Numerical Analysis

Lecture 7: Interpolation & Polynomial Approximation

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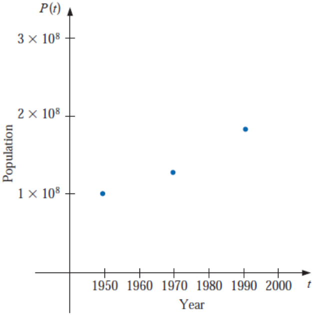
Outline

Motivation

Example: Population Census



Predictions can be obtained by using a function that fits the given data. This process is called interpolation.

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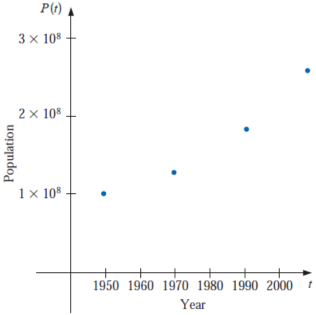

Predictions can be obtained by using a function that fits the given data. This process is called interpolation.

Algebraic Polynomials

Algebraic Polynomials

The algebraic polynomials are the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, \cdots, a_n$ are real constants.

Benefits of Algebraic Polynomials

- They uniformly approximate continuous functions.
- The derivative and indefinite integral of a polynomial are easy to determine and are also polynomials.

Weierstrass Approximation Theorem

Weierstrass Approximation Theorem

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b].$$

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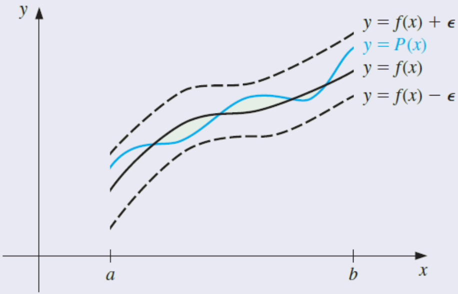


Outline

Taylor Polynomials

Taylor's Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

Here, $P_n(x)$ is called the nth Taylor polynomial for f about x_0, and $R_n(x)$ is called the remainder term (or truncation error) associated with $P_n(x)$.

Taylor Polynomials

Example1: $f(x) = e^x$

Calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$.

$$P_0\mathrm{x} = 1$$

$$P_1(\mathrm{x}) = 1 + \mathrm{x}$$

$$P_2(\mathrm{x}) = 1 + \mathrm{x} + \frac{\mathrm{x}^2}{2}$$

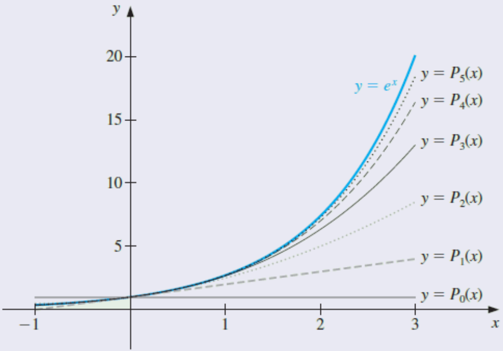
$$P_3(\mathrm{x}) = 1 + \mathrm{x} + \frac{\mathrm{x}^2}{2} + \frac{\mathrm{x}^3}{6}$$

$$P_4(\mathrm{x}) = 1 + \mathrm{x} + \frac{\mathrm{x}^2}{2} + \frac{\mathrm{x}^3}{6} + \frac{\mathrm{x}^4}{24}$$

$$P_5(\mathrm{x}) = 1 + \mathrm{x} + \frac{\mathrm{x}^2}{2} + \frac{\mathrm{x}^3}{6} + \frac{\mathrm{x}^4}{24} + \frac{\mathrm{x}^5}{120}$$

Taylor Polynomials

Example1: $f(x) = e^x$



- Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from zero.

Taylor Polynomials

Example2: $f(x) = 1/x$

Use Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

Taylor Polynomials

Example2: $f(x) = 1/x$

Use Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

The Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

Taylor Polynomials

Example2: $f(x) = 1/x$

Use Taylor polynomials of various degrees for $f(x) = 1/x$ expanded about $x_0 = 1$ to approximate $f(3) = 1/3$.

The Taylor polynomials are

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To approximate $f(3) = 1/3$ by $P_n(3)$

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

Taylor Polynomials

Remarks

- The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.
- However, a good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.
- The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

Outline

Lagrange Interpolating Polynomials

Polynomial Interpolation

Using a polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.

Lagrange Interpolating Polynomial

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

The linear Lagrange Interpolating Polynomial through (x_0, y_0) and (x_1, y_1) is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1)$$

Lagrange Interpolating Polynomial

Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Lagrange Interpolating Polynomial

Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Solution:

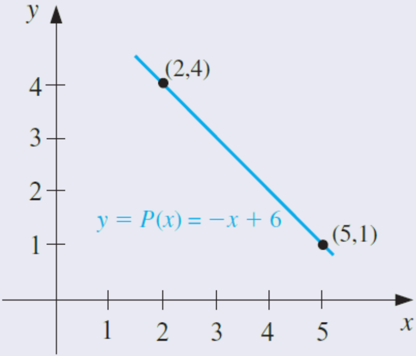
In this case we have

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

so

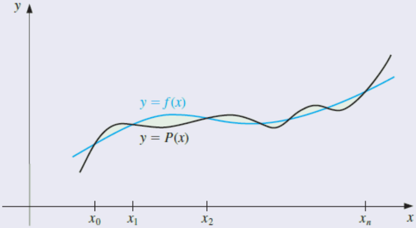
$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x+6.$$

Lagrange Interpolating Polynomial



The linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Lagrange Interpolating Polynomial



Generalization

To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $n+1$ points

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ \cdots,\ (x_n, f(x_n)).$$

Lagrange Interpolating Polynomial

Constructing the Degree n Polynomial

- Construct a function $L_{n,k}(x)$ such that

$$L_{n,k}(x_i) = \begin{cases} 1 & i = k \\ 0 & otherwise \end{cases}$$

Lagrange Interpolating Polynomial

Constructing the Degree n Polynomial

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-

$$\begin{aligned} L_{n,k}(x) &= \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \\ &= \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i} \end{aligned}$$

Lagrange Interpolating Polynomial

Theorem: n-th Lagrange Interpolating Polynomial

If $x_0, x_1, \cdots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \cdots, n.$$

The polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where, for each $k = 0, 1, \cdots, n$, $L_{n,k}(x)$ is defined as previous.

The Lagrange Polynomial: 2nd Degree

Example: $f(x) = 1/x$

1. Use the number (called nodes) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.
2. Use this polynomial to approximate $f(3) = 1/3$.

The Lagrange Polynomial: 2nd Degree

Solution (1/3)

We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$:

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$
$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$
$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75)$$

Also, since $f(x) = 1/x$:

$$f(x_0) = f(2) = 1/2, \quad f(x_1) = f(2.75) = 4/11, \quad f(x_2) = f(4) = 1/4$$

The Lagrange Polynomial: 2nd Degree

Solution (2/3)

Therefore, we obtain

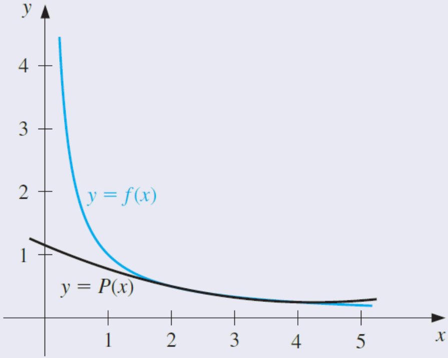
$$
\begin{aligned}
P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\
&= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75) \\
&= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.
\end{aligned}
$$

The Lagrange Polynomial: 2nd Degree

Solution (3/3)

An approximation to $f(3) = \frac{1}{3}$ is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955$$



The Lagrange Polynomial

Theorem: Theoretical Error Bound

Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

The Lagrange Polynomial

Proof: (1/6)

Note first that if $x = x_k$, for any $k = 0, 1, \cdots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in (a, b) yields the result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

if $x \neq x_k$, for all $k = 0, 1, \cdots, n$, define the function g for t in $[a, b]$ by

$$\begin{aligned} g(t) &= f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)} \\ &= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{t - x_i}{x - x_i} \end{aligned}$$

The Lagrange Polynomial

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Proof: (2/6)

Since $f \in C^{(n+1)}[a, b]$, and $P \in C^{\infty}[a, b]$, it follows that $g \in C^{(n+1)}[a, b]$. For $t = x_k$,we have

$$\mathrm{g}(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x_k - x_i}{x - x_i} = 0 - [f(x) - P(x)] \cdot 0 = 0$$

The Lagrange Polynomial

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Proof: (3/6)

We have seen that $\mathrm{g}(x_k) = 0$. Furthermore,

$$\begin{aligned} g(x) &= f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{x - x_i}{x - x_i} \\ &= f(x) - P(x) - [f(x) - P(x)] = 0 \end{aligned}$$

Thus $g \in C^{(n+1)}[a, b]$, and g is zero at the $n + 2$ distinct numbers $x, x_0, x_1, \cdots, x_n$.

The Lagrange Polynomial

Proof: (4/6)

Since $g \in C^{(n+1)}[a,b]$, and g is zero at the $n+2$ distinct numbers $x, x_0, x_1, \cdots, x_n$, by Generalized Rolle's Theorem, there exists a number ξ in(a,b) for which $g^{(n+1)}(\xi) = 0$. So

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) \\ &= f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{t - x_i}{x - x_i} \right]_{t=\xi} \end{aligned}$$

However, $P(x)$ is a polynomial of degree at most n, so the $(n+1)st$ derivative, $P^{(n+1)}(x)$, is identically zero.

The Lagrange Polynomial

Proof: (5/6)

Also, $\prod_{i=0}^{n} \frac{t-x_i}{x-x_i}$ is a polynomial of degree$(n+1)$, so

$$\prod_{i=0}^{n} \frac{t-x_i}{x-x_i} = \left[\frac{1}{\prod_{i=0}^{n}(x-x_i)}\right] t^{n+1} + (\text{lower} - \text{degree terms in t}),$$

and

$$\frac{d^{n+1}}{dt^{n+1}} \prod_{i=0}^{n} \frac{t-x_i}{x-x_i} = \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

The Lagrange Polynomial

Proof: (6/6)

We therefore have :

$$0 = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)] \frac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \frac{t - x_i}{x - x_i} \right]_{t=\xi}$$

$$= f^{(n+1)}(\xi) - 0 - [f(x) - P(x)] \frac{(n+1)!}{\prod_{i=0}^{n} (x - x_i)}$$

and upon solving for $f(x)$, we get the desired result:

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

The Lagrange Polynomial: 2nd Degree

Example: $f(x) = 1/x$

1. Use the number (called nodes) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.
2. Use this polynomial to approximate $f(3) = 1/3$.
3. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

The Lagrange Polynomial: 2nd Degree

Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2},\ \ f''(x) = \frac{2}{x^3},\ \ \textit{and}\ \ f'''(x) = -\frac{6}{x^4}$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2) = -\frac{1}{\xi(x)^4}(x-2)(x-2.75)(x-4)$$

for $\xi(x)$ in$(2,4)$. The maximum value of $\frac{1}{\xi(x)^4}$ on the interval is $\frac{1}{2^4} = \frac{1}{16}$

The Lagrange Polynomial: 2nd Degree

Solution (2/3)

We now need to determine the maximum value on $[2, 4]$ of the absolute value of the polynomial

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

Because

$$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7)$$

the critical points occur at

$$x = \frac{7}{3} \;\; with \;\; g(\frac{7}{3}) = \frac{25}{108} \;\; and \;\; x = \frac{7}{2} \;\; with \;\; g(\frac{7}{2}) = -\frac{9}{16}$$

The Lagrange Polynomial: 2nd Degree

Solution (3/3)

Hence, the maximum error is

$$\begin{aligned}
&\max_{[2,4]}\left|\frac{f'''(\xi(x))}{3!}\right| \cdot \max_{[2,4]}|(x-x_0)(x-x_1)(x-x_2)| \\
&\leq \frac{1}{16}\cdot\frac{9}{16} \\
&= \frac{9}{256} \\
&\approx 0.035156
\end{aligned}$$

The Lagrange Polynomial: 2nd Degree

Example: Tabulated Data

1. Suppose that a table is to be prepared for the function $f(x) = e^x$, for $x \in [0, 1]$.
2. Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h.
3. What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all $x \in [0, 1]$?

The Lagrange Polynomial: 2nd Degree

Solution (1/3)

The step size is h, so $x_j = jh, x_{j+1} = (j+1)h$, and

$$|f(x) - P(x)| \leq \frac{\left|f^{(2)}(\xi)\right|}{2!} \left|(x - jh)(x - (j+1)h\right|$$

Hence

$$|f(x) - P(x)| \leq \frac{\max_{\xi \in [0,1]} e^{\xi}}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j+1)h)|$$

$$\leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j+1)h)|$$

The Lagrange Polynomial: 2nd Degree

Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j+1)h)$, for $jh \leq x \leq (j+1)h$. Because

$$g'(x) = (x - (j+1)h) + (x - jh) = 2(x - jh - \frac{h}{2})$$

The only critical point for g is at $x = jh + \frac{h}{2}$, with

$$g(jh + \frac{h}{2}) = (\frac{h}{2})^2 = \frac{h^2}{4}$$

Since $g(jh) = 0$ and $g((j+1)h) = 0$, the maximum value of $|g(x)|$ in $[jh, (j+1)h]$ must occur at the critical point.

The Lagrange Polynomial: 2nd Degree

Solution (3/3)

This implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}$$

Consequently, to ensure that the error in linear interpolation is bounded by 10^{-6}, it is sufficient for h to be chosen so that $\frac{eh^2}{8} \leq 10^{-6}$. This implies that $h < 1.72 \times 10^{-3}$.
Because $n = \frac{(1-0)}{h}$ must be an integer, a reasonable choice for the step size is $h = 0.001$.

Example

Example: Tabulated Data

The following table

x	1.0	1.3	1.6	1.9	2.2
f(x)	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

lists values of a function f at various points. The approximations to $f(1.5)$ obtained by various Lagrange polynomials that use this data will be compared to try and determine the accuracy of the approximation.

Example

Solution (1/6)

The most appropriate linear polynomial uses $x_0 = 1.3$ and $x_1 = 1.6$ because 1.5 is between 1.3 and 1.6. The value of the interpolating polynomial at 1.5 is

$$P_1(1.5) = \frac{(1.5-1.6)}{(1.3-1.6)}f(1.3) + \frac{(1.5-1.3)}{(1.6-1.3)}f(1.6)$$

$$= \frac{(1.5-1.6)}{(1.3-1.6)}(0.6200860) + \frac{(1.5-1.3)}{(1.6-1.3)}(0.4554022)$$

$$= 0.5102968$$

Example

Solution (2/6)

Two polynomials of degree 2 can reasonably be used, one with $x_0 = 1.3$, $x_1 = 1.6$ and $x_2 = 1.9$, which gives

$$\begin{aligned} P_2(1.5) &= \frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)}(0.6200860) \\ &+ \frac{(1.5-1.3)(1.5-1.9)}{(1.6-1.3)(1.6-1.9)}(0.4554022) \\ &+ \frac{(1.5-1.3)(1.5-1.6)}{(1.9-1.3)(1.9-1.6)}(0.2818186) = 0.5112857 \end{aligned}$$

and one with $x_0 = 1.0$, $x_1 = 1.3$, and $x_2 = 1.6$, which gives $\hat{P}_2(1.5) = 0.5124715$.

Example

Solution (3/6)

- In the third-degree case, there are also two reasonable choices for the polynomial. One with $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$ and $x_4 = 2.2$, which gives $P_3(1.5) = 0.5118302$.
- The second third-degree approximation is obtained with $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$ and $x_3 = 1.9$, which gives $\hat{P}_3(1.5) = 0.5118127$.
- The fourth-degree Lagrange polynomial uses all the entries in the table.
- With $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$ and $x_4 = 2.2$, the approximation is$P_4(1.5) = 0.5118200$.

Example

Solution (4/6)

- Because $P_3(1.5), \hat{P}_3(1.5)$, and $P_4(1.5)$ all agree to within 2×10^{-5} units, we expect this degree of accuracy for these approximations.
- We also expect $P_4(1.5)$ to be the most accurate approximation, since it uses more of the given data.
- The function we are approximating is actually the Bessel function of the first kind of order zero, whose value at 1.5 is known to be 0.5118277.

Example

Solution (5/6)

Therefore, the true accuracies of the approximations are as follows:

$$
\begin{aligned}
|P_1(1.5) - f(1.5)| &\approx 1.53 \times 10^{-3} \\
|P_2(1.5) - f(1.5)| &\approx 5.42 \times 10^{-4} \\
\left|\hat{P}_2(1.5) - f(1.5)\right| &\approx 6.44 \times 10^{-4} \\
|P_3(1.5) - f(1.5)| &\approx 2.5 \times 10^{-6} \\
\left|\hat{P}_3(1.5) - f(1.5)\right| &\approx 1.50 \times 10^{-5} \\
|P_4(1.5) - f(1.5)| &\approx 7.7 \times 10^{-6}
\end{aligned}
$$

Example

Solution (6/6)

- Although$P_3(1.5)$ is the most accurate approximation, if we had no knowledge of the actual value of $f(1.5)$, we would accept $P_4(1.5)$ as the best approximation since it includes the most data about the function.
- The theoretical Lagrange error term cannot be applied here because we have no knowledge of the fourth derivative of f.
- Unfortunately, this is generally the case.

Outline

Introduction to Neville's Method

Definition: Lagrange Polynomial $P_{m_1,m_2,\cdots,m_k}(x)$

- Let f be a function defined at $x_0, x_1, x_2, \cdots, x_n$, and suppose that $m_1, m_2, \cdots, m_k$ are k distinct integers, with $0 \leq m_i \leq n$ for each i.
- The Lagrange polynomial that agrees with $f(x)$ at the k points $x_{m_1}, x_{m_2}, \cdots, x_{m_k}$ is denoted by

$$P_{m_1,m_2,\cdots,m_k}(x)$$

Introduction to Neville's Method

Example: $P_{1,2,4}(x)$

- Suppose that $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 6$, and $f(x) = e^x$.
- Determine the interpolating polynomial denoted $P_{1,2,4}(x)$, and use this polynomial to approximate $f(5)$.

Introduction to Neville's Method

Example: $P_{1,2,4}(x)$

- Suppose that $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 6$, and $f(x) = e^x$.
- Determine the interpolating polynomial denoted $P_{1,2,4}(x)$, and use this polynomial to approximate $f(5)$.

This is the Lagrange polynomial that agrees with $f(x)$ at $x_1 = 2, x_2 = 3$, and $x_4 = 6$. Hence

$$P_{1,2,4}(x) = \frac{(x-3)(x-6)}{(2-3)(2-6)}e^2 + \frac{(x-2)(x-6)}{(3-2)(3-6)}e^3 + \frac{(x-2)(x-3)}{(6-2)(6-3)}e^6$$

$$\begin{aligned} f(5) \approx P(5) &= \frac{(5-3)(5-6)}{(2-3)(2-6)}e^2 + \frac{(5-2)(5-6)}{(3-2)(3-6)}e^3 + \frac{(5-2)(5-3)}{(6-2)(6-3)}e^6 \\ &= -\frac{1}{2}e^2 + e^3 + \frac{1}{2}e^6 \approx 218.105 \end{aligned}$$

Recursive Lagrange Polynomial Approximations

Theorem

Let f be defined at $x_0, x_1, \cdots, x_k$, and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x - x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x - x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i - x_j)}$$

is the kth Lagrange polynomial that interpolates f at the $k+1$ points $x_0, x_1, \cdots, x_k$.

Recursive Lagrange Polynomial Approximations

For ease of notation, let

$$Q \equiv P_{0,1,\ldots,i-1,i+1,\ldots,k} \quad \text{and} \quad \hat{Q} \equiv P_{0,1,\ldots,j-1,j+1,\ldots,k}$$

Since $Q(x)$ and $\hat{Q}(x)$ are polynomials of degree $k-1$ or less, $P(x)$ is of degree at most k.

Proof (1/2)

First note that $\hat{Q}(x_i) = f(x_i)$, implies that

$$P(x_i) = \frac{(x_i - x_j)\hat{Q}(x_i) - (x_i - x_i)Q(x_i)}{x_i - x_j} = \frac{(x_i - x_j)}{(x_i - x_j)} f(x_i) = f(x_i)$$

Similarly, since $Q(x_j) = f(x_j)$, we have $P(x_j) = f(x_j)$.

Recursive Lagrange Polynomial Approximations

Proof (2/2)

In addition, if $0 \le r \le k$ and r is neither i nor j, then $Q(x_r) = \hat{Q}(x_r) = f(x_r)$. So

$$P(x_r) = \frac{(x_r - x_j)\hat{Q}(x_r) - (x_r - x_i)Q(x_r)}{x_i - x_j} = \frac{(x_i - x_j)}{(x_i - x_j)} f(x_r) = f(x_r)$$

But, by definition, $P_{0,1,\ldots,k}(x)$ is the unique polynomial of degree at most k that agrees with f at $x_0, x_1, \ldots, x_k$. Thus, $P \equiv P_{0,1,\ldots,k}$.

Recursive Lagrange Polynomial Approximations

Comments

- This result implies that the interpolating polynomials can be generated recursively.
- For example we have

$$
\begin{aligned}
P_{0,1} &= \frac{1}{x_1 - x_0}\left[(x - x_0)P_1 - (x - x_1)P_0\right] \\
P_{1,2} &= \frac{1}{x_2 - x_1}\left[(x - x_1)P_2 - (x - x_2)P_1\right] \\
P_{0,1,2} &= \frac{1}{x_2 - x_0}\left[(x - x_0)P_{1,2} - (x - x_2)P_{0,1}\right]
\end{aligned}
$$

and so on.

Neville's Method: Recursive Generation

The following table illustrate how the interpolating polynomials can be generated recursively, where each row is completed before the succeeding rows are begun.

x_0	P_0				
x_1	P_1	$P_{0,1}$			
x_2	P_2	$P_{1,2}$	$P_{0,1,2}$		
x_3	P_3	$P_{2,3}$	$P_{1,2,3}$	$P_{0,1,2,3}$	
x_4	P_4	$P_{3,4}$	$P_{2,3,4}$	$P_{1,2,3,4}$	$P_{0,1,2,3,4}$

The procedure that uses the results of the theorem to recursively generate interpolating polynomial approximations is called Neville's method.

Neville's Method: Recursive Generation

To avoid the multiple subscripts, we let $Q_{i,j}(x)$, for $0 \leq j \leq i$, denote the interpolating polynomial of degree j on the $(j+1)$ numbers $x_{i-j}, x_{i-j+1}, \cdots, x_{i-1}, x_i$; that is,

$$Q_{i,j} = P_{i-j,i-j+1,\cdots,i-1,i}.$$

Using this notation provides the following Q notation array

x_0	$P_0 = Q_{0,0}$				
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,\ldots,4} = Q_{4,4}$

Recursive Lagrange Polynomial Approximations

Example

Values of various interpolating polynomials at $x = 1.5$ were obtained in an earlier example using the following data:

x	1.0	1.3	1.6	1.9	2.2
$f(x)$	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

Apply Neville's method to the data by constructing a recursive table in the Q-notation array format.

Solution (1/6)

Let $x_0 = 1.0, x_1 = 1.3, x_2 = 1.6, x_3 = 1.9$, and $x_4 = 2.2$, then $Q_{0,0} = f(1.0), Q_{1,0} = f(1.3), Q_{2,0} = f(1.6), Q_{3,0} = f(1.9)$, and $Q_{4,0} = f(2.2)$.
These are the five polynomials of degree zero (constants) that approximate $f(1.5)$, and are the same as data given in the example table.

Recursive Lagrange Polynomial Approximations

Solution (2/6)

Calculating the first-degree approximation $Q_{1,1}(1.5)$ gives

$$\begin{aligned} Q_{1,1}(1.5) &= \frac{(x-x_0)Q_{1,0}-(x-x_1)Q_{0,0}}{x_1-x_0} \\ &= \frac{(1.5-1.0)Q_{1,0}-(1.5-1.3)Q_{0,0}}{1.3-1.0} \\ &= \frac{0.5(0.6200860)-0.2(0.7651977)}{0.3} \\ &= 0.5233449 \end{aligned}$$

Recursive Lagrange Polynomial Approximations

Solution (3/6)

Similarly

$$Q_{2,1}(1.5) = \frac{(1.5 - 1.3)(0.4554022) - (1.5 - 1.6)(0.6200860)}{1.6 - 1.3}$$
$$= 0.5102968$$
$$Q_{3,1}(1.5) = 0.5132634 \quad \text{and} \quad Q_{4,1}(1.5) = 0.5104270$$

The best linear approximation is expected to be $Q_{2,1}$ because 1.5 is between $x_1 = 1.3$ and $x_2 = 1.6$.

Recursive Lagrange Polynomial Approximations

Solution (4/6)

In a similar manner, approximations using higher-degree polynomials are given by

$$Q_{2,2}(1.5) = \frac{(1.5-1.0)(0.5102968)-(1.5-1.6)(0.5233449)}{1.6-1.0}$$

$$= 0.5124715$$

$$Q_{3,2}(1.5) = 0.5112857$$
$$Q_{4,2}(1.5) = 0.5137361$$

Recursive Lagrange Polynomial Approximations

Solution (5/6)

1.0	0.7651977				
1.3	0.6200860	0.5233449			
1.6	0.4554022	0.5102968	0.5124715		
1.9	0.2818186	0.5132634	0.5112857	0.5118127	
2.2	0.1103623	0.5104270	0.5137361	0.5118302	0.5118200

Recursive Lagrange Polynomial Approximations

Solution (6/6)

- If the latest approximation, $Q_{4,4}$, was not sufficiently accurate, another node, x_5, could be selected, and another row added:

$$x_5 \quad Q_{5,0} \quad Q_{5,1} \quad Q_{5,2} \quad Q_{5,3} \quad Q_{5,4} \quad Q_{5,5}.$$

  Then $Q_{4,4}$, $Q_{5,4}$, and $Q_{5,5}$ could be compared to determine further accuracy.

- The function in this example is the Bessel function of the first kind of order zero, whose value at 2.5 is -0.0483838, and the next row of approximations to $f(1.5)$ is

$$2.5 \quad -0.0483838 \quad 0.4807699 \quad 0.5301984 \quad 0.5119070 \quad 0.5118430 \quad 0.5118277$$

  The final new entry, 0.5118277, is correct to all 7 decimal places.

Recursive Lagrange Polynomial Approximations

Example: 4-Digit Values of *lnx*

The following table lists the values of $f(x) = \ln x$ accurate to the places given.

i	x_i	$\ln x_i$
0	2.0	0.6931
1	2.2	0.7885
2	2.3	0.8329

Use Neville's method and 4 - digit rounding arithmetic to approximate $f(2.1) = \ln 2.1$ by completing the Neville table.

Recursive Lagrange Polynomial Approximations

Solution: (1/2)

Because $x - x_0 = 0.1$, $x - x_1 = -0.1$, $x - x_2 = -0.2$, and we are given $Q_{0,0} = 0.6931$, $Q_{1,0} = 0.7885$, and $Q_{2,0} = 0.8329$, we have

$$Q_{1,1} = \frac{1}{0.2}[(0.1)0.7885 - (-0.1)0.6931] = \frac{0.1482}{0.2} = 0.7410$$

and

$$Q_{2,1} = \frac{1}{0.1}[(-0.1)0.8329 - (-0.2)0.7885] = \frac{0.07441}{0.1} = 0.7441.$$

The final approximation we can obtain from this data is

$$Q_{2,2} = \frac{1}{0.3}[(0.1)0.7441 - (-0.2)0.7410] = \frac{0.2276}{0.3} = 0.7420.$$

Solution: (2/2)

The calculations are summarized in the following table:

i	x_i	$x - x_i$	Q_{i0}	Q_{i1}	Q_{i2}
0	2.0	0.1	0.6931		
1	2.2	−0.1	0.7885	0.7410	
2	2.3	−0.2	0.8329	0.7441	0.7420

Accuracy of 4-Digit Approximations

Absolute Error vs. Error Bound (1/2)

In the preceding example, we have $f(2.1) = \ln 2.1 = 0.7419$ to four decimal places, so the absolute error is

$$|f(2.1) - P_2(2.1)| = |0.7419 - 0.7420| = 10^{-4}$$

However $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, and $f'''(x) = \frac{2}{x^3}$, so the Lagrange error formula gives the error bound

$$\begin{aligned}
|f(2.1) - P_2(2.1)| &= \left|\frac{f'''(\xi(2.1))}{3!}(x - x_0)(x - x_1)(x - x_2)\right| \\
&= \left|\frac{1}{3(\xi(2.1))^3}(0.1)(-0.1)(-0.2)\right| \\
&\leq \frac{0.002}{3(2)^3} = 8.3 \times 10^{-5}
\end{aligned}$$

Accuracy of 4-Digit Approximations

Absolute Error vs. Error Bound (2/2)

- Notice that the actual error, 10^{-4} , exceeds the error bound, 8.3×10^{-5}
- This apparent contradiction is a consequence of finite-digit computations
- We used four-digit rounding arithmetic, whereas the Lagrange error formula assumes infinite-digit arithmetic
- This caused our actual errors to exceed the theoretical error estimate.

Neville's Iterated Interpolation Algorithm

Neville's Iterated Interpolation Algorithm

INPUT numbers $x, x_0, x_1, \ldots, x_n$; values $f(x_0), f(x_1), \ldots, f(x_n)$ as the first column $Q_{0,0}, Q_{1,0}, \ldots, Q_{n,0}$ of Q.

OUTPUT the table Q with $P(x) = Q_{n,n}$.

Step 1 For $i = 1, 2, \ldots, n$

for $j = 1, 2, \ldots, i$

$$\text{set } Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}.$$

Step 2 OUTPUT (Q);

STOP.

Neville's Iterated Interpolation Algorithm

Additional Nodes & Stopping Criteria

- The algorithm can be modified to allow for the addition of new interpolating nodes. For example, the inequality.

$$|Q_{i,j} - Q_{i-1,j-1}| < \varepsilon$$

  can be used as a stopping criterion, where ε is a prescribed error tolerance
- If the inequality is ture, $Q_{i,j}$ is a reasonable approximation to $f(x)$.
- If the inequality is false, a new interpolation point, x_{i+1} is added.

Assignment

- Reading Assignment: Chap 3.1-3.2