Problem 1

a. $h=0.1 \quad x_0 = 1.1$

$f'(1.1) = \frac{1}{2h}(-3f(1.1) + 4f(1.2) - f(1.3)) =$

$f'(1.2) = \frac{1}{h}(-\frac{1}{2}f(1.1) + \frac{1}{2}f(1.3)) =$

$f'(1.3) = \frac{1}{h}(-\frac{1}{2}f(1.2) + \frac{1}{2}f(1.4)) =$

$f'(1.4) = \frac{1}{2h}(3f(1.4) - 4f(1.3) + f(1.2)) =$

b. $h=0.2 \quad x_0 = 8.1$

$f'(8.1) = \frac{1}{2h}[-3f(8.1) + 4f(8.3) - f(8.5)]$

$f'(8.3) = \frac{1}{2h}[-f(8.1) + f(8.5)]$

$f'(8.5) = \frac{1}{2h}[-f(8.3) + f(8.7)]$

$f'(8.7) = \frac{1}{2h}[3f(8.7) - 4f(8.5) + f(8.3)]$

Problem 2.

$M = N(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 \cdots$

$M = N(\frac{h}{3}) + K_1 \frac{h^2}{9} + K_2 \frac{h^4}{81} + K_3 \frac{h^6}{3^6} \cdots$

$M = N(\frac{h}{9}) + K_1 \frac{h^2}{9^2} + K_2 \frac{h^4}{9^4} + K_3 \frac{h^6}{9^6} \cdots$

$8M = [9N(\frac{h}{3}) - N(h)] + K_2(\frac{1}{9}-1)h^4 + K_3(\frac{1}{3^4}-1)h^6$

$M = \frac{1}{8}[9N(\frac{h}{3}) - N(h)] + \frac{1}{8}(\frac{1}{9}-1)K_2 h^4 + \frac{1}{8}(\frac{1}{3^4}-1)K_3 h^6$

$M = \frac{1}{8}[9N(\frac{h}{9}) - N(\frac{h}{3})] + \frac{1}{8}\times[\frac{1}{9^3} - \frac{1}{9^2}] + \frac{1}{8}(\frac{1}{9^5} - \frac{1}{3^6})K_3 h^6$

$(9^2-1)M = \frac{9^2}{8}[9N(\frac{h}{9}) - N(\frac{h}{3})] - \frac{1}{8}[9N(\frac{h}{3}) - N(h)] + O(h^6)$

$M = \frac{1}{640}[9^3 N(\frac{h}{9}) - 90N(\frac{h}{3}) + N(h)] + O(h^6)$

Problem 3.

1) Trapezoidal rule

a. $\int_{-0.25}^{0.25} (\cos x)^2 dx = \frac{0.25-(-0.25)}{2} \times [\cos^2(-0.25) + \cos^2(0.25)] = 0.4694$

b. $\int_{-0.5}^{0} x\ln(x+1)\, dx = \frac{0-(-0.5)}{2} \times [0 + (-0.5)\ln 0.5] = 0.08664$

c. $\int_{0.75}^{1.3} (\sin^2 x - 2x\sin x + 1)\, dx = \frac{1.3-0.75}{2} \times [f(0.75) + f(1.3)] = -0.03702$

d. $\int_{e}^{e+1} \frac{1}{x\ln x}\, dx = \frac{1}{2}\times[f(e) + f(e+1)] = 0.2863$

2) Simpson's rule

a. $\int_{-0.25}^{0.25} f(x)\, dx = \frac{0.25}{3} \times [f(-0.25) + 4f(0) + f(0.25)] = 0.4898$

b. $\int_{-0.5}^{0} f(x)\, dx = \frac{0.25}{3}[f(-0.5) + 4f(-0.25) + f(0)] = 0.05285$

c. $\int_{0.75}^{1.3} f(x)\, dx = \frac{0.55}{6}[f(0.75) + 4f(1.025) + f(1.3)] = -0.02027$

d. $\int_{e}^{e+1} f(x)\, dx = \frac{1}{6}[f(e) + 4f(e+\frac{1}{2}) + f(e+1)] = 0.2727$

Problem 4. (附代码)

设在 $[a, b]$ 区间内，分为 $2^0, 2^1, \cdots 2^k, \cdots 2^n$ 个区间 $h = \frac{b-a}{2^j}$

$$R_{k,1} = \frac{1}{2} \cdot \frac{b-a}{2^{k-1}} \cdot [f(a) + 2\sum_{i=1}^{2^{k-1}-1} f(x_i) + f(b)]$$

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}(R_{k,j-1} - R_{k-1,j-1})$$

a. $R_{1,1} = 2 \times \frac{\cos^2 1 + \cos^2 -1}{2} = 0.5819$

$R_{2,1} = \frac{1}{2} \times \frac{2}{2} \times [\cos^2 -1 + 2\cos^2 0 + \cos^2 1] = 1.2919$

$R_{3,1} = \frac{1}{2} \times \frac{2}{4} \times [\cos^2 -1 + 2\cos^2 -0.5 + 2\cos^2 0 + 2\cos^2 0.5 + \cos^2 1] = 1.4161$

$R_{2,2} = R_{2,1} + \frac{1}{4^1-1}(R_{2,1} - R_{1,1})$

$R_{3,2} = R_{3,1} + \frac{1}{4^1-1}(R_{3,1} - R_{2,1})$

$R_{3,3} = R_{3,2} + \frac{1}{4^2-1}(R_{3,2} - R_{2,2}) = 1.4528$

b. $R_{1,1} = 6.3280$

c. $R_{3,3} = 1.3871$

d. $R_{3,3} = 0.5266$

Problem 5. (附代码)

a. $y' = f(t, y) = \frac{y}{t} - (\frac{y}{t})^2 \quad y(1) = 1 \quad h = 0.1$

$y(1) = 1 \quad \because y_{i+1} = y_i + hf(t_i + y_i) + O(h^2)$

t	y
t = 1.0	1.0000
t = 1.1	1.0000
t = 1.2	1.0083
t = 1.3	1.0217
t = 1.4	1.0385
t = 1.5	1.0577
t = 1.6	1.0785
t = 1.7	1.1004
t = 1.8	1.1233
t = 1.9	1.1467
t = 2.0	1.1707

b. 同理

t	y
1.0	0.0000
1.2	0.2000
1.4	0.4381
1.6	0.7212
1.8	1.0520
2.0	1.4372
2.2	1.8843
2.4	2.4023
2.6	3.0028
2.8	3.7006
3.0	4.5143

Problem 6

线性最小二乘法拟合

① 一阶 $p(x) = a_0 + a_1 x$

$$E = \sum_{i=1}^{4} [y_i - p(x_i)]^2$$

$$= (6-a_0)^2 + (8-a_0-2a_1)^2 + (14-a_0-4a_1)^2 + (20-a_0-5a_1)^2$$

$$\frac{\partial E}{\partial a_0} = 0 \quad \frac{\partial E}{\partial a_1} = 0$$

$\therefore p(x) = 2.712x + 4.542 \quad E = 11.53$

2阶 $p(x) = 0.6432x^2 - 0.4824x + 6.090 \quad E = 0.3618$

3阶 $p(x) = 0.1000x^3 - 0.1000x^2 + 0.8000x + 6.000 \quad E = 0$

三阶可以无误差拟合