

$$1 = \mathbb{E}[Z_{n^*}] = \Pr(\mathcal{A}) \left(\frac{1-p}{p}\right)^{k^*(w)} + \Pr(\mathcal{B}) \left(\frac{1-p}{p}\right)^{-k^*(w)}$$

We also know from the lemma that

$$\Pr(\mathcal{A}) + \Pr(\mathcal{B}) = 1$$

so combining these two facts, we arrive at

$$\Pr(\mathcal{A}) = \frac{p^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}} \quad \text{and} \quad \Pr(\mathcal{B}) = \frac{(1-p)^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}}$$

■

For $k^*(w) = 2$ we have agents who are fully rational and in this case, these probabilities of cascading to A and B coincide with the probabilities that were derived for the benchmark model with fully rational agents. In this setting, it is well-known that the probability of eventually being in a cascade to the better restaurant is $p^2/[p^2 + (1-p)^2]$ the probability of not eventually being in a cascade to either restaurant is zero, and the probability of eventually being in a cascade to the worse restaurant is $(1-p)^2/[p^2 + (1-p)^2]$. We provide a direct heuristic derivation of these formulas in the appendix.

For $k^*(w) > 2$ the agents are motivated reasoners, with the strength of motivation increasing in $k^*(w)$, as $k^*(w)$ itself is increasing in w, the weight put on motivation.

To measure welfare, we take the limiting expected average sum of utilities of the agents; that is, treating the sequence of restaurants $\rho = \{\rho_1, \rho_2, ...\}$ as ex ante random and conditioning on θ being the better restaurant, welfare is

$$W(\theta) = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} u(\rho_i, \theta)\right]$$

This is the limiting fraction of agents in the population who go to the better restaurant. When $k^*(w)$ is finite, a cascade initiates almost surely following a finite number of agents, so welfare is simply

$$W(\theta) = \Pr(\text{correct cascade}) = \frac{p^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}}$$

When $k^*(w) = \infty$, welfare is simply the limiting share of agents who go to the correct restaurant, which by the law of large numbers is just p.

Theorem 5. *On the range $w \in [0, 1/2)$ welfare (and the probability of a correct cascade) is increasing in the strength of motivated reasoning w. Then at $w = 1/2$ welfare discontinuously drops to p and remains p on the range $w \in [1/2, 1]$.*

The probability of cascading to A is strictly increasing in $k^*(w)$, and so welfare is also strictly increasing in $k^*(w)$ with welfare approaching 1 as $k^*(w)$ approaches ∞. At $k^*(w) = \infty$ there is a discontinuity in welfare. In this case, all agents go by their private signals and by the weak law of large numbers only a fraction p of agents go to the correct restaurant, A. Since

$$p < \frac{p^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}}$$

for all $2 \leq k^*(w) < \infty$, we conclude that motivated reasoning enhances information aggregation up to a point. Too much motivated reasoning to the point where all agents always act only on their private information is harmful for information aggregation.

3.3 Finite Population Analysis

Theorem 5 suggests that motivated reasoning can lead to unbounded improvements in information aggregation as w approaches the discontinuity point 1/2. This is because motivated reasoning gives individuals an incentive to delay a cascade and learn more by overweighing their own private signals compared to potentially conflicting public information. However, when the number of agents is finite, the delay does not always improve social welfare since the benefits of learning are proportional to the remaining number of agents – a phenomenon known as a trade-off between exploration and exploitation.

This section studies welfare under motivated reasoning with N agents. Let $Y_{k^*(w)}$ be a random variable that indicates the number of periods passed until a cascade occurs when the threshold is $k^*(w)$. The total utility is

$$\sum_{i=1}^{N} u(\rho_i, \theta) \overset{d}{=} \sum_{n=1}^{N} \left(A_n + (N-n) \cdot \mathrm{Bern}\big(\tilde{p}_{k^*(w)}\big)\right) \cdot \mathbf{1}_{\min(Y_{k^*(w)}, N) = n}$$

where A_n is the number of correct signals up to the n-th agent as defined in Theorem 4 and $\tilde{p}_{k^*(w)} = \frac{p^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}}$. If conditioned on $Y_{k^*(w)} < N$, $A_n = \frac{Y_{k^*(w)} + k}{2}$ with probability

$\tilde{p}_{k^*(w)}$ and $\frac{Y_{k^*(w)}-k}{2}$ with probability $1-\tilde{p}_{k^*(w)}$. The welfare is its averaged expectation:

$$W_N(\theta) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} u(\rho_i,\, \theta)\right] = \mathbb{E}\left[\sum_{j=1}^{\infty} r_j \cdot \mathbf{1}_{Y_{k^*(w)}=j}\right]$$

where

$$r_j = \begin{cases} \tilde{p}_{k^*(w)} - \big(j-k^*(w)\big)\big(2\tilde{p}_{k^*(w)}-1\big)/(2N) & (j \le N) \\ A_N/N & (j \ge N) \end{cases}$$

Note that $j = N$ admits both expressions, which will serve as a key trick in the proofs that follow.

The first term in r_j when $j \le N$ captures the potential benefits from exploration. It increases in $k^*(w)$, indicating that longer exploration improves the welfare if every agent could reap its benefits.

The second term reflects the costs of exploration. The expectation of this term depends on the conditional expectation $\mathbb{E}[Y_{k^*(w)} \,|\, Y_{k^*(w)} \le N]$, which increases in $k^*(w)$ since Y_k stochastically dominates Y_k' for every $k > k'$. We also have $\tilde{p}_{k^*(w)} > 1/2$. This implies that longer exploration diminishes the potential benefits from exploration in proportion to the number of exploring agents $Y_{k^*(w)}$.

The benefits and costs of exploration will depend on both population size and motivation. Intuitively, the marginal benefits of exploration should not decrease as the population grows if motivation remains constant. In contrast, the effect of varying motivation will be more ambiguous. When motivation is low, increasing it will encourage beneficial exploration as in the infinite population case. When motivation is already high, further increases will amplify the costs of exploration.

A wrinkle to this intuition arises when the welfare when the cascade does not occur until the last agent. While r_j is easy to analyze when $j \le N$, it becomes more challenging when $j \ge N$ due to the absence of closed-form representation. The next two theorems show that those borderline cases also conform to our intuition through appropriate matching.

Lemma 2. $\mathbb{E}[Y_{k^*(w)}]$ *is finite. Specifically,*

$$\mathbb{E}\big[Y_{k^*(w)}\big] = \frac{k^*(w)}{2p-1} \cdot \frac{p^{k^*(w)} - (1-p)^{k^*(w)}}{p^{k^*(w)} + (1-p)^{k^*(w)}}.$$

Proof. Appendix. ■

Theorem 6. *$W_N(\theta)$ increases in N, and converges to $W(\theta)$ in the limit.*

Proof.

$$\begin{aligned} W_N(\theta) &= \mathbb{E}\left[\left[\tilde{p}_{k^*(w)} - \frac{Y_{k^*(w)} - k^*(w)}{2N} \cdot \left(2\tilde{p}_{k^*(w)} - 1\right)\right] \cdot \mathbf{1}_{Y_{k^*(w)} \le N} + \frac{A_N}{N} \cdot \mathbf{1}_{Y_{k^*(w)} > N}\right] \\ &< \mathbb{E}\left[\left[\tilde{p}_{k^*(w)} - \frac{Y_{k^*(w)} - k^*(w)}{2(N+1)} \cdot \left(2\tilde{p}_{k^*(w)} - 1\right)\right] \cdot \mathbf{1}_{Y_{k^*(w)} < N+1} + \frac{A_N}{N} \cdot \mathbf{1}_{Y_{k^*(w)} \ge N+1}\right]. \end{aligned}$$

$\mathbb{E}\big[A_{N+1} \cdot \mathbf{1}_{Y_{k^*(w)} \ge N+1}\big] = \mathbb{E}\big[(A_N + p) \cdot \mathbf{1}_{Y_{k^*(w)} \ge N+1}\big]$ since the difference solely depends on the private signal of the $(N+1)$-th agent. Using this equation, we can derive that

$$\mathbb{E}\left[\frac{A_N}{N} \cdot \mathbf{1}_{Y_{k^*(w)} \ge N+1}\right] \le \mathbb{E}\left[\frac{A_{N+1}}{N+1} \cdot \mathbf{1}_{Y_{k^*(w)} \ge N+1}\right]$$

holds if $\mathbb{E}\big[A_N \cdot \mathbf{1}_{Y_{k^*(w)} \ge N}\big] \le \mathbb{E}\big[Np \cdot \mathbf{1}_{Y_{k^*(w)} \ge N}\big]$, or equivalently, $\mathbb{E}\big[A_N \,\big|\, Y_{k^*(w)} \ge N\big] \le Np$. It suffices to show that

$$\mathbb{E}\big[A_{n+1} \,\big|\, Y_{k^*(w)} \ge n+1\big] \le \mathbb{E}\big[A_n \,\big|\, Y_{k^*(w)} \ge n\big] + p \tag{2}$$

for all n.

Case 1— n and $k^*(w)$ have the same parity: $Y_{k^*(w)} \ge n+1$ is equivalent to $Y_{k^*(w)} \ge n$. The equality holds from $\mathbb{E}\big[A_{n+1} - A_n \,\big|\, X_n\big] = p$.

Case 2— n and $k^*(w)$ have different parities: $\mathbb{E}\big[A_{n+1} - A_n \,\big|\, X_n\big] = p$ when $|X_n| < k^*(w)$ likewise, so $\mathbb{E}\big[A_{n+1} \,\big|\, Y_{k^*(w)} \ge n+1\big] = \mathbb{E}\big[A_n \,\big|\, Y_{k^*(w)} \ge n,\, |X_n| < k^*(w)\big] + p$. Note that

$$\mathbb{E}\big[A_n \,\big|\, Y_{k^*(w)} \ge n,\, |X_n| = x\big] = \frac{k^*(w)}{2} + \frac{x \cdot (2\tilde{p}_x - 1)}{2}$$

given that $n > k^*(w)$ and x has the same parity with $k^*(w)$, and this conditional expectation increases in x. This implies $\mathbb{E}\big[A_n \,\big|\, Y_{k^*(w)} \ge n,\, |X_n| < k^*(w)\big] \le \mathbb{E}\big[A_n \,\big|\, Y_{k^*(w)} \ge n\big]$, proving the monotonicity.

For convergence, observe

$$\begin{aligned} W_N(\theta) = &\left[\tilde{p}_{k^*(w)} - \frac{\mathbb{E}[Y_{k^*(w)} \,|\, Y_{k^*(w)} \le N] - k^*(w)}{2N} \cdot \left(2\tilde{p}_{k^*(w)} - 1\right)\right] \cdot \mathrm{P}\big[Y_{k^*(w)} \le N\big] \\ &+ \mathbb{E}\left[\frac{A_N}{N} \,\middle|\, Y_{k^*(w)} > N\right] \cdot \mathrm{P}\big[Y_{k^*(w)} > N\big]. \end{aligned}$$

The process stops in finite time almost surely (Lemma 1) and the stopping time has a finite first moment (Lemma 2), so $\lim_{N\to\infty} \frac{\mathbb{E}[Y_{k^*(w)} \mid Y_{k^*(w)} \le N] - k^*(w)}{2N} = \lim_{N\to\infty} \mathrm{P}\big[Y_{k'} > N\big] = 0$. This implies $\lim_{N\to\infty} W_N(\theta) = \tilde{p}_{k^*(w)} = W(\theta)$. ■

The above theorem supports our intuition that a larger N cannot harm welfare as it only amplifies the potential benefits of longer exploration. The proof constructs matches between borderline cases of population size N and $N+1$ by conditioning on $|X_n|$. We will return to this when studying asymptotics.

Another interesting question is the comparative static under a fixed N. We use the extended notation $W_N(\theta,\, w)$ to state the result.

Theorem 7. *Define $w^*(N) = \arg\max_w W_N(\theta,\, w)$ for $N \ge 4$.*
(a) $w^*(N)$ is bounded.
(b) $w^*(N)$ weakly increases in N.
(c) $\lim_{N\to\infty} \big[\min_{w\in w^*(N)} w\big] = 1/2$.

Proof. (a) $W_N(\theta,\, w) = p$ if $k^*(w)$ exceeds N, and

$$W_N(\theta,\, 0) \ge W_4(\theta,\, 0) = \frac{4p^2 + 3\cdot 2p^3(1-p) + 2\cdot 4p^2(1-p)^2 + 2p(1-p)^3}{4} > p.$$

The maximum of $W_N(\theta,\, \cdot)$ is achieved among the first N values of $k^*(w)$, hence $w^*(N)$ bounded.

(b) Let $k^*(w) = k < k' = k^*(w')$. Rewrite welfare as

$$N \cdot W_N(\theta,\, w) = \mathbb{E}\big[\big[A_{Y_k} + (N - Y_k)\cdot \tilde{p}_k\big]\cdot \mathbf{1}_{Y_k \le N} + A_N \cdot \mathbf{1}_{Y_k > N}\big]$$

and

$$N \cdot W_N(\theta,\, w') = \mathbb{E}\big[\big[A_{Y_{k'}} + (N - Y_{k'})\cdot \tilde{p}_{k'}\big]\cdot \mathbf{1}_{Y_{k'} \le N} + A_N \cdot \mathbf{1}_{Y_{k'} > N}\big].$$

Since $\{Y_k \le N\} \supset \{Y_{k'} \le N\}$,

$$\begin{aligned}
&N \cdot \big[W_N(\theta,\, w') - W_N(\theta,\, w)\big] \\
=&\mathbb{E}\big[\big[\big[A_{Y_{k'}} + (N - Y_{k'})\cdot \tilde{p}_{k'}\big]\cdot \mathbf{1}_{Y_{k'} \le N} + A_N \cdot \mathbf{1}_{Y_{k'} > N} \\
&\qquad\qquad\qquad\qquad - A_{Y_k} - (N - Y_k)\cdot \tilde{p}_k\big]\cdot \mathbf{1}_{Y_k \le N}\big] \\
=&\mathbb{E}\big[\big[\big[(A_{Y_{k'}} - A_{Y_k}) + (N - Y_{k'})\cdot \tilde{p}_{k'}\big]\cdot \mathbf{1}_{Y_{k'} \le N} + (A_N - A_{Y_k})\cdot \mathbf{1}_{Y_{k'} > N} \\
&\qquad\qquad\qquad\qquad - (N - Y_k)\cdot \tilde{p}_k\big]\cdot \mathbf{1}_{Y_k \le N}\big].
\end{aligned}$$

Assume the following inequalities: $\mathbb{E}\big[(A_{Y_{k'}} - A_{Y_k}) \cdot \mathbf{1}_{Y_{k'} \le N}\big] \le \mathbb{E}\big[\tilde{p}_{k'}(Y_{k'} - Y_k) \cdot \mathbf{1}_{Y_{k'} \le N}\big]$ and $\mathbb{E}\big[(A_N - A_{Y_k}) \cdot \mathbf{1}_{Y_{k'} > N,\, Y_k \le N}\big] \le \mathbb{E}\big[p(N - Y_k) \cdot \mathbf{1}_{Y_{k'} > N,\, Y_k \le N}\big]$. If we plug these in the above expectation, then we obtain that $W_N(\theta, w') \ge W_N(\theta, w)$ implies

$$\mathbb{E}\big[\big[\tilde{p}_{k'} \cdot \mathbf{1}_{Y_{k'} \le N} + p \cdot \mathbf{1}_{Y_{k'} > N} - \tilde{p}_k\big] \cdot \mathbf{1}_{Y_k \le N}\big] \ge 0,$$

which is a rearrangement of

$$\begin{aligned} &\big[(N+1) \cdot W_{N+1}(\theta, w') - N \cdot W_N(\theta, w')\big] \\ &\qquad - \big[(N+1) \cdot W_{N+1}(\theta, w) - N \cdot W_N(\theta, w)\big] \\ =&\mathbb{E}\big[\tilde{p}_{k'} \cdot \mathbf{1}_{Y_{k'} \le N} + p \cdot \mathbf{1}_{Y_{k'} > N}\big] - \mathbb{E}\big[\tilde{p}_k \cdot \mathbf{1}_{Y_k \le N} + p \cdot \mathbf{1}_{Y_k > N}\big]. \end{aligned}$$

Therefore, if $W_N(\theta, w') \ge W_N(\theta, w)$, then $W_{N+1}(\theta, w') \ge W_{N+1}(\theta, w)$. It follows that $w^*(N) \le w^*(N+1)$.

We finally justify the two claims that we skipped the proof of. For the first one, due to the symmetry, any sequence of signals that end at $X_{Y_k'} = k'$ is $\tilde{p}_k'$ times more likely than the symmetric sequence that ends at $X_{Y_k'} = -k'$. For any choice of $\{\omega_i\}_{i=1}^N$, there exists r such that

$$\begin{aligned} \mathbb{E}\big[A_{Y_{k'}} - A_{Y_k} \mid \{\omega_i\}_{i=1}^N \cup \{\neg\, \omega_i\}_{i=1}^N\big] &= \tilde{p}_{k'} \cdot r + (1 - \tilde{p}_{k'}) \cdot (Y_{k'} - Y_k - r) \\ &\le \tilde{p}_{k'}(Y_{k'} - Y_k). \end{aligned}$$

Taking the expectation over $\{\{\omega_i\}_{i=1}^N : Y_{k'} \le N\}$ proves the claim.

For the second one, we can show that fixing $Y_k = y\,(\le N)$, $\mathbb{E}\big[A_{n+1} - A_{Y_k} \,\big|\, Y_{k'} > n+1,\, Y_k = y\big] \le \mathbb{E}\big[A_n - A_{Y_k} \,\big|\, Y_{k'} > n,\, Y_k = y\big] + p$ for all $n \ge y$ by repeating the proof of inequality (2) in the proof of Theorem 6. Taking the expectation over $\{y \le N\}$ proves the claim.

(c) Fix k. $\lim_{N\to\infty} W_N(\theta, w) = \tilde{p}_{k^*(w)}$ by Theorem 6. The convergence occurs only through the value of $k^*(w)$, so the uniform convergence holds for all w such that $k^*(w) \le k$. Since $\tilde{p}_k$ strongly increases in k, there exists N_k such that for any $N \ge N_k$, $W_N(\theta, w') \approx \tilde{p}_{k'} < \tilde{p}_k \approx W_N(\theta, w_0)$ for all w', w_0 and k such that $k^*(w') = k' < k = k^*(w_0)$. Therefore, the maximum cannot be attained when $k^*(w) < k$ for these values of N. Sending $k \to \infty$ proves the claim. ■

Welfare in Theorem 5 kept increasing as motivation w approaches the tipping point, but it dropped back to p when $w = 1/2$. Theorem 7 says that this discontinuity does not appear in finite populations. It is because the benefits of exploration are unbounded in the infinite population, whereas they are limited when the population is finite. This result formalizes the intuition behind the trade-off between exploration and exploitation.

Part (b) implies that the calculus of the cost and benefit of exploration is monotonic in N. That is, if the benefit of longer exploration due to higher motivation outweights the cost at some population size, so does for all larger population sizes. This result simplifies the analysis of optimal motivation in finite populations. The proof uses similar matching techniques as the one in Theorem 6.

Part (c) shows that the optimal motivation approaches the tipping point as the population size grows. For small values of w, the cost of exploration becomes ignorable and finite-population welfares uniformly converge to their respective infinite-population welfares that increase in w. Therefore, if the population size exceeds a certain threshold, finite-population welfare increases at least up to the maximal w in the range.

3.4 Asymptotics

How quickly does information aggregate in the sequential model as the population size grows? We can examine the information aggregation in two aspects. First, how quickly does cascading happen? Second, how fast does welfare converge to its limit? These questions were not separately considered in the previous section because agents always adopt the same behavior. In contrast, the social learning model admits two possible limiting outcomes, allowing us to study the two convergence rates separately.

To find those convergence rates, we first need to understand the asymptotic behavior of $Y_{k^*(w)}$. Feller (1950, p.350) characterizes the distribution of $Y_{k^*(w)}$ using generating functions, but we derive the full probability mass function of X using the transition matrix.

Assume even k. Let $p_{n,i} = \mathrm{P}[X_n = i]$ and p_n be a length $k+1$ vector $(p_{n,-k}\ p_{n,-k+2} \cdots\ p_{n,k})^\top$. By definition, p_0 puts a unit mass on $p_{1,0}$. The process follows

$$p_{2n+2} = \begin{pmatrix} 0 & p^2 & 0 & 0 & 0 & \cdots \\ 0 & 2p(1-p) & p^2 & 0 & 0 & \cdots \\ 0 & (1-p)^2 & 2p(1-p) & p^2 & 0 & \cdots \\ 0 & 0 & (1-p)^2 & 2p(1-p) & p^2 & \cdots \\ \vdots & & & & & \end{pmatrix} \times p_{2n}.$$

$p_{n,k} + p_{n,-k}$ is the probability that a cascade happens exactly in the n-th period, or $\mathrm{P}[Y_k = n]$. Denote the transition matrix as T. The analysis simplifies if T is diagonalizable.

Lemma 3. *$T = V\Lambda V^{-1}$ where*

$$\Lambda = 4p(1-p) \cdot \mathrm{diag}\left(\cos^2 \frac{\pi}{2k},\, \cos^2 \frac{2\pi}{2k},\, \cdots,\, \cos^2 \frac{(k-1)\pi}{2k},\, 0,\, 0 \right),$$

$$V = \begin{pmatrix} \mathbf{1}_{k-1}^\top & 1 & 0 \\ A & \mathbf{0}_{k-1} & \mathbf{0}_{k-1} \\ b^\top & 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \mathbf{0}_{k-1} & A^{-1} & \mathbf{0}_{k-1} \\ 1 & -\mathbf{1}_{k-1}^\top A^{-1} & 0 \\ 0 & -b^\top A^{-1} & 1 \end{pmatrix}$$

for a $(k-1) \times (k-1)$ matrix A and its inverse A^{-1} whose (i, j)-th entries are

$$A_{ij} = 2 \cot \frac{\pi j}{2k} \sin \frac{\pi i j}{k} \left(\frac{1-p}{p} \right)^i, \quad (A^{-1})_{ij} = \frac{1}{2} \tan \frac{\pi i}{2k} \sin \frac{\pi i j}{k} \left(\frac{p}{1-p} \right)^j$$

and a length-$(k-1)$ vector b whose i-th entry is

$$b_i = \left(\frac{1-p}{p} \right)^k \cdot (-1)^{i+1}.$$

Proof is via straightforward calculation using the fact that $\sum_{r=1}^{k-1} \sin \frac{\pi i r}{k} \sin \frac{\pi j r}{k} = \frac{k-1}{2} \cdot \delta_{ij}$. Then, $p_{2n} = V\Lambda^n V^{-1} p_0$. For odd k, the construction is identical except for using p_1 for the initial value instead of p_0.

The above diagonalization provides useful intuitions. First, $Y_{k^*(w)}$ is subexponential.

$$\begin{aligned} \mathrm{P}[Y_{k^*(w)} \geq 2N] &= (e_1 + e_{k^*(w)+1})^\top V \left[\sum_{i=N}^{\infty} \Lambda^i \right] V^{-1} p_0 \\ &= (e_1 + e_{k^*(w)+1})^\top V \mathrm{diag}\left(\frac{\lambda_i^N}{1 - \lambda_i} \right) V^{-1} p_0 \end{aligned}$$

where $e_1 + e_{k+1}$ extracts the first and last elements of the following vector and $\lambda_i = \Lambda_{ii}$. For a large t, $\mathrm{P}[|Y_{k^*(w)} - \mathbb{E}[Y_{k^*(w)}]| \geq t] = \mathrm{P}[Y_{k^*(w)} \geq t + \mathbb{E}[Y_{k^*(w)}]] = \Theta(\lambda_1^{N/2})$.

Second, p_{2n} can be approximated with the leading eigenvalue and eigenvector:

$$p_{2n+2} \approx \lambda_1^{n+1} \cdot V_1 V_1^{-1} p_0 \approx \lambda_1 \cdot p_{2n}$$

where V_1 is the first column of V and V_1^{-1} is the first row of V^{-1}. Under the approximation, the conditional probability of X_n given that the process does not end until the n-th period converges to a stable value. The conditional probability means the limiting behavior of agents under motivated reasoning in our substantive application. The above approximation holds at the rate of $1-(\frac{\lambda_2}{\lambda_1})^n$. This decreases in k, so the convergence occurs more slowly as the threshold $k^*(w)$ increases.

Corollary 1. *The followings hold:*
(a) $1-\mathrm{P}[Y_{k^*(w)} \leq N] = \Theta(\lambda_w^{N/2})$ *for* $\lambda_w = 4p(1-p)\cdot\cos^2(\pi/2k^*(w))$.
(b) $\tilde{p}_{k^*(w)} - W_N(\theta) = \Theta(1/N)$.

Proof. (a) This uses the above expression of $\mathrm{P}[Y_{k^*(w)} \geq 2N]$ together with the explicit form of the largest eigenvalue.

(b) $\tilde{p}_{k^*(w)} - W_N(\theta,\, w)$ is

$$\left(2\tilde{p}_{k^*(w)}-1\right)\sum_{j=1}^{N}\frac{(j-k^*(w))\cdot\mathbf{1}_{Y_{k^*(w)}=j}}{2N}+\mathbb{E}\left[\left(\tilde{p}_{k^*(w)}-\frac{A_N}{N}\right)\cdot\mathbf{1}_{Y_{k^*(w)>N}}\right].$$

The first term is $\Theta(\mathbb{E}[Y_{k^*(w)}\cdot\mathbf{1}_{Y_{k^*(w)}\leq N}]/N)$, and the second term is $O(\mathrm{P}[Y_{k^*(w)} > N])$. Note that $\mathrm{P}[Y_{k^*(w)} = k] < \mathbb{E}[Y_{k^*(w)}\cdot\mathbf{1}_{Y_{k^*(w)}\leq N}] < \mathbb{E}[Y_{k^*(w)}]$. The lower bound is $\Theta(1)$ as it does not depend on N, and the upper bound is $\Theta(1)$ by Lemma 2. Meanwhile, we have derived above that $\mathrm{P}[Y_{k^*(w)>N}] = O(\lambda_1^{N/2}) = o(1/N)$ for the largest eigenvalue λ_1 associated with the threshold $k^*(w)$. The overall expression reduces to $\Theta(1/N)$. ■

The above corollary shows that the cascade occurs much faster than the welfare converges. This is because the permanent benefit or penalty resulting from agents cascading to either choice pulls welfare in both directions. Compared to the Condorcet jury model, welfare converges more slowly to its limit in the sequential learning model, but the latter achieves a higher worst-case welfare.

Appendix

BHW Model Review

Since the model is symmetric, assume without loss that A is the better restaurant. The probability of being in a cascade to A after two agents is p^2, while the probability of being in a cascade to B after two agents is $(1-p)^2$. The probability of being in a cascade after three agents conditional on not being in a cascade after two agents is 0. In general, the probability of being in a cascade after an odd number n of agents conditional on not being in a cascade after $n-1$ agents is 0. So now only consider even numbers n of agents.

The probability of being in a cascade to A after four agents conditional on not being in a cascade (to either A or B) after two agents is the probability of realizing either the sequence $ABAA$ or $BAAA$. That conditional probability is $2p(1-p)p^2$. Thus, the probability of being in a cascade to A after four agents is the sum of the probability of being in a cascade to A after two agents and the probability of being in a cascade to A after four agents, or $p^2+2p(1-p)p^2$. The probability of being in a cascade to A after six agents conditional on not being in a cascade to A after four agents is the probability of realizing the sequence $ABABAA$ or $BABAAA$ or $ABBAAA$ or $BAABAA$, which is $4p^2(1-p)^2p^2$. The aggregate probability of being in a cascade to A after six agents is the sum of this probability and the probability of being in a cascade to A after four agents; that sum is $p^2 + 2p(1-p)p^2 + 4p^2(1-p)^2p^2$. The probability of being in a cascade to A after eight agents is the probability of being in a cascade to A after six agents plus the probability of realizing the sequence $ABABABAA$ or $ABABBAAA$ or $BABAABAA$ or $BABABAAA$ or $ABBAABAA$ or $ABBABAAA$ or $BAABABAA$ or $BAABBAAA$, which is $p^2 + 2p(1-p)p^2 + 4p^2(1-p)^2p^2 + 8p^3(1-p)^3p^2$. More generally, the probability of being in a cascade to A after n (even) agents is

$$p^2 \sum_{i=1}^{n/2} [2p(1-p)]^{i-1}$$

and the limit of this probability as $n \to \infty$ is

$$\frac{p^2}{p^2 + (1-p)^2}.$$

This is the probability of eventually being in a cascade to A. Since the probability of not eventually being in a cascade to either one of the restaurants is zero, the probability of eventually being in a cascade to B is $(1-p)^2/[p^2 + (1-p)^2]$.

Proof of Lemma 2

Let $a_{k,i}$ be the expected number of remaining periods until a cascade occurs when $X_n = i$ in the notation of the proof of Theorem 4. The expectations are well-defined by Lemma 1. Since the process immediately ends when X_n hits k or $-k$, we have $a_{k,k} = a_{k,-k} = 0$. If $X_n = i$, then $X_{n+1} = i + 1$ with probability p and $i - 1$ with probability $1 - p$. This recurrent relation suggests

$$a_{k,i} = 1 + pa_{k,i+1} + (1-p)a_{k,i-1}$$

for $-k + 1 \leq i \leq k$. It suffices to verify that

$$a_{k,i} = \frac{2k\big(1 - (\frac{1-p}{p})^{i+k}\big)}{(2p-1)\big(1 - (\frac{1-p}{p})^{2k}\big)} - \frac{i+k}{2p-1} \quad (-k \leq i \leq k)$$

satisfy the relations by plugging in the expression. $i = 0$ proves the lemma.

References

Feller, William. 1950. *An introduction to probability theory and its applications.* Vol. 1 John Wiley & Sons.