Alexander Bondarenko

MATHEMATICAL MODELING OF A PYROELECTRIC DETECTOR

THE STEP BY STEP GUIDE OF THE THERMAL TO ELECTRICAL MODEL

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Математичне моделювання піроелектричного детектора.
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В монографії представлено покрокове рішення десяти задач теплоелектричної моделі піроелектричного детектора. Перші вісім задач пов’язані з процесами, що протікають безпосередньо в самому чутливому елементі. Останні дві розкривають процес перетворення вхідних електричних зарядів у вихідну напругу, коли чутливий елемент підключений до високоомної електроніки. Кожна задача починається із рівняння закону збереження енергії і закінчується рівнянням перехідного процесу.

Призначено для спеціалістів в галузі проектування піроелектричних детекторів, а також для широкого кола наукових робітників, аспірантів та студентів технічних вузів.

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Review

The book provides solutions to ten problems that explain the operating principle of the thermal-to-electrical model of a pyroelectric detector. The problems addressed in the literature concerning this model qualify as heating problems and are limited to the detector's response to sinusoidal heat flows. However, it is quite difficult to ensure a sinusoidal input of heat flow on the detector because the shape and arrangement of the sensitive element require optical matching with the modulation disk. Well-known radiometers (for instance, a series of absolute radiometers produced by Laser Precision Corp.) use a modulation disk that generates a rectangular pulse of heat with slightly smoothed rising and falling edges. Accordingly, the prevailing description for the thermal-to-electrical model of a pyroelectric detector based on the assumption that heat flows are sinusoidal is not universally applicable.

The author has worked out the problems to illustrate how the pyroelectric detector responds to heat flows with both a sinusoidal shape and a unit step input. On this basis, it is possible to capture transient characteristics of the detector in response to any form of signal that lends itself to mathematical approximation. Apart from this, the author provides solutions to problems both on heating and cooling of the detector. Additionally, in order to explain transient characteristics of the tertiary pyroelectric effect, he examines some problems never stated thus before. The author further indicates that a change to the temperature profile of the detector affects the polarity of the pyroelectric current rather than output voltage, which is another omission in theory.

The distinguishing feature of this book is that it guides you through the problems stated and offers both experts and beginners a deeper insight into the operating principle of the thermal-to-electrical model of the pyroelectric detector.

In addition to the detector's thermal-to-electrical model, the author's greatest achievement is developing a software product that simulates transient characteristics of pyroelectric detectors in real time and does not require costly experimentation. The book concludes with the interface screenshot from a simulator's demo version. It may vary depending on the product edition.

Opinion. I recommend Mathematical Modeling of a Pyroelectric Detector by Alexander Bondarenko for publication.

Peter Eliseyev

Reviewer

PhD in Engineering, expert in mathematical modeling

Department of Theoretical Engineering and Industrial Enterprises

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About the author

Alexander Bondarenko (Gajvoronsky, Polish family roots) was born on November 9, 1980, in Severodonetsk, Ukraine. In 2003, he was awarded a Bachelor and, in 2004, Specialist Degree at the Department of Control Theory in the School of Mechanics at Severodonetsk Technological Institute. His major was mathematical modeling of processes in the chemical industry. In 2013, he received a Master of Science Degree at Kharkiv Aerospace University at the Department of Electrical Engineering in the School of Aircraft Flight Control Systems. The aim of the thesis was to research and apply pyroelectric detectors based on lithium tantalate single crystals. In 2016, he graduated from Kharkiv National University of Radioelectronics with a PhD, although for some reasons beyond the author's control failed to receive it in that year. The new thesis on mathematical modeling of a pyroelectric detector and its transient responses is written.

Message

This is my first book. As I write it, I am currently busy on my thesis. I am looking for scientists or engineers who have passion for and skill in the field of pyroelectric detectors, including those who want to gain some knowledge from me. That is important because very few people today understand how pyroelectric detectors work. “Engineers are scared to use an electro-optical system they don’t understand”. It was Hans J. Keller, a man with greatest experience in pyroelectric detectors, who made this point in 2000 in his brilliant paper titled ‘30 Years of Passive Infrared Motion Detectors – a Technology Review’ free available on http://kube.ch/downloads/pdf/kube_irs2paper.pdf. According to my own sources, his statement still holds water. Moreover, I was not able to find a paper or thesis that would explain the transient responses of the detectors in detail. Therefore, I made up my mind to explore this problem independently. I hope this book will help any amateur, student, engineer, or researcher understand the operating principle of the thermal-to-electrical model of a pyroelectric detector. Unlike most books, this one is not overloaded with theory. I have worked particularly on this book for two years. Please, respect my contribution and support me by buying only the original paper book or ebook. Please, do not download it anywhere. Thank you for understanding. I appreciate it.

I welcome appropriate experts to translate this book into their native languages. Before they start, they should contact me first. Undergraduate and graduate students who work closely with their supervisors are a top priority.

You can view my portfolio here at https://www.linkedin.com/in/pyrodetector/

Kind regards, Alexander Bondarenko

Acknowledgments

I would like to thank Rosen Dukov from Sofia, Bulgaria, a professional digital designer. I have known him to be an excellent professional and responsive friend in hard times. His works are available at http://rosendukov.com/

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I am deeply thankful to Marion Weinreich from Hamburg, Germany, a friend, a teacher, deputy GodMother, the one has become a part of my life in recent years. I need to say a heartfelt: thank you for being in my life. I love you, Marion.

I am also indebted to my family, my parents who have loved and supported me financially, and never stopped believing in my success. I love you, my dear father and mother, Yuri and Lyudmila, and my brothers, Viktor and Anatoly, my old grandmother Valentina and my aunt Tatyana with her husband Ivan.

I extend my heartfelt appreciation to my friends Igor, Andrei, Aleksander, Mikhail, Aleksander, Konstantin, Roman, Andrei, Pavel, Andrei, and also Anna, Lena (Alena), Laura and others who have stayed with me for years.

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I sincerely wish to thank I.P. Vnukov (blessed be his memory), my supervisor from the Aerospace University for his invaluable assistance and support when I was his student.

I am especially grateful to I.I. Klyuchnik, my supervisor from the University of Radioelectronics who was the first to contribute to my education in the field of pyroelectric detectors.

Many thanks to the Department of Design Automation led by Dr. Vladimir Hahanov and Dr. Svetlana Chumachenko for assistance in setting up classes with students who were the first ever to solve those problems with me.

Thanks are also due to great scientists from the Technical University of Dresden and InfraTec, GmbH, such as H. Budzier, G. Gerlach, M. Schossig, Y. Querner, D. Schvedov, M. Ebermann, N. Neumann, and others who made great contribution to designing state-of-the-art pyroelectric detectors. Their books and papers have proved to be of immense interest, particularly, those about micro Fabry-Perot interferometers, black coatings, etc.

I would like to take this opportunity to thank Linus Torvalds for his products like LINUX and GIT. I enjoy using both in everyday life.

My extra thank to Jean Michel Jarre for his fantastic music. I love it. It really helps solve mathematics. I hope he gives the name to his next album “Pyroelecrtic charges” (as he did “Magnetic fields” in 1981).

Thanks a million to Pavel Ignatenko, a friend, the publisher, for his priceless support and publishing this book.

I thank Anna for inspiring me in writing this book, and much more...

Finally, I thank other teachers who made their contribution to my education in Severodonetsk and Kharkiv. I will cherish the memory of you.

Kind regards, Alexander Bondarenko

The world's first team of students mentioned above from the Department of Design Automation at Kharkiv National University of Radioelectronics, May 2017

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http://pyrodetector.com/reviewers.html

From left to right: Denis Egorov, Oleksandra Bublyk, Oleksandr Oliinyk, Kateryna Tenditnyk, Valentin Orlik, and Alexander Bondarenko in the centre

I wish you peace, love, good health, professional success, prosperity, and a long and happy life.

Sincerely, Alexander

Abstract

The book offers a step-by-step guide to mathematical modeling of the thermal-to-electrical model of a pyroelectric detector. It contains the solutions to ten problems. The first eight problems are related to processes running in the body of the sensitive element from heating or cooling to generating electrical charges in response. The last two problems examine the transformation of input electrical charges to output voltage when the sensitive element is connected to high-megohm electronics. Every solution starts with the equation for the law of conservation of energy and ends with that of the transient response. In order to make reading easier, the author provides almost every equation with corresponding units of measurement which are extremely useful not only for beginners, but also for advanced readers. The book can be recommended to amateurs, undergraduate and graduate students, teachers, engineers, who want to develop advanced knowledge better concerning to the thermal-to-electrical model of pyroelectric detectors.

Table of Contents

Units of measurement

Table 1 Symbols and units of measurement

#	Symbol	Unit of measurement	Description
01	E	J	Energy
02	Φ	W	Heat Flow
03	C_T	J/K	Heat Capacity
04	G_T	W/K	Heat Losses
05	T	K	Temperature
06	Ψ	K/s	Rate of Temperature
07	p_{pyro}	$(A\ s)/(m^2\ K)$	Pyroelectric Coefficient
08	A_{pyro}	m^2	Sensitive Element Square
09	I_{pyro}	A	Pyroelectric current
10	R_E	Ω	Electrical Resistance
11	C_E	F	Electrical Capacitance
12	U_{OUTPUT}	V	Output Voltage
13	τ_T	s	Thermal Time Constant
14	τ_E	s	Electrical Time Constant
15	t	s	Time

Table 2 Indicies

#	Index	Description
01	INPUT	Flow of energy from source
02	DIST	Flow of energy from ambient
03	VOL.T	Thermal energy in sensitive element
04	VOL.E	Electrical energy in sensitive element
05	VOL.TE	Energy transformed from thermal to electrical
06	OUTPUT	Energy going out of the sensitive element

"1" relates to the primary pyroelectric effect, response to dT/dt;
"3" relates to the tertiary pyroelectric effect, response to T.

Introduction

A pyroelectric detector converts thermal energy to electrical energy. It consists of a thin film pyroelectric material used as a sensitive element, a pair of electrodes deposited onto the flat surfaces of the element, black coating deposited onto the face electrode to improve absorbance and, at least, one leg which the element is mounted on. When the sensitive element absorbs or gives back thermal energy, its temperature changes with different rates. The rate of temperature change is proportional to the pyroelectric curent caused by the primary pyroelectric effect. The pyroelectric curent caused by the tertiary pyroelectric effect, is proportional to the temperature change of the sensitive element, probably, to the thermal gradient within the body, but this question is under discussion. These things are different and can not be confused. The secondary pyroelectric effect gives a piezoelectric contribution to output signals. If the pyroelectric detector has no vibration, then the piezoelectric contribution will occur in the sensitive element due to its thermal expansion. The secondary pyroelectric effect caused by the thermal expansion, contributes marginally to output and can be omitted for small signal applications which cover the bulk part of the market. For this reason, it is not considered in this book.

The pyroelectric detector can be examined in accordance with its logical data model (see **Fig. 1**). This model is based on the law of conservation of energy, namely the change in the amount of energy coming to the sensitive element with the heat flow to be measured, x, called "*input*", and the heat flow not to be measured, z, called "*disturbance*", equals the sum of change in the amount of energy, dy, stored therein, and the change in the amount of energy, y, called "*output*", going out. The variable z is considered, for example, to be the heat flow caused by changes in the ambient temperature. It is required to be as close to zero as possible and is not extensively examined in the relevant literature.

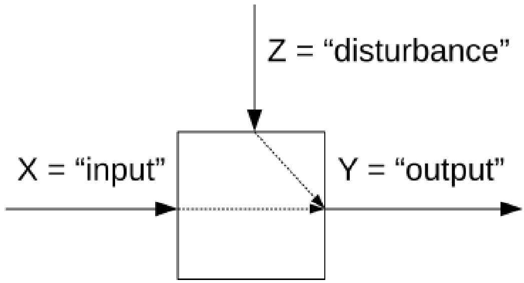


Fig. 1 logical data model

Table **1** consists of symbols and units of measurement. Table **2** includes indices.

Further details of the theory of the pyroelectric detector can be found in almost every book. For example, a list of the references recommended to reading by the author, is available at the link below

http://pyrodetector.com/references.html

1 Temperature change (heating)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{1.1}$$

with

$$dE_{INPUT} = \Phi_{INPUT}\, dt\{J\} \tag{1.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = \Phi_{DIST}\, dt\{J\} \tag{1.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = C_T\, dT\ \{J\} \tag{1.4}$$

being the amount of energy stored in the volume of the sensitive element;

$$dE_{OUTPUT} = G_T\, T dt\{J\} \tag{1.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **1.2-1.5**, we set up an equation for power for which one has to divide each parameter by *dt*

$$\Phi_{INPUT} + \Phi_{DIST} = C_T \frac{dT}{dt} + G_T\, T\{W\}. \tag{1.6}$$

Equation **1.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\Phi_{INPUT} = \Phi_{INPUT\,0} + \Delta\Phi_{INPUT}\{W\}; \tag{1.7}$$

$$\Phi_{DIST} = \Phi_{DIST\,0} + \Delta\Phi_{DIST}\{W\}; \tag{1.8}$$

$$C_T\, dT = [C_T \frac{dT_0}{dt}] + C_T \frac{d\Delta T}{dt}\{W\}; \tag{1.9}$$

$$G_T T = G_T T_0 + G_T \Delta T \{W\}. \qquad (1.10)$$

We extract static terms from the family of equations **1.7-1.10**

$$\Phi_{INPUT\,0} + \Phi_{DIST\,0} = G_T T_0 \{W\}. \qquad (1.11)$$

Equation **1.11** is a static model. We extract equation **1.11** from the family of equations **1.7-1.10**

$$\Delta\Phi_{INPUT} + \Delta\Phi_{DIST} = C_T \frac{d\Delta T}{dt} + G_T \Delta T \{W\}. \qquad (1.12)$$

Equation **1.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **1.12** by G_T

$$\frac{\Delta\Phi_{INPUT}}{G_T} + \frac{\Delta\Phi_{DIST}}{G_T} = \frac{C_T}{G_T} \frac{d\Delta T}{dt} + \Delta T \{K\}. \qquad (1.13)$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}}{G_T} \frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}} + \frac{\Phi_{DIST}}{G_T} \frac{\Delta\Phi_{DIST}}{\Phi_{DIST}} = \qquad (1.14)$$

$$= \frac{C_T}{G_T} T \frac{d\frac{\Delta T}{T}}{dt} + T \frac{\Delta T}{T} \{K\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}} = x\,;\ \frac{\Delta\Phi_{DIST}}{\Phi_{DIST}} = z\,;\ \frac{\Delta T}{T} = y\ \{Unit\}.$$

We insert the variables obtained above into equation **1.14**

$$\frac{\Phi_{INPUT}}{G_T} x + \frac{\Phi_{DIST}}{G_T} z = \frac{C_T}{G_T} T \frac{dy}{dt} + T\,y \{K\}. \qquad (1.15)$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}}{G_T T} x + \frac{\Phi_{DIST}}{G_T T} z = \frac{C_T}{G_T} \frac{dy}{dt} + y \{Unit\}. \tag{1.16}$$

We apply the Laplace Transform to the variables determined above

$$x \to X(S); z \to Z(S); y \to Y(S); \frac{d}{dt} \to s.$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}}{G_T T} = K_{INPUT} \{Unit\}; \tag{1.17}$$

$$\frac{\Phi_{DIST}}{G_T T} = K_{DIST} \{Unit\};$$

$$\frac{C_T}{G_T} = \tau_T \{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s + 1) = K_{INPUT} X(S) + K_{DIST} Z(S). \tag{1.18}$$

We write the transfer functions for the “*input* → *output*” channel

$$W_{x \to y}(S) = \frac{Y(S)}{X(S)} = \frac{K_{INPUT}}{\tau_T s + 1} \{Unit\}, \tag{1.19}$$

and “*disturbance* → *output*” channel

$$W_{z \to y}(S) = \frac{Y(S)}{Z(S)} = \frac{K_{DIST}}{\tau_T s + 1} \{Unit\}. \tag{1.20}$$

We replace the Laplace variable *s* with *jω*

$$s \to j\omega.$$

We apply the replaced variable *jω* to equations **1.19** and **1.20**

$$W(j\omega) = \frac{K(j\omega)}{D(j\omega)}, \tag{1.21}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{1.22}$$

being the numerator for equations **1.19** and **1.20**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{1.23}$$

being the denominator for equations **1.19** and **1.20**. We define the corresponding terms *R*, *I*, *M* and *N*

$$R(\omega)=1\,;I(\omega)=0\,;M(\omega)=1\,;N(\omega)=\omega\,\tau_T\,.$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{1.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{1}{\sqrt{1+\omega^2\,\tau_T^2}}\,. \tag{1.25}$$

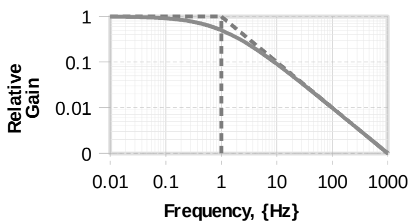


Fig. 1.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **1.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{1.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{1+0}{1+\omega^2\tau_T^2}, \tag{1.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0-\omega\tau_T}{1+\omega^2\tau_T^2}. \tag{1.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=-\arctan\omega\tau_T. \tag{1.29}$$

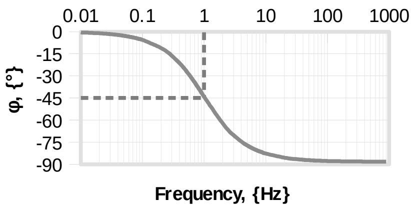


Fig. 1.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **1.19** and **1.20** for which the former and the latter must be divided by *s*

$$L^{-1}(W_{\substack{x\to y\\ z\to y}}(S))=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{s(\tau_T s+1)}\{s\}. \tag{1.30}$$

We resolve equation **1.30** into partial fractions

$$\frac{K_{INPUT/DIST}}{s(\tau_T s+1)}=\frac{A}{s}+\frac{B}{\tau_T s+1}. \tag{1.31}$$

Upon application of some mathematical transformations (Equations **1.31-1.36** where each variable is unitless), such as

$$A\tau_T s + A + Bs = K_{INPUT/DIST}, \tag{1.32}$$

and

$$s^1: A\tau_T + B = 0, \tag{1.33}$$

$$s^0: A = K_{INPUT/DIST},$$

with

$$B = -K_{INPUT/DIST}\tau_T. \tag{1.34}$$

We write equation **1.31** in the form

$$\frac{K_{INPUT/DIST}}{s(\tau_T s+1)} = \frac{K_{INPUT/DIST}}{s} - \frac{K_{INPUT/DIST}\tau_T}{\tau_T s+1}, \tag{1.35}$$

and after manipulations

$$\frac{K_{INPUT/DIST}}{s(\tau_T s+1)} = \frac{K_{INPUT/DIST}}{s} - \frac{K_{INPUT/DIST}}{s+\frac{1}{\tau_T}}. \tag{1.36}$$

As can be seen, the poles are $s_1=0$

$$K_{s_1=0} = e^{st} = K_{INPUT/DIST}\{Unit\}, \tag{1.37}$$

and $s_2=-1/\tau_T$

$$K_{s_2=-\frac{1}{\tau_T}} = K_{INPUT/DIST}\, e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{1.38}$$

We write the function *f(t)* as

$$f(t) = K_{INPUT/DIST}\left(1-e^{-\frac{t}{\tau_T}}\right)\{Unit\}. \tag{1.39}$$

Once equation **1.39** has been multiplied by *T*, the temperature change of the sensitive element when heated, as a function of time equals

$$T(t) = \frac{\Phi_{INPUT/DIST}}{G_T}\left(1-e^{-\frac{t}{\tau_T}}\right)\{K\}. \tag{1.40}$$

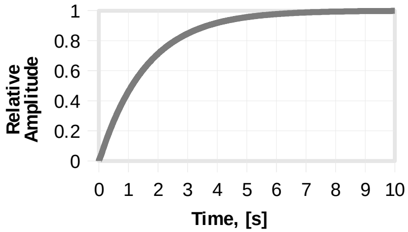


Fig. 1.3 Transient response (τ_T=0.159 s)

2. Temperature change (cooling)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{2.1}$$

with

$$dE_{INPUT} = d\Phi_{INPUT} dt\,\{J\} \tag{2.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = d\Phi_{DIST} dt\{J\} \tag{2.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = G_T dTdt\{J\} \tag{2.4}$$

being the amount of energy stored in the volume of the sensitive element;

$$dE_{OUTPUT} = \frac{G_T^2}{C_T} Tdt^2\{J\} \tag{2.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **2.2-2.5**, we set up an equation for power per second for which one has to divide each parameter by *dt* squared

$$\frac{d\Phi_{INPUT}}{dt}+\frac{d\Phi_{DIST}}{dt}=G_T\frac{dT}{dt}+\frac{G_T^2}{C_T}T\left\{\frac{W}{s}\right\}. \tag{2.6}$$

Equation **2.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\frac{d\Phi_{INPUT}}{dt}=\left[\frac{d\Phi_{INPUT0}}{dt}\right]+\frac{d\Delta\Phi_{INPUT}}{dt}\left\{\frac{W}{s}\right\}; \tag{2.7}$$

$$\frac{d\Phi_{DIST}}{dt}=\left[\frac{d\Phi_{DIST0}}{dt}\right]+\frac{d\Delta\Phi_{DIST}}{dt}\left\{\frac{W}{s}\right\}; \tag{2.8}$$

$$G_T\frac{dT}{dt}=\left[G_T\frac{dT_0}{dt}\right]+G_T\frac{d\Delta T}{dt}\left\{\frac{W}{s}\right\}; \tag{2.9}$$

$$\frac{G_T^2}{C_T}T=\frac{G_T^2}{C_T}T_0+\frac{G_T^2}{C_T}\Delta T\left\{\frac{W}{s}\right\}. \tag{2.10}$$

We extract static terms from the family of equations **2.7-2.10**

$$\frac{G_T^2}{C_T}T_0=0\left\{\frac{W}{s}\right\}. \tag{2.11}$$

Equation **2.11** is a static model. We extract equation **2.11** from the family of equations **2.7-2.10**

$$\frac{d\Delta\Phi_{INPUT}}{dt}+\frac{d\Delta\Phi_{DIST}}{dt}=G_T\frac{d\Delta T}{dt}+\frac{G_T^2}{C_T}\Delta T\left\{\frac{W}{s}\right\}. \tag{2.12}$$

Equation **2.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **2.12** by G^2_T/C_T

$$\frac{C_T}{G_T^2}\frac{d\Delta\Phi_{INPUT}}{dt}+\frac{C_T}{G_T^2}\frac{d\Delta\Phi_{DIST}}{dt}= \tag{2.13}$$

$$=\frac{C_T}{G_T}\frac{d\Delta T}{dt}+\Delta T\,\{K\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{C_T}{G_T^2}\Phi_{INPUT}\frac{d\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}}{dt}+\frac{C_T}{G_T^2}\Phi_{DIST}\frac{d\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}}{dt}= \tag{2.14}$$

$$=\frac{C_T}{G_T}T\frac{d\frac{\Delta T}{T}}{dt}+T\frac{\Delta T}{T}\{K\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}=x\,;\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=z\,;\frac{\Delta T}{T}=y\,\{Unit\}.$$

We insert the variables obtained above into equation **2.14**

$$\frac{C_T}{G_T^2}\Phi_{INPUT}\frac{dx}{dt}+\frac{C_T}{G_T^2}\Phi_{DIST}\frac{dz}{dt}= \tag{2.15}$$

$$=\frac{C_T}{G_T}T\frac{dy}{dt}+T\,y\,\{K\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{C_T}{G_T^2}\frac{\Phi_{INPUT}}{T}\frac{dx}{dt}+\frac{C_T}{G_T^2}\frac{\Phi_{DIST}}{T}\frac{dz}{dt}= \tag{2.16}$$

$$=\frac{C_T}{G_T}\frac{dy}{dt}+y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$x\to X(S);z\to Z(S);y\to Y(S);\frac{d}{dt}\to s.$$

We replace the static terms with static coefficients

$$\frac{C_T}{G_T^2}\frac{\Phi_{INPUT}}{T}=K_{INPUT}\{s\}; \tag{2.17}$$

$$\frac{C_T}{G_T^2}\frac{\Phi_{DIST}}{T}=K_{DIST}\{s\};$$

$$\frac{C_T}{G_T}=\tau_T\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s+1)=K_{INPUT}X(S)s+K_{DIST}Z(S)s. \tag{2.18}$$

We write the transfer functions for the "*input → output*" channel

$$W_{x\to y}(S)=\frac{Y(S)}{X(S)}=\frac{K_{INPUT}s}{\tau_T s+1}\{Unit\}, \tag{2.19}$$

and "*disturbance → output*" channel

$$W_{z\to y}(S)=\frac{Y(S)}{Z(S)}=\frac{K_{DIST}s}{\tau_T s+1}\{Unit\}. \tag{2.20}$$

We replace the Laplace variable s with $j\omega$

$$s\to j\omega.$$

We apply the replaced variable $j\omega$ to equations **2.19** and **2.20**

$$W(j\omega)=\frac{K(j\omega)}{D(j\omega)}, \tag{2.21}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{2.22}$$

being the numerator for equations **2.19** and **2.20**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{2.23}$$

being the denominator for equations **2.19** and **2.20**. We define the corresponding terms *R*, *I*, *M* and *N*

$$R(\omega)=0\,;I(\omega)=\omega\,\tau_T\,;M(\omega)=1\,;N(\omega)=\omega\,\tau_T\,.$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{2.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{\omega\,\tau_T}{\sqrt{1+\omega^2\tau_T^2}}\,. \tag{2.25}$$

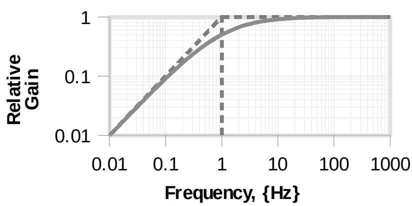


Fig. 2.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **2.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{2.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0+\omega^2\tau_T^2}{1+\omega^2\tau_T^2}, \tag{2.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{\omega\tau_T-0}{1+\omega^2\tau_T^2}. \tag{2.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=\arctan\frac{1}{\omega\tau_T}. \tag{2.29}$$

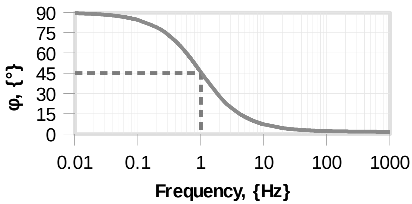


Fig. 2.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **2.19** and **2.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{(\tau_T s+1)}\{s\}. \tag{2.30}$$

In equation **2.30**, we put the τ_T outside the brackets

$$\frac{K_{INPUT/DIST}}{(\tau_T s+1)}=\frac{1}{\tau_T}\frac{K_{INPUT/DIST}}{s+\frac{1}{\tau_T}}\{s\}. \tag{2.31}$$

As can be seen, the pole is s_1=-1/τ_T

$$K_{s_1=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{2.32}$$

We write the function *f(t)* as

$$f(t)=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{2.33}$$

Once equation **2.33** has been multiplied by *T*, the temperature change of the sensitive element when heated, as a function of time equals

$$T(t)=\frac{\Phi_{INPUT/DIST}}{G_T}e^{-\frac{t}{\tau_T}}\{K\}. \tag{2.34}$$

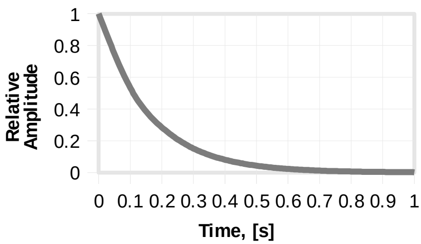


Fig. 2.3 Transient response (τ_T=0.159 s)

3. Rate of temperature change (heating)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{3.1}$$

with

$$dE_{INPUT} = d\Phi_{INPUT} dt \{J\} \tag{3.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = d\Phi_{DIST} dt \{J\} \tag{3.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = C_T d\Psi dt \{J\} \tag{3.4}$$

being the amount of energy coming to the detector from the environment;

$$dE_{OUTPUT} = G_T \Psi dt^2 \{J\} \tag{3.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **3.2-3.5**, we set up an equation for power per second for which one has to divide each parameter by *dt* squared

$$\frac{d\Phi_{INPUT}}{dt} + \frac{d\Phi_{DIST}}{dt} = C_T \frac{d\Psi}{dt} + G_T \Psi \{\frac{W}{s}\}. \tag{3.6}$$

Equation **3.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\frac{d\Phi_{INPUT}}{dt} = [\frac{d\Phi_{INPUT0}}{dt}] + \frac{d\Delta\Phi_{INPUT}}{dt} \{\frac{W}{s}\}; \tag{3.7}$$

$$\frac{d\Phi_{DIST}}{dt} = [\frac{d\Phi_{DIST0}}{dt}] + \frac{d\Delta\Phi_{DIST}}{dt} \{\frac{W}{s}\}; \tag{3.8}$$

$$C_T \frac{d\Psi}{dt} = [C_T \frac{d\Psi_0}{dt}] + C_T \frac{d\Delta\Psi}{dt} \{\frac{W}{s}\}; \tag{3.9}$$

$$G_T \Psi = G_T \Psi_0 + G_T \Delta\Psi \{\frac{W}{s}\}. \tag{3.10}$$

We extract static terms from the family of equations **3.7-3.10**

$$G_T \Psi_0 = 0\{\frac{W}{s}\}. \tag{3.11}$$

Equation **3.11** is a static model. We extract equation **3.11** from the family of equations **3.7-3.10**

$$\frac{d\Delta\Phi_{INPUT}}{dt} + \frac{d\Delta\Phi_{DIST}}{dt} = C_T \frac{d\Delta\Psi}{dt} + G_T \Delta\Psi\{\frac{W}{s}\}. \tag{3.12}$$

Equation **3.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **3.12** by G_T

$$\frac{1}{G_T}\frac{d\Delta\Phi_{INPUT}}{dt} + \frac{1}{G_T}\frac{d\Delta\Phi_{DIST}}{dt} = \tag{3.13}$$

$$= \frac{C_T}{G_T}\frac{d\Delta\Psi}{dt} + \Delta\Psi\{\frac{K}{s}\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}}{G_T}\frac{d\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}}{dt} + \frac{\Phi_{DIST}}{G_T}\frac{d\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}}{dt} = \tag{3.14}$$

$$= \frac{C_T}{G_T}\Psi\frac{d\frac{\Delta\Psi}{\Psi}}{dt} + \Psi\frac{\Delta\Psi}{\Psi}\{\frac{K}{s}\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}} = x; \frac{\Delta\Phi_{DIST}}{\Phi_{DIST}} = z; \frac{\Delta\Psi}{\Psi} = y\{Unit\}.$$

We insert the variables obtained above into equation **3.14**

$$\frac{\Phi_{INPUT}}{G_T}\frac{dx}{dt}+\frac{\Phi_{DIST}}{G_T}\frac{dz}{dt}=\frac{C_T}{G_T}\Psi\frac{dy}{dt}+\Psi\, y\left\{\frac{K}{s}\right\}. \tag{3.15}$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}}{G_T\Psi}\frac{dx}{dt}+\frac{\Phi_{DIST}}{G_T\Psi}\frac{dz}{dt}=\frac{C_T}{G_T}\frac{dy}{dt}+y\{Unit\}. \tag{3.16}$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt}\to s;\, x\to X(S);\, z\to Z(S);\, y\to Y(S).$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}}{G_T\Psi}=K_{INPUT}\{s\}; \tag{3.17}$$

$$\frac{\Phi_{DIST}}{G_T\Psi}=K_{DIST}\{s\};$$

$$\frac{C_T}{G_T}=\tau_T\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s+1)= \tag{3.18}$$

$$=K_{INPUT}X(S)s+K_{DIST}Z(S)s\{Unit\}.$$

We write the transfer functions for the "*input → output*" channel

$$W_{x\to y}(S)=\frac{Y(S)}{X(S)}=\frac{K_{INPUT}s}{\tau_T s+1}\{Unit\}, \tag{3.19}$$

and "*disturbance → output*" channel

$$W_{z\to y}(S)=\frac{Y(S)}{Z(S)}=\frac{K_{DIST}\,s}{\tau_T\,s+1}\{Unit\}. \tag{3.20}$$

We replace the Laplace variable s with $j\omega$

$$s\to j\omega.$$

We apply the replaced variable $j\omega$ to equations **3.19** and **3.20**

$$W(j\omega)=\frac{K(j\omega)}{D(j\omega)}, \tag{3.21}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{3.22}$$

being the numerator for equations **3.19** and **3.20**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{3.23}$$

being the denominator for equations **3.19** and **3.20**. We define the corresponding terms R, I, M and N

$$R(\omega)=0;\,I(\omega)=\omega\,\tau_T;\,M(\omega)=1;\,N(\omega)=\omega\,\tau_T.$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{3.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{\omega\,\tau_T}{\sqrt{1+\omega^2\tau_T^2}}. \tag{3.25}$$

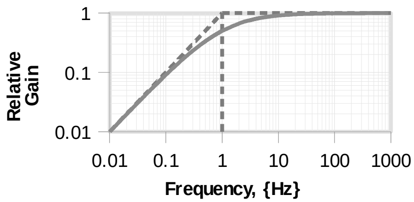


Fig. 3.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **3.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{3.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0+\omega^2\tau_T^2}{1+\omega^2\tau_T^2}, \tag{3.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{\omega\tau_T-0}{1+\omega^2\tau_T^2}. \tag{3.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=\arctan\frac{1}{\omega\tau_T}. \tag{3.29}$$

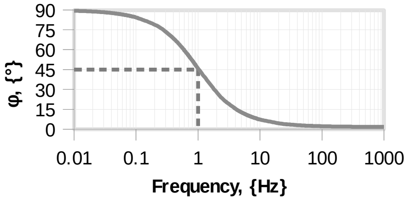


Fig. 3.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **3.19** and **3.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{(\tau_T s+1)}\{s\}. \tag{3.30}$$

In equation **3.30**, we put the τ_T outside the brackets

$$\frac{K_{INPUT/DIST}}{(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{\tau_T}\frac{1}{s+\frac{1}{\tau_T}}\{s\}. \tag{3.31}$$

As can be seen, the pole is s_1=-1/τ_T

$$K_{s=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{3.32}$$

We write the function *f(t)* as

$$K_{s=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{3.33}$$

Once equation **3.33** has been multiplied by *T*, the temperature change of the sensitive element when heated, as a function of time equals

$$\Psi(t)=\frac{\Phi_{INPUT/DIST}}{C_T}e^{-\frac{t}{\tau_T}}\{\frac{K}{s}\}. \tag{3.34}$$

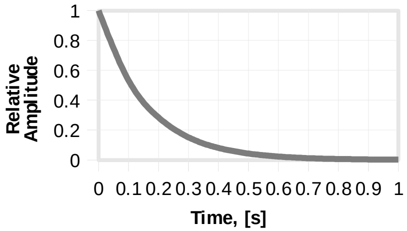


Fig. 3.3 Transient response (τ_T=0.159 s)

4. Rate of temperature change (cooling)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{4.1}$$

with

$$dE_{INPUT} = d^2\Phi_{INPUT}\,dt\,\{J\} \tag{4.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = d^2\Phi_{DIST}\,dt\{J\} \tag{4.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = G_T\,d\,\Psi\,dt^2\{J\} \tag{4.4}$$

being the amount of energy coming to the detector from the environment;

$$dE_{OUTPUT} = \frac{G_T^2}{C_T}\Psi\,dt^3\{J\} \tag{4.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **4.2-4.5**, we set up an equation for power per second squared for which one has to divide each parameter by *dt* cubed

$$\frac{d^2\Phi_{INPUT}}{dt^2}+\frac{d^2\Phi_{DIST}}{dt^2}=G_T\frac{d\Psi}{dt}+\frac{G_T^2}{C_T}\Psi\{\frac{W}{s^2}\}. \quad (4.6)$$

Equation **4.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\frac{d^2\Phi_{INPUT}}{dt^2}=[\frac{d^2\Phi_{INPUT0}}{dt^2}]+\frac{d^2\Delta\Phi_{INPUT}}{dt^2}\{\frac{W}{s^2}\}; \quad (4.7)$$

$$\frac{d^2\Phi_{DIST}}{dt^2}=[\frac{d^2\Phi_{DIST0}}{dt^2}]+\frac{d^2\Delta\Phi_{DIST}}{dt^2}\{\frac{W}{s^2}\}; \quad (4.8)$$

$$G_T\frac{d\Psi}{dt}=[G_T\frac{d\Psi_0}{dt}]+G_T\frac{d\Delta\Psi}{dt}\{\frac{W}{s^2}\}; \quad (4.9)$$

$$\frac{G_T^2}{C_T}\Psi=\frac{G_T^2}{C_T}\Psi_0+\frac{G_T^2}{C_T}\Delta\Psi\{\frac{W}{s^2}\}. \quad (4.10)$$

We extract static terms from the family of equations **4.7-4.10**

$$\frac{G_T^2}{C_T}\Psi_0=0\{\frac{W}{s^2}\}. \quad (4.11)$$

Equation **4.11** is a static model. We extract equation **4.11** from the family of equations **4.7-4.10**

$$\frac{d^2\Delta\Phi_{INPUT}}{dt^2}+\frac{d^2\Delta\Phi_{DIST}}{dt^2}=G_T\frac{d\Delta\Psi}{dt}+\frac{G_T^2}{C_T}\Delta\Psi\{\frac{W}{s^2}\}. \quad (4.12)$$

Equation **4.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **4.12** by G_T^2/C_T

$$\frac{C_T}{G_T^2}\frac{d^2\Delta\Phi_{INPUT}}{dt^2}+\frac{C_T}{G_T^2}\frac{d^2\Delta\Phi_{DIST}}{dt^2}= \quad (4.13)$$

$$=\frac{C_T}{G_T}\frac{d\Delta\Psi}{dt}+\Delta\Psi\left\{\frac{K}{s}\right\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{C_T}{G_T^2}\Phi_{INPUT}\frac{d^2\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}}{dt^2}+\frac{C_T}{G_T^2}\Phi_{DIST}\frac{d^2\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}}{dt^2}= \tag{4.14}$$

$$=\frac{C_T}{G_T}\Psi\frac{d\frac{\Delta\Psi}{\Psi}}{dt}+\Psi\frac{\Delta\Psi}{\Psi}\left\{\frac{K}{s}\right\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}=x\,;\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=z\,;\frac{\Delta\Psi}{\Psi}=y\{Unit\}.$$

We insert the variables obtained above into equation **4.14**

$$\frac{C_T}{G_T^2}\Phi_{INPUT}\frac{d^2x}{dt^2}+\frac{C_T}{G_T^2}\Phi_{DIST}\frac{d^2z}{dt^2}= \tag{4.15}$$

$$=\frac{C_T}{G_T}\Psi\frac{dy}{dt}+\Psi\,y\left\{\frac{K}{s}\right\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{C_T}{G_T^2}\frac{\Phi_{INPUT}}{\Psi}\frac{d^2x}{dt^2}+\frac{C_T}{G_T^2}\frac{\Phi_{DIST}}{\Psi}\frac{d^2z}{dt^2}=\frac{C_T}{G_T}\frac{dy}{dt}+y\{Unit\}. \tag{4.16}$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt}\rightarrow s;\frac{d^2}{dt^2}\rightarrow s^2;x\rightarrow X(S);z\rightarrow Z(S);y\rightarrow Y(S).$$

We replace the static terms with static coefficients

$$\frac{C_T}{G_T^2}\frac{\Phi_{INPUT}}{\Psi}=K_{INPUT}\{s^2\}; \tag{4.17}$$

$$\frac{C_T}{G_T^2}\frac{\Phi_{DIST}}{\Psi}=K_{DIST}\{s^2\};$$

$$\frac{C_T}{G_T}=\tau_T\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s+1)= \tag{4.18}$$

$$=K_{INPUT}X(S)s^2+K_{DIST}Z(S)s^2\{Unit\}.$$

We write the transfer functions for the "*input* → *output*" channel

$$W_{x\rightarrow y}(S)=\frac{Y(S)}{X(S)}=\frac{K_{INPUT}s^2}{\tau_T s+1}\{Unit\}, \tag{4.19}$$

and "*disturbance* → *output*" channel

$$W_{z\rightarrow y}(S)=\frac{Y(S)}{Z(S)}=\frac{K_{DIST}s^2}{\tau_T s+1}\{Unit\}. \tag{4.20}$$

We replace the Laplace variable *s* with $j\omega$

$$s\rightarrow j\omega.$$

We apply the replaced variable $j\omega$ to equations **4.19** and **4.20**

$$W(j\omega)=\frac{K(j\omega)}{D(j\omega)}, \tag{4.21}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{4.22}$$

being the numerator for equations **4.19** and **4.20**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{4.23}$$

being the denominator for equations **4.19** and **4.20**. We define the corresponding terms *R*, *I*, *M* and *N*

$$R(\omega)=-\omega^2\tau_T^2;\, I(\omega)=0;\, M(\omega)=1;\, N(\omega)=\omega\tau_T.$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{4.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{\omega^2\tau_T^2}{\sqrt{1+\omega^2\tau_T^2}}. \tag{4.25}$$

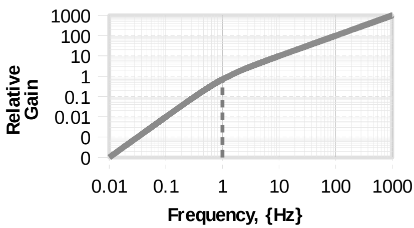


Fig. 4.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **4.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{4.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{-\omega^2\tau_T^2+0}{1+\omega^2\tau_T^2}, \tag{4.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0+\omega^3\tau_T^3}{1+\omega^2\tau_T^2}. \tag{4.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=-\arctan\omega\tau_T. \tag{4.29}$$

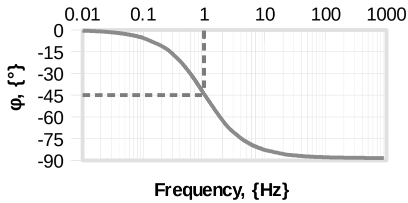


Fig. 4.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **4.19** and **4.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}\,s}{(\tau_T s+1)}\{s\}. \tag{4.30}$$

In equation **4.30**, we put the τ_T outside the brackets

$$\frac{K_{INPUT/DIST}\,s}{(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{\tau_T}\frac{s}{s+\frac{1}{\tau_T}}\{s\}. \tag{4.31}$$

As can be seen, the pole is s_1=-1/τ_T

$$K_{s=-\frac{1}{\tau_T}} = -\frac{1}{\tau_T^2} K_{INPUT/DIST} e^{-\frac{t}{\tau_T}} \{Unit\}. \tag{4.32}$$

We write the function *f(t)* as

$$f(t) = -\frac{K_{INPUT/DIST}}{\tau_T^2} e^{-\frac{t}{\tau_T}} \{Unit\}. \tag{4.33}$$

Once equation **4.33** has been multiplied by *T*, the temperature change of the sensitive element when heated, as a function of time equals

$$\Psi(t) = -\frac{\Phi_{INPUT/DIST}}{C_T} e^{-\frac{t}{\tau_T}} \left\{\frac{K}{s}\right\}. \tag{4.34}$$

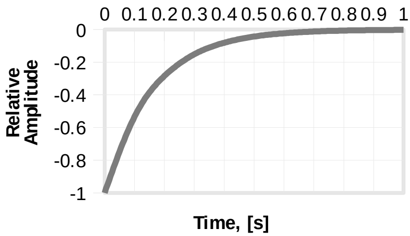


Fig. 4.3 Transient response (τ_T=0.159 s)

5. Alternating pyroelectric current (heating)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{5.1}$$

with

$$dE_{INPUT} = d\Phi_{INPUT} dt \{J\} \tag{5.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = d\,\Phi_{DIST}\,dt\{J\} \tag{5.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = \frac{C_T}{p_{pyro\,1} A_{pyro}} dI_{pyro\,1}\,dt\{J\} \tag{5.4}$$

being the amount of energy coming to the detector from the environment;

$$dE_{OUTPUT} = \frac{G_T}{p_{pyro\,1} A_{pyro}} I_{pyro\,1}\,dt^2\{J\} \tag{5.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **5.2-5.5**, we set up an equation for power per second for which one has to divide each parameter by *dt* squared

$$\frac{d\Phi_{INPUT}}{dt} + \frac{d\Phi_{DIST}}{dt} = \tag{5.6}$$

$$= \frac{C_T}{p_{pyro\,1} A_{pyro}} \frac{dI_{pyro\,1}}{dt} + \frac{G_T}{p_{pyro\,1} A_{pyro}} I_{pyro\,1}\left\{\frac{W}{s}\right\}.$$

Equation **5.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\frac{d\Phi_{INPUT}}{dt} = \left[\frac{d\Phi_{INPUT\,0}}{dt}\right] + \frac{d\,\Delta\Phi_{INPUT}}{dt}\left\{\frac{W}{s}\right\}; \tag{5.7}$$

$$\frac{d\Phi_{DIST}}{dt} = \left[\frac{d\Phi_{DIST\,0}}{dt}\right] + \frac{d\,\Delta\Phi_{DIST}}{dt}\left\{\frac{W}{s}\right\}; \tag{5.8}$$

$$\frac{C_T}{p_{pyro1} A_{pyro}} \frac{dI_{pyro1}}{dt} = \tag{5.9}$$

$$= [\frac{C_T}{p_{pyro1} A_{pyro}} \frac{dI_{pyro10}}{dt}] + \frac{C_T}{p_{pyro1} A_{pyro}} \frac{d\Delta I_{pyro1}}{dt} \{\frac{W}{s}\};$$

$$\frac{G_T}{p_{pyro1} A_{pyro}} I_{pyro1} = \tag{5.10}$$

$$= \frac{G_T}{p_{pyro1} A_{pyro}} I_{pyro10} + \frac{G_T}{p_{pyro1} A_{pyro}} \Delta I_{pyro1} \{\frac{W}{s}\}.$$

We extract static terms from the family of equations **5.7-5.10**

$$\frac{G_T}{p_{pyro} A_{pyro}} I_{pyro10} = 0 \{\frac{W}{s}\}. \tag{5.11}$$

Equation **5.11** is a static model. We extract equation **5.11** from the family of equations **5.7-5.10**

$$\frac{d\Delta\Phi_{INPUT}}{dt} + \frac{d\Delta\Phi_{DIST}}{dt} = \tag{5.12}$$

$$= \frac{C_T}{p_{pyro1} A_{pyro}} \frac{d\Delta I_{pyro1}}{dt} + \frac{G_T}{p_{pyro1} A_{pyro}} \Delta I_{pyro1} \{\frac{W}{s}\}.$$

Equation **5.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **5.12** by G_T/($p_{pyro1}A_{pyro}$)

$$\frac{p_{pyro\,1} A_{pyro}}{G_T}\frac{d\,\Delta\,\Phi_{INPUT}}{dt}+\frac{p_{pyro\,1} A_{pyro}}{G_T}\frac{d\,\Delta\,\Phi_{DIST}}{dt}= \tag{5.13}$$

$$=\frac{C_T}{G_T}\frac{d\,\Delta\,I_{pyro\,1}}{dt}+\Delta\,I_{pyro\,1}\{A\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}\,p_{pyro\,1} A_{pyro}}{G_T}\frac{d\frac{\Delta\,\Phi_{INPUT}}{\Phi_{INPUT}}}{dt}+ \tag{5.14}$$

$$+\frac{\Phi_{DIST}\,p_{pyro\,1} A_{pyro}}{G_T}\frac{d\frac{\Delta\,\Phi_{DIST}}{\Phi_{DIST}}}{dt}=$$

$$=\frac{C_T}{G_T}I_{pyro\,1}\frac{d\frac{\Delta\,I_{pyro\,1}}{I_{pyro\,1}}}{dt}+I_{pyro\,1}\frac{\Delta\,I_{pyro\,1}}{I_{pyro\,1}}\{A\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\,\Phi_{INPUT}}{\Phi_{INPUT}}=x\,;\frac{\Delta\,\Phi_{DIST}}{\Phi_{DIST}}=z\,;\frac{\Delta\,I_{pyro\,1}}{I_{pyro\,1}}=y\{Unit\}.$$

We insert the variables obtained above into equation **5.14**

$$\frac{\Phi_{INPUT}\,p_{pyro\,1} A_{pyro}}{G_T}\frac{dx}{dt}+\frac{\Phi_{DIST}\,p_{pyro\,1} A_{pyro}}{G_T}\frac{dz}{dt}= \tag{5.15}$$

$$=\frac{C_T}{G_T} I_{pyro1} \frac{dy}{dt} + I_{pyro1}\, y\{A\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}\, p_{pyro1} A_{pyro}}{G_T I_{pyro1}} \frac{dx}{dt} + \frac{\Phi_{DIST}\, p_{pyro1} A_{pyro}}{G_T I_{pyro1}} \frac{dz}{dt} = \tag{5.16}$$

$$=\frac{C_T}{G_T} \frac{dy}{dt} + y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt} \to s;\, x \to X(S);\, z \to Z(S);\, y \to Y(S).$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}\, p_{pyro1} A_{pyro}}{G_T I_{pyro1}} = K_{INPUT}\{s\}; \tag{5.17}$$

$$\frac{\Phi_{DIST}\, p_{pyro1} A_{pyro}}{G_T I_{pyro1}} = K_{DIST}\{s\};$$

$$\frac{C_T}{G_T} = \tau_T\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s + 1) = K_{INPUT} X(S) s + K_{DIST} Z(S) s. \tag{5.18}$$

We write the transfer functions for the "*input → output*" channel

$$W_{x\to y}(S) = \frac{Y(S)}{X(S)} = \frac{K_{INPUT}\, s}{\tau_T s + 1}\{Unit\}, \tag{5.19}$$

and "*disturbance* → *output*" channel

$$W_{z \to y}(S) = \frac{Y(S)}{Z(S)} = \frac{K_{DIST}\, s}{\tau_T\, s + 1}\{Unit\}. \tag{5.20}$$

We replace the Laplace variable s with $j\omega$

$$s \to j\omega.$$

We apply the replaced variable $j\omega$ to equations **5.19** and **5.20**

$$W(j\omega) = \frac{K(j\omega)}{D(j\omega)}, \tag{5.21}$$

with

$$K(j\omega) = R(\omega) + jI(\omega) \tag{5.22}$$

being the numerator for equations **5.19** and **5.20**;

$$D(j\omega) = M(\omega) + jN(\omega) \tag{5.23}$$

being the denominator for equations **5.19** and **5.20**. We define the corresponding terms R, I, M and N

$$R(\omega) = 0;\ I(\omega) = \omega\tau_T;\ M(\omega) = 1;\ N(\omega) = \omega\tau_T.$$

We define the amplitude-frequency response

$$A(\omega) = \frac{A_{OUTPUT}}{A_{INPUT}} = |W(j\omega)|, \tag{5.24}$$

or

$$A(\omega) = \frac{\sqrt{R^2(\omega) + I^2(\omega)}}{\sqrt{M^2(\omega) + N^2(\omega)}} = \frac{\omega\tau_T}{\sqrt{1 + \omega^2\tau_T^2}}. \tag{5.25}$$

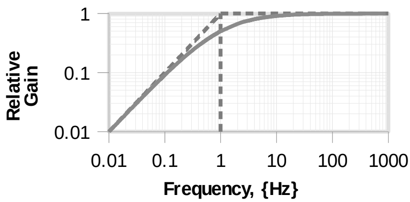


Fig. 5.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **5.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{5.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0+\omega^2\tau_T^2}{1+\omega^2\tau_T^2}, \tag{5.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{\omega\tau_T-0}{1+\omega^2\tau_T^2}. \tag{5.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=\arctan\frac{1}{\omega\tau_T}. \tag{5.29}$$

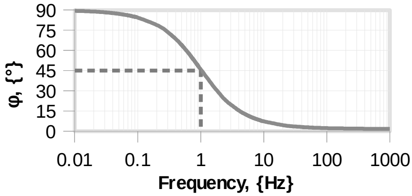


Fig. 5.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **5.19** and **5.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{(\tau_T s+1)}\{s\}. \tag{5.30}$$

In equation **5.30**, we put the τ_T outside the brackets

$$\frac{K_{INPUT/DIST}}{(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{\tau_T}\frac{1}{s+\frac{1}{\tau_T}}\{s\}. \tag{5.31}$$

As can be seen, the pole is s_1=-1/τ_T

$$K_{s=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{5.32}$$

We write the function *f(t)* as

$$f(t)=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{5.33}$$

Once equation **5.33** has been multiplied by I_{pyro}, the temperature change of the sensitive element when heated, as a function of time equals

$$I_{pyro1}(t)=\frac{\Phi_{INPUT/DIST}\,p_{pyro1}\,A_{pyro}}{C_T}e^{-\frac{t}{\tau_T}}\{A\}. \tag{5.34}$$

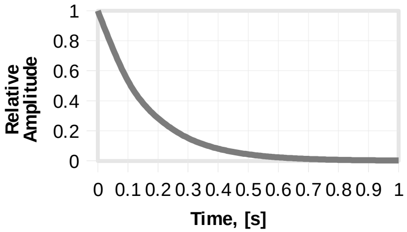


Fig. 5.3 Transient response (τ_T=0.159 s)

6. Alternating pyroelectric current (cooling)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{6.1}$$

with

$$dE_{INPUT} = d^2\Phi_{INPUT}\, dt\,\{J\} \tag{6.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = d^2\Phi_{DIST}\, dt\{J\} \tag{6.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = \frac{G_T}{p_{pyro\,1} A_{pyro}}\, dI_{pyro\,1}\, dt^2\{J\} \tag{6.4}$$

being the amount of energy coming to the detector from the environment;

$$dE_{OUTPUT} = \frac{G_T^2}{C_T\, p_{pyro\,1} A_{pyro}}\, I_{pyro\,1}\, dt^3\{J\} \tag{6.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **6.2-6.5**, we set up an equation for power for which one has to divide each parameter by *dt* cubed

$$\frac{d^2\Phi_{INPUT}}{dt^2}+\frac{d^2\Phi_{DIST}}{dt^2}= \tag{6.6}$$

$$=\frac{G_T}{p_{pyro1}A_{pyro}}\frac{dI_{pyro1}}{dt}+\frac{G_T^2}{C_T p_{pyro1}A_{pyro}}I_{pyro1}\{\frac{W}{s^2}\}.$$

Equation **6.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\frac{d^2\Phi_{INPUT}}{dt^2}=[\frac{d^2\Phi_{INPUT0}}{dt^2}]+\frac{d^2\Delta\Phi_{INPUT}}{dt^2}\{\frac{W}{s^2}\}; \tag{6.7}$$

$$\frac{d^2\Phi_{DIST}}{dt^2}=[\frac{d^2\Phi_{DIST0}}{dt^2}]+\frac{d^2\Delta\Phi_{DIST}}{dt^2}\{\frac{W}{s^2}\}; \tag{6.8}$$

$$\frac{G_T}{p_{pyro1}A_{pyro}}\frac{dI_{pyro1}}{dt}= \tag{6.9}$$

$$=[\frac{G_T}{p_{pyro1}A_{pyro}}\frac{dI_{pyro10}}{dt}]+\frac{G_T}{p_{pyro1}A_{pyro}}\frac{d\Delta I_{pyro1}}{dt}\{\frac{W}{s^2}\};$$

$$\frac{G_T^2}{C_T p_{pyro1}A_{pyro}}I_{pyro1}= \tag{6.10}$$

$$=\frac{G_T^2}{C_T p_{pyro1}A_{pyro}}I_{pyro10}+\frac{G_T^2}{C_T p_{pyro1}A_{pyro}}\Delta I_{pyro1}\{\frac{W}{s^2}\}.$$

We extract static terms from the family of equations **6.7-6.10**

$$\frac{G_T^2}{C_T p_{pyro1} A_{pyro}} I_{pyro\,10} = 0 \left\{ \frac{W}{s^2} \right\}. \tag{6.11}$$

Equation **6.11** is a static model. We extract equation **6.11** from the family of equations **6.7-6.10**

$$\frac{d^2 \Delta \Phi_{INPUT}}{dt^2} + \frac{d^2 \Delta \Phi_{DIST}}{dt^2} = \tag{6.12}$$

$$= \frac{G_T}{p_{pyro1} A_{pyro}} \frac{d \Delta I_{pyro1}}{dt} + \frac{G_T^2}{C_T p_{pyro1} A_{pyro}} \Delta I_{pyro1} \left\{ \frac{W}{s^2} \right\}.$$

Equation **6.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **6.12** by $G_T^2/(C_T p_{pyro1} A_{pyro})$

$$\frac{C_T p_{pyro1} A_{pyro}}{G_T^2} \frac{d^2 \Delta \Phi_{INPUT}}{dt^2} + \tag{6.13}$$

$$\frac{C_T p_{pyro1} A_{pyro}}{G_T^2} \frac{d^2 \Delta \Phi_{DIST}}{dt^2} =$$

$$= \frac{C_T}{G_T} \frac{d \Delta I_{pyro1}}{dt} + \Delta I_{pyro1} \{A\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\,\Phi_{INPUT}\,\frac{d^2\dfrac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}}{dt^2}+ \tag{6.14}$$

$$+\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\,\Phi_{DIST}\,\frac{d^2\dfrac{\Delta\Phi_{DIST}}{\Phi_{DIST}}}{dt^2}=$$

$$=\frac{C_T}{G_T}\,I_{pyro1}\,\frac{d\dfrac{\Delta I_{pyro1}}{I_{pyro1}}}{dt}+I_{pyro1}\,\frac{\Delta I_{pyro1}}{I_{pyro1}}\{A\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}=x;\ \frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=z;\ \frac{\Delta I_{pyro1}}{I_{pyro1}}=y\{Unit\}.$$

We insert the variables obtained above into equation **6.14**

$$\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\,\Phi_{INPUT}\,\frac{d^2x}{dt^2}+ \tag{6.15}$$

$$+\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\,\Phi_{DIST}\,\frac{d^2z}{dt^2}=$$

$$=\frac{C_T}{G_T}\,I_{pyro1}\,\frac{dy}{dt}+I_{pyro1}\,y\{A\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\frac{\Phi_{INPUT}}{I_{pyro1}}\frac{d^2 x}{dt^2}+ \tag{6.16}$$

$$+\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\frac{\Phi_{DIST}}{I_{pyro1}}\frac{d^2 z}{dt^2}=$$

$$=\frac{C_T}{G_T}\frac{dy}{dt}+y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt}\rightarrow s;\frac{d^2}{dt^2}\rightarrow s^2;x\rightarrow X(S);z\rightarrow Z(S);y\rightarrow Y(S).$$

We replace the static terms with static coefficients

$$\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\frac{\Phi_{INPUT}}{I_{pyro1}}=K_{INPUT}\{s^2\}; \tag{6.17}$$

$$\frac{C_T\, p_{pyro1}\, A_{pyro}}{G_T^2}\frac{\Phi_{DIST}}{I_{pyro1}}=K_{DIST}\{s^2\};$$

$$\frac{C_T}{G_T}=\tau_T\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s+1)=K_{INPUT}\,X(S)s^2+K_{DIST}\,Z(S)s^{2.} \tag{6.18}$$

We write the transfer functions for the “*input → output*” channel

$$W_{x\to y}(S)=\frac{Y(S)}{X(S)}=\frac{K_{INPUT}\,s^2}{\tau_T s+1}\{Unit\}, \tag{6.19}$$

and "*disturbance* → *output*" channel

$$W_{z\to y}(S)=\frac{Y(S)}{Z(S)}=\frac{K_{DIST}\,s^2}{\tau_T s+1}\{Unit\}. \tag{6.20}$$

We replace the Laplace variable *s* with *jω*

$$s\to j\omega.$$

We apply the replaced variable *jω* to equations **6.19** and **6.20**

$$W(j\omega)=\frac{K(j\omega)}{D(j\omega)}, \tag{6.21}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{6.22}$$

being the numerator for equations **6.19** and **6.20**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{6.23}$$

being the denominator for equations **6.19** and **6.20**. We define the corresponding terms *R*, *I*, *M* and *N*

$$R(\omega)=-\omega^2\tau_T^2;\,I(\omega)=0;\,M(\omega)=1;\,N(\omega)=\omega\,\tau_T.$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{6.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{\omega^2\tau_T^2}{\sqrt{1+\omega^2\tau_T^2}}. \tag{6.25}$$

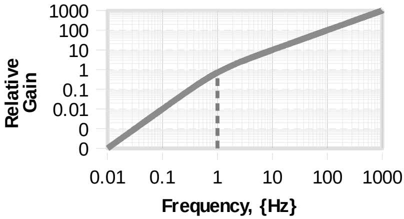


Fig. 6.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **6.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{6.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{-\omega^2\tau_T^2+0}{1+\omega^2\tau_T^2}, \tag{6.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0+\omega^3\tau_T^3}{1+\omega^2\tau_T^2}. \tag{6.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=-\arctan\omega\tau_T. \tag{6.29}$$

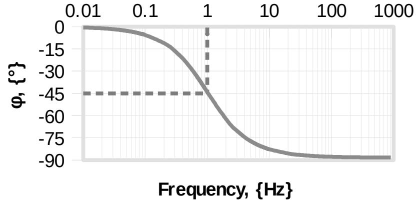


Fig. 6.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **6.19** and **6.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}\,s}{(\tau_T s+1)}\{s\}. \tag{6.30}$$

In equation **6.30**, we put the τ_T outside the brackets

$$\frac{K_{INPUT/DIST}\,s}{(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{\tau_T}\frac{s}{s+\frac{1}{\tau_T}}\{s\}. \tag{6.31}$$

As can be seen, the pole is s_1=-1/τ_T

$$K_{s=-\frac{1}{\tau_T}}=-\frac{1}{\tau_T^2}K_{INPUT/DIST}\,e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{6.32}$$

We write the function *f(t)* as

$$f(t)=-\frac{1}{\tau_T^2}K_{INPUT/DIST}\,e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{6.33}$$

Once equation **6.33** has been multiplied by *T*, the temperature change of the sensitive element when heated, as a function of time equals

$$I_{pyro\,1}(t)=-\frac{\Phi_{INPUT/DIST}\,p_{pyro\,1}A_{pyro}}{C_T}e^{-\frac{t}{\tau_T}}\{A\}. \tag{6.34}$$

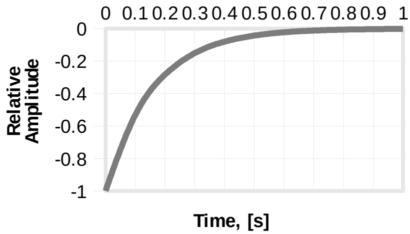


Fig. 6.3 Transient response (τ_T=0.159 s)

7. Direct pyroelectric current (heating)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{7.1}$$

with

$$dE_{INPUT} = \Phi_{INPUT}\, dt\{J\} \tag{7.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = \Phi_{DIST}\, dt\{J\} \tag{7.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = \frac{C_T^2}{G_T\, p_{pyro3}\, A_{pyro}} dI_{pyro3}\{J\} \tag{7.4}$$

being the amount of energy stored in the volume of the sensitive element;

$$dE_{OUTPUT} = \frac{C_T}{p_{pyro3}\, A_{pyro}} I_{pyro3}\, dt\{J\} \tag{7.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **7.2-7.5**, we set up an equation for power for which one has to divide each parameter by *dt*

$$\Phi_{INPUT} + \Phi_{DIST} = \tag{7.6}$$

$$= \frac{C_T^2}{G_T p_{pyro3} A_{pyro}} \frac{dI_{pyro3}}{dt} + \frac{C_T}{p_{pyro3} A_{pyro}} I_{pyro3} \{W\}.$$

Equation **7.6** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\Phi_{INPUT} = \Phi_{INPUT\,0} + \Delta\,\Phi_{INPUT} \{W\};\tag{7.7}$$

$$\Phi_{DIST} = \Phi_{DIST\,0} + \Delta\,\Phi_{DIST} \{W\};\tag{7.8}$$

$$\frac{C_T^2}{G_T p_{pyro\,3} A_{pyro}} \frac{dI_{pyro3}}{dt} = [\frac{C_T^2}{G_T p_{pyro3} A_{pyro}} \frac{dI_{pyro30}}{dt}] + \tag{7.9}$$

$$+ \frac{C_T^2}{G_T p_{pyro\,3} A_{pyro}} \frac{d\,\Delta\, I_{pyro\,3}}{dt} \{W\};$$

$$\frac{C_T}{p_{pyro\,3} A_{pyro}} I_{pyro3} = \tag{7.10}$$

$$= \frac{C_T}{p_{pyro3} A_{pyro}} I_{pyro30} + \frac{C_T}{p_{pyro3} A_{pyro}} \Delta\, I_{pyro\,3} \{W\}.$$

We extract static terms from the family of equations **7.7**-**7.10**

$$\frac{C_T}{p_{pyro\,3} A_{pyro}} I_{pyro30} = \Phi_{INPUT\,0} + \Phi_{DIST0} \{W\}.\tag{7.11}$$

Equation **7.11** is a static model. We extract equation **7.11** from the family of equations **7.7-7.10**

$$\Delta\Phi_{INPUT}+\Delta\Phi_{DIST}=\frac{C_T^2}{G_T\,p_{pyro3}A_{pyro}}\frac{d\Delta I_{pyro3}}{dt}+ \tag{7.12}$$

$$+\frac{C_T}{p_{pyro3}A_{pyro}}\Delta I_{pyro3}\{W\}.$$

Equation **7.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **7.12** by $C_T/(p_{pyro3}A_{pyro})$

$$\frac{p_{pyro3}A_{pyro}}{C_T}\Delta\Phi_{INPUT}+\frac{p_{pyro3}A_{pyro}}{C_T}\Delta\Phi_{DIST}= \tag{7.13}$$

$$=\frac{C_T}{G_T}\frac{d\Delta I_{pyro3}}{dt}+\Delta I_{pyro3}\{A\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}\,p_{pyro3}A_{pyro}}{C_T}\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}+ \tag{7.14}$$

$$+\frac{\Phi_{DIST}\,p_{pyro3}A_{pyro}}{C_T}\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=$$

$$=\frac{C_T}{G_T}I_{pyro3}\frac{d\dfrac{\Delta I_{pyro3}}{I_{pyro3}}}{dt}+I_{pyro3}\frac{\Delta I_{pyro3}}{I_{pyro3}}\{A\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}=x;\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=z;\frac{\Delta I_{pyro3}}{I_{pyro3}}=y\{Unit\}.$$

We insert the variables obtained above into equation **7.14**

$$\frac{\Phi_{INPUT}\,p_{pyro3}A_{pyro}}{C_T}x+\frac{\Phi_{DIST}\,p_{pyro3}A_{pyro}}{C_T}z= \tag{7.15}$$

$$=\frac{C_T}{G_T}I_{pyro3}\frac{dy}{dt}+I_{pyro3}\,y\{A\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}\,p_{pyro3}A_{pyro}}{C_T I_{pyro3}}x+\frac{\Phi_{DIST}\,p_{pyro3}A_{pyro}}{C_T I_{pyro3}}z= \tag{7.16}$$

$$=\frac{C_T}{G_T}\frac{dy}{dt}+y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt}\rightarrow s;x\rightarrow X(S);z\rightarrow Z(S);y\rightarrow Y(S).$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}\,p_{pyro3}A_{pyro}}{C_T I_{pyro3}}=K_{INPUT}\{Unit\}; \tag{7.17}$$

$$\frac{\Phi_{DIST}\,p_{pyro3}A_{pyro}}{C_T I_{pyro3}}=K_{DIST}\{Unit\};$$

$$\frac{C_T}{G_T} = \tau_T \{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s + 1) = K_{INPUT} X(S) + K_{DIST} Z(S). \tag{7.18}$$

We write the transfer functions for the "*input* → *output*" channel

$$W_{x \to y}(S) = \frac{Y(S)}{X(S)} = \frac{K_{INPUT}}{\tau_T s + 1}\{Unit\}, \tag{7.19}$$

and "*disturbance* → *output*" channel

$$W_{z \to y}(S) = \frac{Y(S)}{Z(S)} = \frac{K_{DIST}}{\tau_T s + 1}\{Unit\}. \tag{7.20}$$

We replace the Laplace variable s with $j\omega$

$$s \to j\omega.$$

We apply the replaced variable $j\omega$ to equations **7.19** and **7.20**

$$W(j\omega) = \frac{K(j\omega)}{D(j\omega)}, \tag{7.21}$$

with

$$K(j\omega) = R(\omega) + jI(\omega), \tag{7.22}$$

being the numerator for equations **7.19** and **7.20**;

$$D(j\omega) = M(\omega) + jN(\omega), \tag{7.23}$$

being the denominator for equations **7.19** and **7.20**. We define the corresponding terms R, I, M and N

$$R(\omega) = 1;\ I(\omega) = 0;\ M(\omega) = 1;\ N(\omega) = \omega\tau_T.$$

We define the amplitude-frequency response

$$A(\omega) = \frac{A_{OUTPUT}}{A_{INPUT}} = |W(j\omega)|, \tag{7.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{1}{\sqrt{1+\omega^2\tau_T^2}}. \tag{7.25}$$

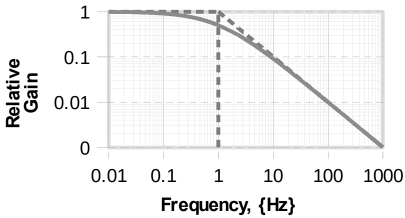


Fig. 7.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator of equation **7.21** by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{7.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{1+0}{1+\omega^2\tau_T^2}, \tag{7.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0-\omega\tau_T}{1+\omega^2\tau_T^2}. \tag{7.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=-\arctan\omega\tau_T. \tag{7.29}$$

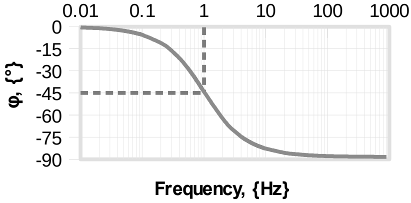


Fig. 7.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **7.19** and **7.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{s(\tau_T s+1)}\{s\}. \tag{7.30}$$

We resolve equation **7.30** into partial fractions

$$\frac{K_{INPUT/DIST}}{s(\tau_T s+1)}=\frac{A}{s}+\frac{B}{\tau_T s+1}. \tag{7.31}$$

Upon application of some mathematical transformations (Equations **7.31-7.36** where each variable is unitless), such as

$$A\tau_T s+A+Bs=K_{INPUT/DIST}, \tag{7.32}$$

and

$$s^1: A\tau_T+B=0, \tag{7.33}$$

$$s^0: A=K_{INPUT/DIST},$$

with

$$B=-K_{INPUT/DIST}\tau_T. \tag{7.34}$$

We write equation **7.31** in the form

$$\frac{K_{INPUT/DIST}}{s(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{s}-\frac{K_{INPUT/DIST}\tau_T}{\tau_T s+1}. \tag{7.35}$$

and after manipulations

$$\frac{K_{INPUT/DIST}}{s(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{s}-\frac{K_{INPUT/DIST}}{s+\frac{1}{\tau_T}}. \tag{7.36}$$

As can be seen, the poles are s_1=0

$$K_{s_1=0}=e^{st}=K_{INPUT/DIST}\{Unit\}, \tag{7.37}$$

and s_2=-1/τ_T

$$K_{s_2=-\frac{1}{\tau_T}}=K_{INPUT/DIST}\,e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{7.38}$$

We write the function *f(t)* as

$$f(t)=K_{INPUT/DIST}\left(1-e^{-\frac{t}{\tau_T}}\right)\{Unit\}. \tag{7.39}$$

Once equation **7.39** has been multiplied by I_{pyro}, the temperature change of the sensitive element when heated, as a function of time equals

$$I_{pyro3}(t)=\frac{\Phi_{INPUT}\,p_{pyro3}\,A_{pyro}}{C_T}\left(1-e^{-\frac{t}{\tau_T}}\right)\{A.\} \tag{7.40}$$

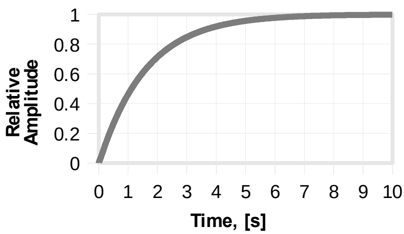


Fig. 7.3 Transient response (τ_T=0.159 s)

8. Direct pyroelectric current (cooling)

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT} + dE_{DIST} - dE_{VOL} - dE_{OUTPUT} = 0\{J\}, \tag{8.1}$$

with

$$dE_{INPUT} = d\Phi_{INPUT} dt\{J\} \tag{8.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST} = d\Phi_{DIST} dt\{J\} \tag{8.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL} = \frac{C_T}{p_{pyro3} A_{pyro}} dI_{pyro3} dt\{J\} \tag{8.4}$$

being the amount of energy stored in the volume of the sensitive element;

$$dE_{OUTPUT} = \frac{G_T}{p_{pyro3} A_{pyro}} I_{pyro3} dt^2\{J\} \tag{8.5}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **8.2-8.5**, we set up an equation for power for which one has to divide each parameter by *dt* squared

$$\frac{d\Phi_{INPUT}}{dt} + \frac{d\Phi_{DIST}}{dt} = \tag{8.6}$$

$$= \frac{C_T}{p_{pyro3} A_{pyro}} \frac{dI_{pyro3}}{dt} + \frac{G_T}{p_{pyro3} A_{pyro}} I_{pyro3} \left\{\frac{W}{s}\right\}.$$

Equation **8.6** is a non-linear mathematical model. We linearize it by applying a

Taylor series

$$\frac{d\Phi_{INPUT}}{dt}=[\frac{d\Phi_{INPUT0}}{dt}]+\frac{d\Delta\Phi_{INPUT}}{dt}\{\frac{W}{s}\}; \quad (8.7)$$

$$\frac{d\Phi_{DIST}}{dt}=[\frac{d\Phi_{DIST0}}{dt}]+\frac{d\Delta\Phi_{DIST}}{dt}\{\frac{W}{s}\}; \quad (8.8)$$

$$\frac{C_T}{p_{pyro3}A_{pyro}}\frac{dI_{pyro3}}{dt}=[\frac{C_T}{p_{pyro3}A_{pyro}}\frac{dI_{pyro30}}{dt}]+ \quad (8.9)$$

$$+\frac{C_T}{p_{pyro3}A_{pyro}}\frac{d\Delta I_{pyro3}}{dt}\{\frac{W}{s}\};$$

$$\frac{G_T}{p_{pyro3}A_{pyro}}I_{pyro3}= \quad (8.10)$$

$$=\frac{G_T}{p_{pyro3}A_{pyro}}I_{pyro30}+\frac{G_T}{p_{pyro3}A_{pyro}}\Delta I_{pyro3}\{\frac{W}{s}\}.$$

We extract static terms from the family of equations **8.7-8.10**

$$\frac{G_T}{p_{pyro3}A_{pyro}}I_{pyro30}=0\{\frac{W}{s}\}. \quad (8.11)$$

Equation **8.11** is a static model. We extract equation **8.11** from the family of equations **8.7-8.10**

$$\frac{d\Delta\Phi_{INPUT}}{dt}+\frac{d\Delta\Phi_{DIST}}{dt}=\frac{C_T}{p_{pyro3}A_{pyro}}\frac{d\Delta I_{pyro3}}{dt}+ \quad (8.12)$$

$$+\frac{G_T}{p_{pyro\,3}A_{pyro}}\Delta I_{pyro3}\left\{\frac{W}{s}\right\}.$$

Equation **8.12** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **8.12** by $G_T/(p_{pyro3}A_{pyro})$

$$\frac{p_{pyro\,3}A_{pyro}}{G_T}\frac{d\,\Delta\,\Phi_{INPUT}}{dt}+\frac{p_{pyro3}A_{pyro}}{G_T}\frac{d\,\Delta\,\Phi_{DIST}}{dt}= \tag{8.13}$$

$$=\frac{C_T}{G_T}\frac{d\,\Delta\,I_{pyro3}}{dt}+\Delta I_{pyro\,3}\{A\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}\,p_{pyro3}A_{pyro}}{G_T}\frac{d\,\dfrac{\Delta\,\Phi_{INPUT}}{\Phi_{INPUT}}}{dt}+ \tag{8.14}$$

$$+\frac{\Phi_{DIST}\,p_{pyro\,3}A_{pyro}}{G_T}\frac{d\,\dfrac{\Delta\,\Phi_{DIST}}{\Phi_{DIST}}}{dt}=$$

$$=\frac{C_T}{G_T}I_{pyro3}\frac{d\,\dfrac{\Delta\,I_{pyro3}}{I_{pyro\,3}}}{dt}+I_{pyro3}\frac{\Delta\,I_{pyro3}}{I_{pyro3}}\{A\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}=x\,;\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=z\,;\frac{\Delta I_{pyro3}}{I_{pyro3}}=y\{Unit\}.$$

We insert the variables obtained above into equation **8.14**

$$\frac{\Phi_{INPUT}\,p_{pyro3}\,A_{pyro}}{G_T}\frac{dx}{dt}+\frac{\Phi_{DIST}\,p_{pyro3}\,A_{pyro}}{G_T}\frac{dz}{dt}= \tag{8.15}$$

$$=\frac{C_T}{G_T}I_{pyro3}\frac{dy}{dt}+I_{pyro3}\,y\{A\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}\,p_{pyro3}\,A_{pyro}}{G_T\,I_{pyro3}}\frac{dx}{dt}+\frac{\Phi_{DIST}\,p_{pyro3}\,A_{pyro}}{G_T\,I_{pyro3}}\frac{dz}{dt}= \tag{8.16}$$

$$=\frac{C_T}{G_T}\frac{dy}{dt}+y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt}\rightarrow s\,;x\rightarrow X(S);z\rightarrow Z(S);y\rightarrow Y(S).$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}\,p_{pyro3}\,A_{pyro}}{G_T\,I_{pyro3}}=K_{INPUT}\{s\}; \tag{8.17}$$

$$\frac{\Phi_{DIST}\,p_{pyro3}\,A_{pyro}}{G_T\,I_{pyro3}}=K_{DIST}\{s\};$$

$$\frac{C_T}{G_T}=\tau_T\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T s+1)=K_{INPUT}X(S)s+K_{DIST}Z(S)s. \tag{8.18}$$

We write the transfer functions for the "*input* → *output*" channel

$$W_{x\to y}(S)=\frac{Y(S)}{X(S)}=\frac{K_{INPUT}s}{\tau_T s+1}\{Unit\}, \tag{8.19}$$

and "*disturbance* → *output*" channel

$$W_{z\to y}(S)=\frac{Y(S)}{Z(S)}=\frac{K_{DIST}s}{\tau_T s+1}\{Unit\}. \tag{8.20}$$

We replace the Laplace variable *s* with *jω*

$$s\to j\omega.$$

We apply the replaced variable *jω* to equations **8.19** and **8.20**

$$W(j\omega)=\frac{K(j\omega)}{D(j\omega)}, \tag{8.21}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{8.22}$$

being the numerator for equations **8.19** and **8.20**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{8.23}$$

being the denominator for equations **8.19** and **8.20**. We define the corresponding terms *R*, *I*, *M* and *N*

$$R(\omega)=0;\, I(\omega)=\omega\tau_T;\, M(\omega)=1;\, N(\omega)=\omega\tau_T.$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{8.24}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{\omega\,\tau_T}{\sqrt{1+\omega^2\tau_T^2}}. \tag{8.25}$$

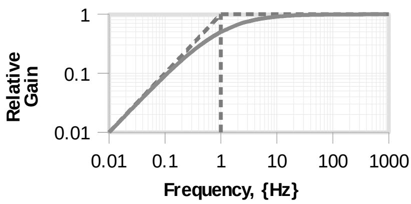


Fig. 8.1 Amplitude-frequency response (τ_T=0.159 s)

We multiply the numerator and denominator by a conjugate $M(\omega)-jN(\omega)$

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{8.26}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{0+\omega^2\tau_T^2}{1+\omega^2\tau_T^2}, \tag{8.27}$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}=\frac{\omega\,\tau_T-0}{1+\omega^2\tau_T^2}. \tag{8.28}$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=\arctan\frac{1}{\omega\,\tau_T}. \tag{8.29}$$

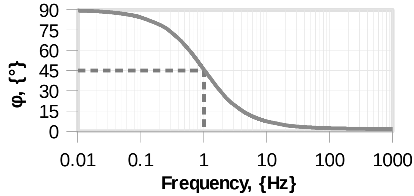


Fig. 8.2 Phase response (τ_T=0.159 s)

We apply the Inverse Laplace Transform to equations **8.19** and **8.20** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{(\tau_T s+1)}\{s\}. \tag{8.30}$$

In equation **8.30**, we put the τ_T outside the brackets

$$\frac{K_{INPUT/DIST}}{(\tau_T s+1)}=\frac{K_{INPUT/DIST}}{\tau_T}\frac{1}{s+\frac{1}{\tau_T}}\{s\}. \tag{8.31}$$

As can be seen, the pole is s_1=-1/τ_T

$$K_{s=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{8.32}$$

We write the function *f(t)* as

$$f(t)=\frac{K_{INPUT/DIST}}{\tau_T}e^{-\frac{t}{\tau_T}}\{Unit\}. \tag{8.33}$$

Once equation **8.33** has been multiplied by *T*, the temperature change of the sensitive element when heated, as a function of time equals

$$I_{pyro\,3}(t)=\frac{\Phi_{INPUT/DIST}\,p_{pyro\,3}\,A_{pyro}}{C_T}e^{-\frac{t}{\tau_T}}\{A\}. \qquad (8.34)$$

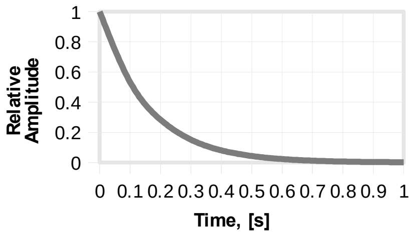


Fig. 8.3 Transient response (τ_T=0.159 s)

9. Alternating pyroelectric voltage

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT}+dE_{DIST}-dE_{OUTPUT}- \qquad (9.1)$$

$$-dE_{VOL.T}-dE_{VOL.E}-dE_{VOL.TE}=0,$$

with

$$dE_{INPUT}=d\,\Phi_{INPUT}\,dt\,\{J\} \qquad (9.2)$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST}=d\,\Phi_{DIST}\,dt\,\{J\} \qquad (9.3)$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL.TE} = \frac{C_T C_E}{p_{pyro1} A_{pyro}} d^2 U_{OUTPUT1} \{J\} \tag{9.4}$$

being the amount of energy that transforms from thermal to electrical, being stored in the volume of the sensitive element;

$$dE_{VOL.T} = \frac{C_T}{p_{pyro1} A_{pyro} R_E} dU_{OUTPUT1} dt \{J\} \tag{9.5}$$

being the amount of thermal energy stored in the volume of the sensitive element;

$$dE_{VOL.E} = \frac{G_T C_E}{p_{pyro1} A_{pyro}} dU_{OUTPUT1} dt \{J\} \tag{9.6}$$

being the amount of electrical energy stored in the volume of the sensitive element;

$$dE_{OUTPUT} = \frac{G_T}{p_{pyro1} A_{pyro} R_E} U_{OUTPUT1} dt^2 \{J\} \tag{9.7}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **9.2-9.7**, we set up an equation for power for which one has to divide each parameter by *dt* squared

$$\frac{d\Phi_{INPUT}}{dt} + \frac{d\Phi_{DIST}}{dt} = \tag{9.8}$$

$$= \frac{C_T C_E}{p_{pyro1} A_{pyro}} \frac{d^2 U_{OUTPUT1}}{dt^2} +$$

$$+\frac{C_T}{p_{pyro\,1}A_{pyro}R_E}\frac{dU_{OUTPUT\,1}}{dt}+$$

$$+\frac{G_T C_E}{p_{pyro\,1}A_{pyro}}\frac{dU_{OUTPUT\,1}}{dt}+$$

$$+\frac{G_T}{p_{pyro\,1}A_{pyro}R_E}U_{OUTPUT\,1}\left\{\frac{W}{s}\right\}.$$

Equation **9.8** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\frac{d\Phi_{INPUT}}{dt}=\left[\frac{d\Phi_{INPUT\,0}}{dt}\right]+\frac{d\Delta\Phi_{INPUT}}{dt}\left\{\frac{W}{s}\right\}; \tag{9.9}$$

$$\frac{d\Phi_{DIST}}{dt}=\left[\frac{d\Phi_{DIST\,0}}{dt}\right]+\frac{d\Delta\Phi_{DIST}}{dt}\left\{\frac{W}{s}\right\}; \tag{9.10}$$

$$\frac{C_T C_E}{p_{pyro\,1}A_{pyro}}\frac{d^2U_{OUTPUT\,1}}{dt^2}= \tag{9.11}$$

$$=\left[\frac{C_T C_E}{p_{pyro\,1}A_{pyro}}\frac{d^2U_{OUTPUT\,10}}{dt^2}\right]+$$

$$+\frac{C_T C_E}{p_{pyro\,1}A_{pyro}}\frac{d^2\Delta U_{OUTPUT\,1}}{dt^2}\left\{\frac{W}{s}\right\};$$

$$\frac{C_T}{p_{pyro\,1} A_{pyro} R_E} \frac{d\,U_{OUTPUT\,1}}{dt} = \tag{9.12}$$

$$= [\frac{C_T}{p_{pyro1} A_{pyro} R_E} \frac{d\,U_{OUTPUT\,10}}{dt}] +$$

$$+ \frac{C_T}{p_{pyro\,1} A_{pyro} R_E} \frac{d\,\Delta U_{OUTPUT\,1}}{dt} \{\frac{W}{s}\};$$

$$\frac{G_T C_E}{p_{pyro\,1} A_{pyro}} \frac{d\,U_{OUTPUT\,1}}{dt} = \tag{9.13}$$

$$= [\frac{G_T C_E}{p_{pyro1} A_{pyro}} \frac{d\,U_{OUTPUT\,10}}{dt}] +$$

$$+ \frac{G_T C_E}{p_{pyro\,1} A_{pyro}} \frac{d\,\Delta U_{OUTPUT\,1}}{dt} \{\frac{W}{s}\};$$

$$\frac{G_T}{p_{pyro\,1} A_{pyro} R_E} U_{OUTPUT\,1} = \tag{9.14}$$

$$= \frac{G_T}{p_{pyro1} A_{pyro} R_E} U_{OUTPUT\,10} +$$

$$+\frac{G_T}{p_{pyro\,1} A_{pyro} R_E} \Delta U_{OUTPUT\,1} \left\{\frac{W}{s}\right\}.$$

We extract static terms from the family of equations **9.9-9.14**

$$\frac{G_T}{p_{pyro\,1} A_{pyro} R_E} U_{OUTPUT\,10} = 0 \left\{\frac{W}{s}\right\}. \qquad (9.15)$$

Equation **9.15** is a static model. We extract equation **9.15** from the family of equations **9.9-9.14**

$$\frac{d\,\Delta\Phi_{INPUT}}{dt} + \frac{d\,\Delta\Phi_{DIST}}{dt} = \frac{C_T C_E}{p_{pyro\,1} A_{pyro}} \frac{d^2 \Delta U_{OUTPUT\,1}}{dt^2} + \qquad (9.16)$$

$$+\frac{C_T}{p_{pyro\,1} A_{pyro} R_E} \frac{d\,\Delta U_{OUTPUT\,1}}{dt} +$$

$$+\frac{G_T C_E}{p_{pyro\,1} A_{pyro}} \frac{d\,\Delta U_{OUTPUT\,1}}{dt} +$$

$$+\frac{G_T}{p_{pyro\,1} A_{pyro} R_E} \Delta U_{OUTPUT\,1} \left\{\frac{W}{s}\right\}.$$

Equation **9.16** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **9.16** by $G_T/(p_{pyro} A_{pyro} R_E)$

$$\frac{p_{pyro\,1} A_{pyro} R_E}{G_T} \frac{d\,\Delta\Phi_{INPUT}}{dt} + \qquad (9.17)$$

$$+\frac{p_{pyro\,1}A_{pyro}R_E}{G_T}\frac{d\Delta\Phi_{DIST}}{dt}=$$

$$=\frac{C_T C_E R_E}{G_T}\frac{d^2\Delta U_{OUTPUT\,1}}{dt^2}+$$

$$+\frac{C_T}{G_T}\frac{d\Delta U_{OUTPUT\,1}}{dt}+R_E C_E\frac{d\Delta U_{OUTPUT\,1}}{dt}+$$

$$+\Delta U_{OUTPUT\,1}\{V\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}\,p_{pyro1}A_{pyro}R_E}{G_T}\frac{d\dfrac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}}{dt}+ \tag{9.18}$$

$$+\frac{\Phi_{DIST}\,p_{pyro\,1}A_{pyro}R_E}{G_T}\frac{d\dfrac{\Delta\Phi_{DIST}}{\Phi_{DIST}}}{dt}=$$

$$=\frac{C_T R_E C_E}{G_T}U_{OUTPUT1}\frac{d^2\dfrac{\Delta U_{OUTPUT\,1}}{U_{OUTPUT\,1}}}{dt^2}+$$

$$+U_{OUTPUT\,1}\frac{C_T}{G_T}\frac{d\dfrac{\Delta U_{OUTPUT\,1}}{U_{OUTPUT\,1}}}{dt}+$$

$$+U_{OUTPUT\,1}R_E C_E\frac{d\dfrac{\Delta U_{OUTPUT}}{U_{OUTPUT}}}{dt}+$$

$$+U_{OUTPUT\,1}\frac{\Delta U_{OUTPUT\,1}}{U_{OUTPUT\,1}}\{V\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}=x;\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=z;\frac{\Delta U_{OUTPUT\,1}}{U_{OUTPUT\,1}}=y\{Unit\}.$$

We insert the variables obtained above into equation **9.18**

$$\frac{\Phi_{INPUT}\,p_{pyro\,1}A_{pyro}R_E}{G_T}\frac{dx}{dt}+ \tag{9.19}$$

$$+\frac{\Phi_{DIST}\,p_{pyro\,1}A_{pyro}R_E}{G_T}\frac{dz}{dt}=$$

$$=\frac{C_T R_E C_E}{G_T}U_{OUTPUT\,1}\frac{d^2y}{dt^2}+$$

$$+U_{OUTPUT\,1}\frac{C_T}{G_T}\frac{dy}{dt}+U_{OUTPUT\,1}R_E C_E\frac{dy}{dt}+$$

$$+U_{OUTPUT\,1}\,y\{V\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}\,p_{pyro\,1}\,A_{pyro}\,R_E}{G_T\,U_{OUTPUT\,1}}\frac{dx}{dt}+ \tag{9.20}$$

$$+\frac{\Phi_{DIST}\,p_{pyro\,1}\,A_{pyro}\,R_E}{G_T\,U_{OUTPUT\,1}}\frac{dz}{dt}=$$

$$=\frac{C_T\,R_E\,C_E}{G_T}\frac{d^2y}{dt^2}+\frac{C_T}{G_T}\frac{dy}{dt}+R_E\,C_E\frac{dy}{dt}+y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt}\rightarrow s;\frac{d^2}{dt^2}\rightarrow s^2;x\rightarrow X(S);z\rightarrow Z(S);y\rightarrow Y(S).$$

We apply the variables obtained above to equation **9.20**

$$s\frac{\Phi_{INPUT}\,p_{pyro1}\,A_{pyro}R_E}{G_T\,U_{OUTPUT\,1}}X(S)+ \tag{9.21}$$

$$+s\frac{\Phi_{DIST}\,p_{pyro\,1}\,A_{pyro}\,R_E}{G_T\,U_{OUTPUT\,1}}Z(S)=$$

$$=s^2\frac{C_T\,R_E\,C_E}{G_T}Y(S)+s\frac{C_T}{G_T}Y(S)+$$

$$+sR_E C_E Y(S)+Y(S)\{Unit\}.$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}\, p_{pyro1} A_{pyro} R_E}{G_T U_{OUTPUT1}} = K_{INPUT}\{s\}; \tag{9.22}$$

$$\frac{\Phi_{DIST}\, p_{pyro1} A_{pyro} R_E}{G_T U_{OUTPUT1}} = K_{DIST}\{s\};$$

$$\frac{C_T R_E C_E}{G_T} = \tau_T \tau_E \{s^2\}; \frac{C_T}{G_T} = \tau_T\{s\};$$

$$R_E C_E = \tau_E\{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T \tau_E s^2 + \tau_T s + \tau_E s + 1) = \tag{9.23}$$

$$= K_{INPUT} X(S) s + K_{DIST} Z(S) s \{Unit\}.$$

We write the transfer functions for the “*input → output*” channel

$$W_{x\to y}(S) = \frac{Y(S)}{X(S)} = \frac{K_{INPUT}\, s}{(\tau_T \tau_E s^2 + \tau_T s + \tau_E s + 1)} = \tag{9.24}$$

$$= \frac{K_{INPUT}\, s}{(\tau_T s + 1)(\tau_E s + 1)} \{Unit\},$$

and “*disturbance → output*” channel

$$W_{z\to y}(S) = \frac{Y(S)}{Z(S)} = \frac{K_{DIST}\, s}{(\tau_T \tau_E s^2 + \tau_T s + \tau_E s + 1)} = \tag{9.25}$$

$$=\frac{K_{DIST}\, s}{(\tau_T s+1)(\tau_E s+1)}\{Unit\}.$$

We replace the Laplace variable *s* with *jω*

$$s \rightarrow j\omega.$$

We apply the replaced variable *jω* to equations **9.24** and **9.25**

$$W(j\omega)=\frac{K(j\omega)}{D(j\omega)}, \tag{9.26}$$

with

$$K(j\omega)=R(\omega)+jI(\omega) \tag{9.27}$$

being the numerator for equations **9.24** and **9.25**;

$$D(j\omega)=M(\omega)+jN(\omega) \tag{9.28}$$

being the denominator for equations **9.19** and **9.20**. We define the corresponding terms *R*, *I*, *M* and *N*

$$R(\omega)=0\,;I(\omega)=\omega\,\tau_T\,;$$

$$M(\omega)=1-\tau_T\,\tau_E\,\omega^2\,;N(\omega)=\omega(\tau_T+\tau_E).$$

We define the amplitude-frequency response

$$A(\omega)=\frac{A_{OUTPUT}}{A_{INPUT}}=|W(j\omega)|, \tag{9.29}$$

or

$$A(\omega)=\frac{\sqrt{R^2(\omega)+I^2(\omega)}}{\sqrt{M^2(\omega)+N^2(\omega)}}=\frac{\omega\,\tau_T}{\sqrt{(1+\omega^2\,\tau_T^2)(1+\omega^2\,\tau_E^2)}}. \tag{9.30}$$

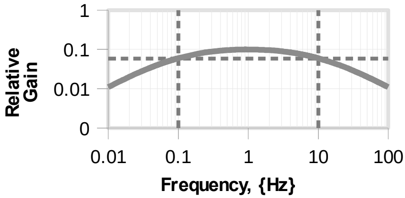


Fig. 9.1 Amplitude-frequency response (τ_T=0.0159 s, τ_E=1.59 s)

We multiply the numerator and denominator by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{9.31}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}= \tag{9.32}$$

$$=\frac{\omega^2\tau_T(\tau_T+\tau_E)}{(1-\tau_T\tau_E\omega^2)^2+\omega^2(\tau_T+\tau_E)^2},$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}= \tag{9.33}$$

$$=\frac{\omega\tau_T(1-\tau_T\tau_E\omega^2)}{(1-\tau_T\tau_E\omega^2)^2+\omega^2(\tau_T+\tau_E)^2}.$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=\arctan\frac{1-\tau_T\tau_E\omega^2}{\omega(\tau_T+\tau_E)}. \tag{9.34}$$

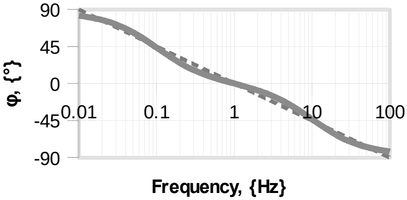


Fig.9.2 Phase response (τ_T=0.0159 s, τ_E=1.59 s)

We apply the Inverse Laplace Transform to equations **9.24** and **9.25** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{(\tau_T s+1)(\tau_E s+1)}\{s\}. \tag{9.35}$$

In equation **9.35**, we put the $\tau_T\tau_E$ outside the brackets

$$\frac{K_{INPUT/DIST}}{(\tau_T s+1)(\tau_E s+1)}=\frac{1}{\tau_T\tau_E}\frac{K_{INPUT/DIST}}{\left(s+\frac{1}{\tau_T}\right)\left(s+\frac{1}{\tau_E}\right)}\{s\}. \tag{9.36}$$

As can be seen, the poles are s_1=-1/τ_T

$$K_{s=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T\tau_E}\frac{e^{-\frac{t}{\tau_T}}}{\frac{1}{\tau_E}-\frac{1}{\tau_T}}\{Unit\}, \tag{9.37}$$

where

$$\frac{1}{\tau_T\tau_E}\frac{1}{\frac{1}{\tau_E}-\frac{1}{\tau_T}}=\frac{1}{\tau_T\tau_E}\frac{1}{\frac{\tau_T}{\tau_E\tau_T}-\frac{\tau_E}{\tau_E\tau_T}}=\frac{1}{\tau_T-\tau_E}\left\{\frac{1}{s}\right\}. \tag{9.38}$$

We write the result

$$K_{s=-\frac{1}{\tau_T}}=\frac{K_{INPUT/DIST}}{\tau_T-\tau_E}e^{-\frac{t}{\tau_T}}\{Unit\}, \quad (9.39)$$

and s_2=-1/τ_E

$$K_{s=-\frac{1}{\tau_E}}=\frac{K_{INPUT/DIST}}{\tau_T\tau_E}\frac{e^{-\frac{t}{\tau_E}}}{\frac{1}{\tau_T}-\frac{1}{\tau_E}}\{Unit\}, \quad (9.40)$$

where

$$\frac{1}{\tau_T\tau_E}\frac{1}{\frac{1}{\tau_T}-\frac{1}{\tau_E}}=\frac{1}{\tau_T\tau_E}\frac{1}{\frac{\tau_E}{\tau_E\tau_T}-\frac{\tau_T}{\tau_E\tau_T}}=\frac{1}{\tau_E-\tau_T}\{\frac{1}{s}\}. \quad (9.41)$$

We write the result

$$K_{s=-\frac{1}{\tau_E}}=\frac{K_{INPUT/DIST}}{\tau_E-\tau_T}e^{-\frac{t}{\tau_E}}\{Unit\}. \quad (9.42)$$

We write the function $f(t)$ as

$$f(t)=K_{INPUT/DIST}(K_{s=-\frac{1}{\tau_T}}+K_{s=-\frac{1}{\tau_E}})\{Unit\}. \quad (9.43)$$

After inserting the corresponding poles, we obtain

$$f(t)=K_{INPUT/DIST}(\frac{e^{-\frac{t}{\tau_T}}}{\tau_T-\tau_E}+\frac{e^{-\frac{t}{\tau_E}}}{\tau_E-\tau_T})\{Unit\}. \quad (9.44)$$

Finally, the function $f(t)$ can be written in the form

$$f(t)=\frac{K_{INPUT/DIST}}{\tau_T-\tau_E}(e^{-\frac{t}{\tau_T}}-e^{-\frac{t}{\tau_E}})\{Unit\}. \quad (9.45)$$

Once equation **9.45** has been multiplied by U_{OUTPUT}, the temperature change of the sensitive element when heated, as a function of time equals

$$U_{OUTPUT1}(t)= \quad (9.46)$$

$$\frac{\Phi_{INPUT/DIST}\, p_{pyro\,1} A_{pyro} R_E}{G_T(\tau_T-\tau_E)}\left(e^{-\frac{t}{\tau_T}}-e^{-\frac{t}{\tau_E}}\right)\{V\}.$$

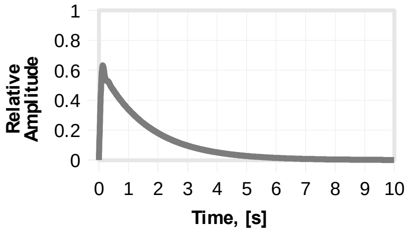


Fig. 9.3 Transient response (τ_T=0.0159 s, τ_E=1.59 s)

10. Direct pyroelectric voltage

In accordance with the logical data model, we set up an equation for the law of conservation of energy

$$dE_{INPUT}+dE_{DIST}-dE_{OUTPUT}- \tag{10.1}$$

$$-dE_{VOL.T}-dE_{VOL.E}-dE_{VOL.TE}=0,$$

with

$$dE_{INPUT}=\Phi_{INPUT}\,dt\,\{J\} \tag{10.2}$$

being the amount of energy coming to the detector from a heat source;

$$dE_{DIST}=\Phi_{DIST}\,dt\,\{J\} \tag{10.3}$$

being the amount of energy coming to the detector from the environment;

$$dE_{VOL.TE}=\frac{C_T^2 C_E}{G_T\, p_{pyro3} A_{pyro}} \frac{d\left(dU_{OUTPUT3}\right)}{dt}\{J\} \tag{10.4}$$

being the amount of energy that transforms from thermal to electrical, being stored in the volume of the sensitive element;

$$dE_{VOL.T}=\frac{C_T^2}{G_T\, p_{pyro3} A_{pyro} R_E} dU_{OUTPUT3}\{J\} \tag{10.5}$$

being the amount of thermal energy stored in the volume of the sensitive element;

$$dE_{VOL.E}=\frac{C_T C_E}{p_{pyro3} A_{pyro}} dU_{OUTPUT3}\{J\} \tag{10.6}$$

being the amount of electrical energy stored in the volume of the sensitive element;

$$dE_{OUTPUT}=\frac{C_T}{p_{pyro3} A_{pyro} R_E} U_{OUTPUT3}\, dt\{J\} \tag{10.7}$$

being the amount of energy going out of the sensitive element to the environment.

Taking into account the parameters determined in equations **10.2-10.7**, we set up an equation for power for which one has to divide each parameter by *dt*

$$\Phi_{INPUT}+\Phi_{DIST}=\frac{C_T^2 C_E}{G_T\, p_{pyro3} A_{pyro}} \frac{d^2 U_{OUTPUT3}}{dt^2}+ \tag{10.8}$$

$$+\frac{C_T^2}{G_T\, p_{pyro3} A_{pyro} R_E} \frac{dU_{OUTPUT3}}{dt}+$$

$$+\frac{C_T C_E}{p_{pyro\,3} A_{pyro}} \frac{dU_{OUTPUT\,3}}{dt} +$$

$$+\frac{C_T}{p_{pyro\,3} A_{pyro} R_E} U_{OUTPUT\,3} \{W\}.$$

Equation **10.8** is a non-linear mathematical model. We linearize it by applying a Taylor series

$$\Phi_{INPUT} = \Phi_{INPUT\,0} + \Delta\,\Phi_{INPUT} \{W\}; \tag{10.9}$$

$$\Phi_{DIST} = \Phi_{DIST\,0} + \Delta\,\Phi_{DIST} \{W\}; \tag{10.10}$$

$$\frac{C_T^2 C_E}{G_T\, p_{pyro\,3} A_{pyro}} \frac{d^2 U_{OUTPUT\,3}}{dt^2} = \tag{10.11}$$

$$= [\frac{C_T^2 C_E}{G_T\, p_{pyro3} A_{pyro}} \frac{d^2 U_{OUTPUT\,30}}{dt^2}] +$$

$$+\frac{C_T^2 C_E}{G_T\, p_{pyro\,3} A_{pyro}} \frac{d^2 \Delta\, U_{OUTPUT\,3}}{dt^2} \{W\};$$

$$\frac{C_T^2}{G_T\, p_{pyro\,3} A_{pyro} R_E} \frac{d\,U_{OUTPUT\,3}}{dt} = \tag{10.12}$$

$$=[\frac{C_T^2}{G_T\, p_{pyro3} A_{pyro} R_E} \frac{d U_{OUTPUT30}}{dt}]+$$

$$+\frac{C_T^2}{G_T\, p_{pyro3} A_{pyro} R_E} \frac{d \Delta U_{OUTPUT3}}{dt} \{W\};$$

$$\frac{C_T C_E}{p_{pyro3} A_{pyro}} \frac{d U_{OUTPUT3}}{dt} = \qquad (10.13)$$

$$=[\frac{C_T C_E}{p_{pyro3} A_{pyro}} \frac{d U_{OUTPUT30}}{dt}]+$$

$$+\frac{C_T C_E}{p_{pyro3} A_{pyro}} \frac{d \Delta U_{OUTPUT3}}{dt} \{W\};$$

$$\frac{C_T}{p_{pyro3} A_{pyro} R_E} U_{OUTPUT3} = \qquad (10.14)$$

$$=\frac{C_T}{p_{pyro3} A_{pyro} R_E} U_{OUTPUT30} +$$

$$=\frac{C_T}{p_{pyro3} A_{pyro} R_E} \Delta U_{OUTPUT3} \{W\}.$$

We extract static terms from the family of equations **10.9-10.14**

$$\frac{C_T}{p_{pyro\,3} A_{pyro} R_E} U_{OUTPUT\,30} = \Phi_{INPUT\,0} + \Phi_{DIST\,0} \{W\}. \tag{10.15}$$

Equation **10.15** is a static model. We extract equation **10.15** from the family of equations **10.9-10.14**

$$\Delta\Phi_{INPUT} + \Delta\Phi_{DIST} = \frac{C_T^2 C_E}{G_T\, p_{pyro\,3} A_{pyro}} \frac{d^2 \Delta U_{OUTPUT\,3}}{dt^2} + \tag{10.16}$$

$$+ \frac{C_T^2}{G_T\, p_{pyro\,3} A_{pyro} R_E} \frac{d \Delta U_{OUTPUT\,3}}{dt} +$$

$$+ \frac{C_T C_E}{p_{pyro\,3} A_{pyro}} \frac{d \Delta U_{OUTPUT\,3}}{dt} +$$

$$+ \frac{C_T}{p_{pyro\,3} A_{pyro} R_E} \Delta U_{OUTPUT\,3} \{W\}.$$

Equation **10.16** is a dynamic model. We find the proper unit of measurement for which one has to divide both left and right terms of equation **10.16** by $C_T/(p_{pyro3} A_{pyro} R_E)$

$$\frac{p_{pyro\,3} A_{pyro} R_E}{C_T} \Delta\Phi_{INPUT} + \tag{10.17}$$

$$+ \frac{p_{pyro\,3} A_{pyro} R_E}{C_T} \Delta\Phi_{DIST} =$$

$$=\frac{C_T C_E R_E}{G_T}\frac{d^2 \Delta U_{OUTPUT3}}{dt^2}+$$

$$+\frac{C_T}{G_T}\frac{d\Delta U_{OUTPUT3}}{dt}+R_E C_E \frac{d\Delta U_{OUTPUT3}}{dt}+$$

$$+\Delta U_{OUTPUT3}\{V\}.$$

We multiply and divide both left and right terms by corresponding variables

$$\frac{\Phi_{INPUT}\, p_{pyro3} A_{pyro} R_E}{C_T}\frac{\Delta\Phi_{INPUT}}{\Phi_{INPUT}}+ \tag{10.18}$$

$$+\frac{\Phi_{DIST}\, p_{pyro3} A_{pyro} R_E}{C_T}\frac{\Delta\Phi_{DIST}}{\Phi_{DIST}}=$$

$$=\frac{C_T R_E C_E}{G_T} U_{OUTPUT3}\frac{d^2 \dfrac{\Delta U_{OUTPUT3}}{U_{OUTPUT3}}}{dt^2}+$$

$$+\frac{C_T}{G_T} U_{OUTPUT3}\frac{d\dfrac{\Delta U_{OUTPUT3}}{U_{OUTPUT3}}}{dt}+$$

$$+R_E C_E U_{OUTPUT\,3} \frac{d \dfrac{\Delta U_{OUTPUT\,3}}{U_{OUTPUT\,3}}}{dt} +$$

$$+U_{OUTPUT\,3} \frac{\Delta U_{OUTPUT\,3}}{U_{OUTPUT\,3}} \{V\}.$$

We replace relative increments with variables of *input*, *disturbance*, and *output*

$$\frac{\Delta \Phi_{INPUT}}{\Phi_{INPUT}} = x\,; \frac{\Delta \Phi_{DIST}}{\Phi_{DIST}} = z\,; \frac{\Delta U_{OUTPUT\,3}}{U_{OUTPUT\,3}} = y \{Unit\}.$$

We insert the variables obtained above into equation **10.18**

$$\frac{\Phi_{INPUT}\, p_{pyro\,3} A_{pyro} R_E}{C_T} x + \tag{10.19}$$

$$+\frac{\Phi_{DIST}\, p_{pyro\,3} A_{pyro} R_E}{C_T} z =$$

$$= \frac{C_T R_E C_E}{G_T} U_{OUTPUT\,3} \frac{d^2 y}{dt^2} +$$

$$+U_{OUTPUT\,3} \frac{C_T}{G_T} \frac{dy}{dt} + U_{OUTPUT\,3} R_E C_E \frac{dy}{dt} +$$

$$+U_{OUTPUT\,3}\, y \{V\}.$$

We divide both left and right terms by the variable that determines the unit of measurement

$$\frac{\Phi_{INPUT}\, p_{pyro3}\, A_{pyro}\, R_E}{C_T\, U_{OUTPUT\,3}}\, x + \tag{10.20}$$

$$+\frac{\Phi_{DIST}\, p_{pyro\,3}\, A_{pyro}\, R_E}{C_T\, U_{OUTPUT3}}\, z =$$

$$= \frac{C_T\, R_E\, C_E}{G_T}\frac{d^2 y}{dt^2} + \frac{C_T}{G_T}\frac{dy}{dt} + R_E\, C_E\frac{dy}{dt} + y\{Unit\}.$$

We apply the Laplace Transform to the variables determined above

$$\frac{d}{dt} \rightarrow s;\ \frac{d^2}{dt^2} \rightarrow s^2;\ x \rightarrow X(S);\ z \rightarrow Z(S);\ y \rightarrow Y(S).$$

We apply the variables obtained above to equation **10.20**

$$s\frac{\Phi_{INPUT}\, p_{pyro3}\, A_{pyro}\, R_E}{C_T\, U_{OUTPUT3}}\, X(S) + \tag{10.21}$$

$$+s\frac{\Phi_{DIST}\, p_{pyro\,3}\, A_{pyro}\, R_E}{C_T\, U_{OUTPUT\,3}}\, Z(S) =$$

$$= s^2\frac{C_T\, R_E\, C_E}{G_T}\, Y(S) + s\frac{C_T}{G_T}\, Y(S) +$$

$$+ sR_E C_E\, Y(S) + Y(S)\{Unit\}.$$

We replace the static terms with static coefficients

$$\frac{\Phi_{INPUT}\, p_{pyro\,3} A_{pyro} R_E}{C_T U_{OUTPUT\,3}} = K_{INPUT} \{Unit\};$$ (10.22)

$$\frac{\Phi_{DIST}\, p_{pyro\,3} A_{pyro} R_E}{C_T U_{OUTPUT\,3}} = K_{DIST} \{Unit\};$$

$$\frac{C_T R_E C_E}{G_T} = \tau_T \tau_E \{s^2\}; \frac{C_T}{G_T} = \tau_T \{s\};$$

$$R_E C_E = \tau_E \{s\}.$$

We write an equation for the law of conservation of energy in terms of the Laplace variables

$$Y(S)(\tau_T \tau_E s^2 + \tau_T s + \tau_E s + 1) =$$ (10.23)

$$= K_{INPUT} X(S) + K_{DIST} Z(S) \{Unit\}.$$

We write the transfer functions for the "*input → output*" channel

$$W_{x \to y}(S) = \frac{Y(S)}{X(S)} = \frac{K_{INPUT}}{(\tau_T \tau_E s^2 + \tau_T s + \tau_E s + 1)} =$$ (10.24)

$$= \frac{K_{INPUT}}{(\tau_T s + 1)(\tau_E s + 1)} \{Unit\},$$

and "*disturbance → output*" channel

$$W_{z \to y}(S) = \frac{Y(S)}{Z(S)} = \frac{K_{DIST}}{(\tau_T \tau_E s^2 + \tau_T s + \tau_E s + 1)} =$$ (10.25)

$$= \frac{K_{DIST}}{(\tau_T s + 1)(\tau_E s + 1)} \{Unit\}.$$

We replace the Laplace variable s with $j\omega$

$$s \rightarrow j\omega .$$

We apply the replaced variable $j\omega$ to equations **10.24** and **10.25**

$$W(j\omega) = \frac{K(j\omega)}{D(j\omega)}, \tag{10.26}$$

with

$$K(j\omega) = R(\omega) + jI(\omega) \tag{10.27}$$

being the numerator for equations **10.24** and **10.25**;

$$D(j\omega) = M(\omega) + jN(\omega) \tag{10.28}$$

being the denominator for equations **10.19** and **10.20**. We define the corresponding terms R, I, M and N

$$R(\omega) = 1;\, I(\omega) = 0;$$

$$M(\omega) = 1 - \tau_T \tau_E \omega^2;\, N(\omega) = \omega(\tau_T + \tau_E).$$

We define the amplitude-frequency response

$$A(\omega) = \frac{A_{OUTPUT}}{A_{INPUT}} = |W(j\omega)|, \tag{10.29}$$

or

$$A(\omega) = \frac{\sqrt{R^2(\omega) + I^2(\omega)}}{\sqrt{M^2(\omega) + N^2(\omega)}} = \frac{1}{\sqrt{(1+\omega^2\tau_T^2)(1+\omega^2\tau_E^2)}}. \tag{10.30}$$

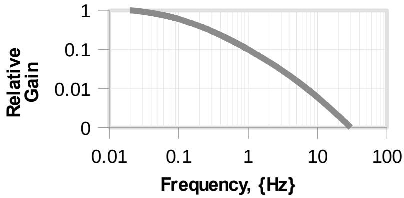


Fig. 10.1 Amplitude-frequency response (τ_T=0.0159 s, τ_E=1.59 s)

We multiply the numerator and denominator by a conjugate *M(ω)-jN(ω)*

$$W(j\omega)=P(\omega)+jQ(\omega), \tag{10.31}$$

with

$$P(\omega)=\frac{R(\omega)M(\omega)+I(\omega)N(\omega)}{M^2(\omega)+N^2(\omega)}= \tag{10.32}$$

$$=\frac{1-\tau_T\tau_E\omega^2}{(1-\tau_T\tau_E\omega^2)^2+\omega^2(\tau_T+\tau_E)^2},$$

and

$$Q(\omega)=\frac{I(\omega)M(\omega)-N(\omega)R(\omega)}{M^2(\omega)+N^2(\omega)}= \tag{10.33}$$

$$=-\frac{\omega(\tau_T+\tau_E)}{(1-\tau_T\tau_E\omega^2)^2+\omega^2(\tau_T+\tau_E)^2}.$$

We define the phase response

$$\varphi=\arctan\frac{Q(\omega)}{P(\omega)}=-\arctan\frac{\omega(\tau_T+\tau_E)}{1-\tau_T\tau_E\omega^2}. \tag{10.34}$$

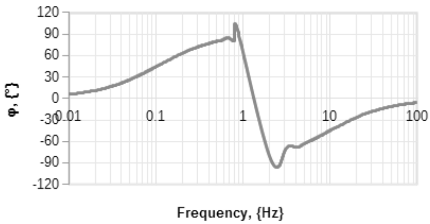


Fig. 10.2 Phase response (τ_T=0.0159 s, τ_E=1.59 s)

We apply the Inverse Laplace Transform to equations **10.24** and **10.25** for which the former and the latter must be divided by *s*

$$L^{-1}\left(W_{\substack{x\to y\\ z\to y}}(S)\right)=\frac{W_{\substack{x\to y\\ z\to y}}(S)}{s}=\frac{K_{INPUT/DIST}}{s(\tau_T s+1)(\tau_E s+1)}\{s\}. \tag{10.35}$$

We resolve equation **10.35** into partial fractions temporary removing $K_{INPUT/DIST}$ for simplification

$$\frac{1}{s(\tau_T s+1)(\tau_E s+1)}=\frac{A}{s}+\frac{B}{\tau_T s+1}+\frac{C}{\tau_E s+1}\{s\}. \tag{10.36}$$

Upon application of some mathematical transformations (Equations **10.36-10.45** where each variable is unitless), such as

$$\frac{A}{s}+\frac{B}{\tau_T s+1}+\frac{C}{\tau_E s+1}= \tag{10.37}$$

$$\frac{A(\tau_T s+1)(\tau_E s+1)+Bs(\tau_E s+1)+Cs(\tau_T s+1)}{s(\tau_T s+1)(\tau_E s+1)},$$

and

$$A(\tau_T s+1)(\tau_E s+1)+Bs(\tau_E s+1)+Cs(\tau_T s+1)=1, \tag{10.38}$$

and

$$A\tau_T\tau_E s^2 + A\tau_T s + A\tau_E s + A + B\tau_E s^2 + Bs + \tag{10.39}$$

$$+C\tau_T s^2 + Cs = 1,$$

and

$$s^2 : A\tau_T\tau_E + B\tau_E + C\tau_T = 0; \tag{10.40}$$

$$s^1 : A\tau_T + A\tau_E + B + C = 0;$$

$$s^0 : A = 1,$$

and

$$\tau_T\tau_E + B\tau_E + C\tau_T = 0; \tag{10.41}$$

$$\tau_T + \tau_E + B + C = 0,$$

and

$$B\tau_E + C\tau_T = -\tau_T\tau_E; \tag{10.42}$$

$$B + C = -(\tau_T + \tau_E).$$

We define matrix determinants

$$\Delta = \begin{vmatrix} \tau_E & \tau_T \\ 1 & 1 \end{vmatrix} = \tau_E - \tau_T, \tag{10.43}$$

and

$$\Delta_B = \begin{vmatrix} -\tau_T\tau_E & \tau_T \\ -(\tau_T + \tau_E) & 1 \end{vmatrix} = \tau_T(\tau_T + \tau_E) - \tau_T\tau_E = \tau_T^2, \tag{10.44}$$

and

$$\Delta_C = \begin{vmatrix} \tau_E & -\tau_T\tau_E \\ 1 & -(\tau_T + \tau_E) \end{vmatrix} = \tau_T\tau_E - \tau_T(\tau_T + \tau_E) = -\tau_E^2. \tag{10.45}$$

We define coefficient B

$$B = \frac{\Delta_B}{\Delta} = \frac{\tau_T(\tau_T + \tau_E) - \tau_T\tau_E}{\tau_E - \tau_T} = \frac{\tau_T^2}{\tau_E - \tau_T}, \tag{10.46}$$

and coefficient C

$$C = \frac{\Delta_C}{\Delta} = \frac{\tau_T \tau_E - \tau_E(\tau_T + \tau_E)}{\tau_E - \tau_T} = -\frac{\tau_E^2}{\tau_E - \tau_T}. \tag{10.47}$$

As can be seen, the poles are $s_1=0$

$$K_{s=0} = \frac{A}{s} = 1, \tag{10.48}$$

$s_2=-1/\tau_T$

$$K_{s=-\frac{1}{\tau_T}} = \frac{B}{\tau_T s + 1} = \frac{B}{\tau_T} e^{-\frac{t}{\tau_T}}, \tag{10.49}$$

where

$$\frac{\tau_T(\tau_T + \tau_E) - \tau_T \tau_E}{(\tau_E - \tau_T)\tau_T} e^{-\frac{t}{\tau_T}} = \frac{\tau_T}{\tau_E - \tau_T} e^{-\frac{t}{\tau_T}}, \tag{10.50}$$

and $s_3=-1/\tau_E$

$$K_{s=-\frac{1}{\tau_E}} = \frac{C}{\tau_E s + 1} = \frac{C}{\tau_E} e^{-\frac{t}{\tau_E}}, \tag{10.51}$$

where

$$\frac{\tau_T \tau_E - \tau_E(\tau_T + \tau_E)}{(\tau_E - \tau_T)\tau_E} e^{-\frac{t}{\tau_E}} = -\frac{\tau_E}{\tau_E - \tau_T} e^{-\frac{t}{\tau_E}}. \tag{10.52}$$

We write the function *f(t)* as

$$f(t) = K_{INPUT/DIST}\left(K_{s=0} + K_{s=-\frac{1}{\tau_T}} + K_{s=-\frac{1}{\tau_E}}\right)\{Unit\}. \tag{10.53}$$

After inserting poles determined above, we obtain

$$f(t) = K_{INPUT/DIST}\left(1 - \frac{1}{\tau_E - \tau_T}\left(\tau_E e^{\frac{-t}{\tau_E}} - \tau_T e^{\frac{-t}{\tau_T}}\right)\right)$$
$$\{Unit\}. \tag{10.54}$$

Once equation **10.54** has been multiplied by U_{OUTPUT}, the temperature change of the sensitive element when heated, as a function of time equals

$$U_{OUTPUT3}(t)=\frac{\Phi_{INPUT}\, p_{pyro3} A_{pyro} R_E}{C_T}\times \tag{10.55}$$

$$\left(1-\frac{1}{\tau_E-\tau_T}\left(\tau_E e^{\frac{-t}{\tau_E}}-\tau_T e^{\frac{-t}{\tau_T}}\right)\right)\{V\}.$$

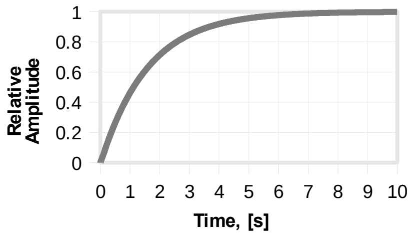
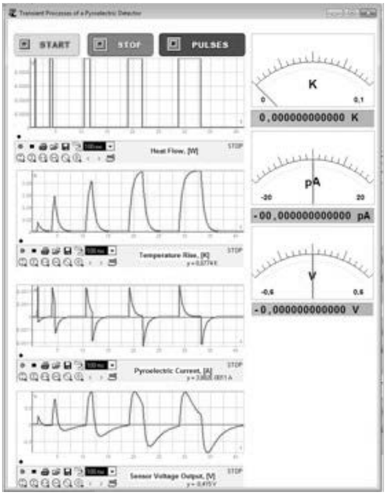


Fig. 10.3 Transient response (τ_T=0.0159 s, τ_E=1.59 s)

Simulator of a pyroelectric detector



Ten mathematical models have been applied to the simulator of a pyroelectric detector. This simulator is a simple program that imitates transient responses for thermal and electrical processes in the detector. A sample view is in the right screenshot. The software is available in free and commercial editions. Free edition can be downloaded directly from the site Commercial editions (please, see details on http://pyrodetector.com/simulator.html or ask me on email) offer an option to change input data and export the simulated results to an external file like .txt to plot graphs. Such graphs can be used by researchers in their papers.

Please, keep in mind that copying these simulators from a third person or downloading them elsewhere online puts you at potential risk of getting a virus.

I guarantee that my products of the best quality and free from viruses. I use https://www.virustotal.com to analyze the output .exe files.

Heartfelt donations would be a good honor and best inspiration for the author to keep doing his works further http://pyrodetector.com/donate.html

Thank you.

Kind regards, Alexander Bondarenko

The end

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Монографія

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