Numerical Analysis

Lecture 12: Solution of Differential Equations

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Outline

Introduction

Initial-Value Problems for ODEs

Approximating the solution $y(t)$ to a problem of the form

$$\frac{dy}{dt} = f(t, y), \quad \text{for} \quad a \le t \le b$$

subject to an initial condition $y(a) = \alpha$.

- The methods do not produce a continuous approximation to the solution of the initial-value problem.
- Rather, approximations are found at certain specified, and often equally spaced, points.
- Some method of interpolation, is used if intermediate values are needed.

Definition: Lipschitz Condition

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a Lipschitz constant for f.

Elementary Theory of IVPs: Lipschitz Condition

Example

Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) | 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\}$.

Solution

For each pair of points (t, y_1) and (t, y_2) in D, we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|||y_1| - |y_2|| \leq 2|y_1 - y_2|.$$

Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$, because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|.$$

Elementary Theory of IVPs: Lipschitz Condition

Example

Show that $f(x) = \sqrt{x}$ does not satisfy a Lipschitz condition on the interval $D = \{x | 0 \le x \le 1\}$.

Proof: Given L, $x_1 = 0$, $x_2 = \dfrac{1}{4L^2}$, $|f(x_1) - f(x_2)| > L|x_1 - x_2|$.

Example

Show that $f(x) = \dfrac{1}{x}$ does not satisfy a Lipschitz condition on the interval $D = \{x | 0 \le x \le 1\}$.

Proof: Given L^2, $x_1 = \dfrac{1}{L}$, $x_2 = \dfrac{1}{2L}$, $|f(x_1) - f(x_2)| > L^2|x_1 - x_2|$.
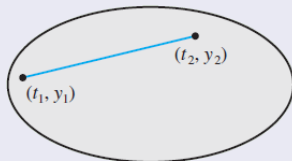
Elementary Theory of IVPs: Convex Set
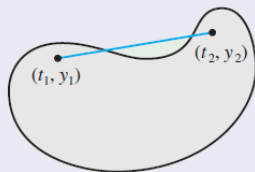
Convex Set

A set $D \subset \mathbb{R}^2$ is said to be convex if whenever (t_1, y_1) and (t_2, y_2) belong to D, then

$$((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2)$$

also belongs to D for every λ in [0, 1].

Convex   Not convex

Theorem: Sufficient Conditions

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \ \text{ for all } \ (t, y) \in D$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Elementary Theory of IVPs: Existence & Uniqueness

Theorem: Existence & Uniqueness

Suppose that $D = \{(t,y) | a \leq t \leq b$ and $-\infty < y < \infty\}$ and that $f(t,y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t,y), \ \ a \leq t \leq b, \ \ y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$.

Elementary Theory of IVPs: Existence & Uniqueness

Example

Use the Existence & Uniqueness Theorem to show that there is a unique solution to the initial-value problem

$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \quad y(0) = 0$$

Solution (1/2)

Holding t constant and applying the Mean Value Theorem to the function

$$f(t, y) = 1 + t\sin(ty)$$

we find that when $y_1 < y_2$, a number ξ in (y_1, y_2) exists with

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \frac{\partial}{\partial y}f(t, \xi) = t^2\cos(\xi t)$$

Elementary Theory of IVPs: Existence & Uniqueness

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t, \xi) = t^2 \cos(\xi t)$$

Solution (2/2)

- Thus

$$|f(t, y_2) - f(t, y_1)| = |y_2 - y_1||t^2 \cos(\xi t)| \leq 4|y_2 - y_1|$$

and f satisfies a Lipschitz condition in the variable y with lipschitz constant $L = 4$.

- Additionally, $f(t, y)$ is continuous when $0 \leq t \leq 2$ and $-\infty < y < \infty$, so the Existence & Uniqueness Theorem implies that a unique solution exists to this initial-value problem.

Elementary Theory of IVPs: Well-posed Problems

Definition: Well-Posed Problem

The initial-value problem

$$y'(t) = f(t, y), \ \ a \le t \le b, \ \ y(a) = \alpha,$$

is said to be a well-posed problem if the following 2 conditions are satisfied:

Elementary Theory of IVPs: Well-posed Problems

Definition: Well-Posed Problem (cont'd)

1. A unique solution, $y(t)$, to the problem exists, and

2. There exist constants $\epsilon_0 > 0$ and $k > 0$ such that for any ϵ, with $\epsilon_0 > \epsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \epsilon$ for all t in $[a, b]$, and when $|\delta_0| < \epsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t, z) + \delta(t), \ \ a \le t \le b, \ \ z(a) = \alpha + \delta_0,$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\epsilon \ \text{ for all } t \text{ in } [a, b].$$

Theorem: Well-Posed Problem

Suppose $D = \{(t,y) | a \leq t \leq b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$y'(t) = f(t,y), \ \ a \leq t \leq b, \ \ y(a) = \alpha,$$

is well-posed.

Example

Show that the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

is well posed on $D = \{(t, y) | 0 \le t \le 2 \text{ and } -\infty < y < \infty\}$.

Solution (1/3)

- Because
$$\left|\frac{\partial(y - t^2 + 1)}{\partial y}\right| = 1$$

  the Lipschitz Condition Theorem implies that
  $f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in y on D
  with Lipschitz constant 1.

- Since f is continuous on D, the Theorem on Well-Posed
  Problems implies that the problem is well-posed.

Solution (2/3)

- As an illustration, consider the solution to the perturbed problem

$$\frac{dz}{dt} = z - t^2 + 1 + \delta, \quad 0 \le t \le 2, \quad z(0) = 0.5 + \delta_0$$

where δ and δ_0 are constants.

- The solutions to the original problem and this perturbed problem are

$$
\begin{aligned}
y(t) &= (t+1)^2 - 0.5e^t \\
\text{and} \quad z(t) &= (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta
\end{aligned}
$$

respectively.

Elementary Theory of IVPs: Well-posed Problems

Solution (3/3)

- Suppose that ϵ is a positive number. If $|\delta| < \epsilon$ and $|\delta_0| < \epsilon$, then

$$|y(t) - z(t)| = |(\delta + \delta_0)e^t - \delta| \leq |\delta + \delta_0|e^2 + |\delta| \leq (2e^2 + 1)\epsilon$$

for all t.

- This implies that the original problem is well-posed with $k(\epsilon) = 2e^2 + 1$ for all $\epsilon > 0$.

Outline

Euler's Method

Objective

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$.

Euler's Method

Derivation

- Let $t_i = a + ih$, for $i = 0, 1, \cdots, N$,
  in which $h = (b - a)/N = t_{i+1} - t_i$ is called the step size.
- By Taylor's Theorem, we have

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$

  for some number ξ_i in (t_i, t_{i+1}).
- Substituting $h = t_{i+1} - t_i$ and $y' = f(t, y)$, we get

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i),$$

Euler's Method

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i),$$

Euler's Method

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \cdots, N$, by deleting the remainder term. Thus Euler's method is

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \cdots, N-1
\end{aligned}
$$

This equation is called the difference equation associated with Euler's method.

Euler's Method: Example

Example

Prior to introducing an algorithm for Euler's Method, we will illustrate the steps in the technique to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

at $t = 2$ using a step size of $h = 0.5$.

Euler's Method: Example

Solution

For this problem $f(t, y) = y - t^2 + 1$, so

$$
\begin{aligned}
w_0 &= y(0) = 0.5 \\
w_1 &= w_0 + 0.5 \left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25 \\
w_2 &= w_1 + 0.5 \left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25 \\
w_3 &= w_2 + 0.5 \left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375
\end{aligned}
$$

and

$$
y(2) \approx w_4 = w_3 + 0.5 \left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375
$$

Euler's Method: Algorithm

Algorithm

INPUT   endpoints a, b; integer N; initial condition α.

OUTPUT   approximation w to y at the $(N + 1)$ values of t.

Step 1   Set $h = (b - a)/N$

          $t = a$

          $w = \alpha$

          OUTPUT (t, w)

Step 2   For $i = 1, 2, \ldots, N$ do Steps 3 & 4

          Step 3 Set $w = w + hf(t, w)$;   (*Compute w_i*)

           $t = a + ih$.   (*Compute t_i*)
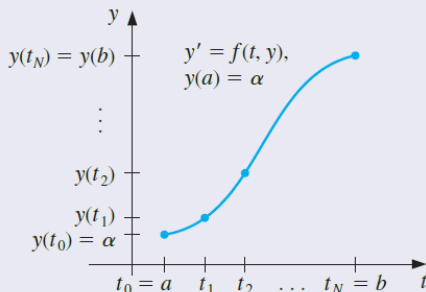
          Step 4 OUTPUT (t, w)

Step 5   STOP

Euler's Method: Geometric Interpretation

To interpret Euler's method geometrically, note that when w_i is a close approximation to $y(t_i)$, the assumption that the problem is well-posed implies that
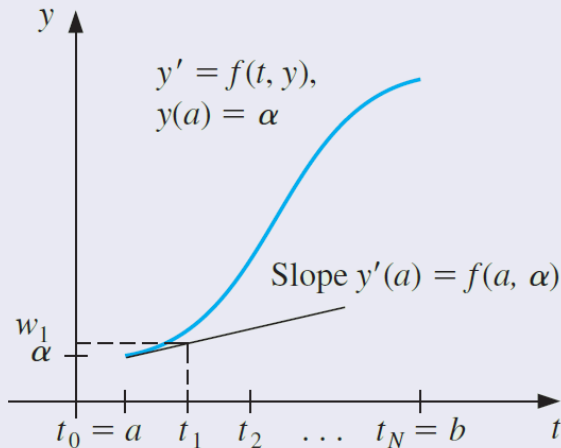
$$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$$

The graph of the function highlighting $y(t_i)$ is shown below

Euler's Method: Geometric Interpretation

One step in Euler's method

$$y' = f(t, y),$$
$$y(a) = \alpha$$

Slope $y'(a) = f(a, \alpha)$

w_1
α

$t_0 = a \quad t_1 \quad t_2 \quad \ldots \quad t_N = b \quad t$

Euler's Method: Geometric Interpretation

A series of steps in Euler's method

$y' = f(t, y),$
$y(a) = \alpha$
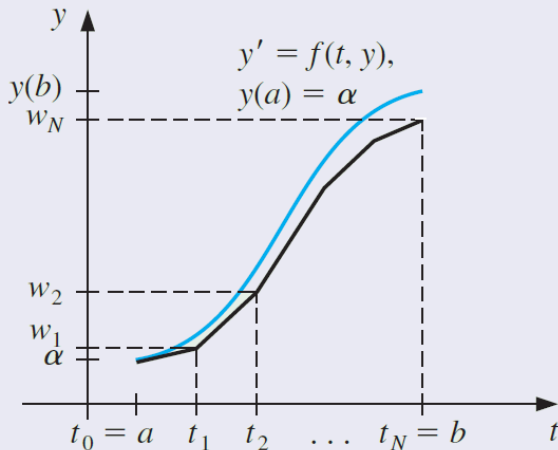
Euler's Method: Numerical Example

Example

Use the algorithm for Euler's method with $N = 10$ to determine approximations to the solution to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

and compare these with the exact values given by

$$y(t) = (t + 1)^2 - 0.5e^t$$

Euler's Method: Numerical Example

Solution

With $N = 10$, we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, so that:

$$
\begin{aligned}
w_{i+1} &= w_i + h(w_i - t_i^2 + 1) \\
&= w_i + 0.2[w_i - 0.04i^2 + 1] \\
&= 1.2w_i - 0.008i^2 + 0.2
\end{aligned}
$$

for $i = 0, 1, \ldots, 9$. So

$$
\begin{aligned}
w_1 &= 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8 \\
w_2 &= 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152
\end{aligned}
$$

and so on.

Euler's Method: Numerical Example

Results for $y' = y - t^2 + 1$, $\quad 0 \leq t \leq 2$, $\quad y(0) = 0.5$

t_i	w_i	$y_i = y(t_i)$	$|y_i - w_i|$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Euler's Method: Numerical Example

Comments

- Note that the error grows slightly as the value of *t* increases.
- This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.

Error Bounds for Euler's Method–Theorem

Suppose f is continuous and satisfies a Lipschitz condition on

$$D = \{(t,y)|a \le t \le b \text{ and } -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(t)| \le M, \quad \text{for all } t \in [a,b]$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha$$

Let $w_0, w_1, \cdots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then, for each $i = 0, 1, 2, \cdots, N$

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]$$

Error Bounds for Euler's Method

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + hf(t_i, w_i), \quad \text{for each } i = 0, 1, \cdots, N-1
\end{aligned}
$$

Round-off Error

$$
\begin{aligned}
u_0 &= \alpha + \delta_0 \\
u_{i+1} &= u_i + hf(t_i, u_i) + \delta_{i+1}, \quad \text{for each } i = 0, 1, \cdots, N-1
\end{aligned}
$$

Error Bounds for Euler's Method

The Error Bound with Round-off Error

$$|y(t_i) - u_i| \leq \frac{1}{L} \left(\frac{hM}{2} + \frac{\delta}{h} \right) \left[e^{L(t_i-a)} - 1 \right] + |\delta_0| e^{L(t_i-a)}$$

The error bound is no longer linear and it approaches infinity when $h \approx 0$. In fact, the optimal h can be achieved when

$$h = \sqrt{\frac{2\delta}{M}}.$$

Error Bounds for Euler's Method

Comments on the Theorem

- The weakness of the error-bound theorem lies in the requirement that a bound be known for the second derivative of the solution.

- Although this condition often prohibits us from obtaining a realistic error bound, it should be noted that if $\partial f / \partial t$ and $\partial f / \partial y$ both exist, the chain rule for partial differentiation implies that

$$y''(t) = \frac{dy'}{dt}(t) = \frac{df}{dt}(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))$$

- So it is at times possible to obtain an error bound for $y''(t)$ without explicitly knowing $y(t)$

Local Truncation Error

Informal Definition of LTE

The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation at that step.

Note

- We really want to know how well the approximations generated by the methods satisfy the differential equation, not the other way around.

- However, we don't know the exact solution so we cannot generally determine this, and the local truncation will serve quite well to determine not only the local error of a method but the actual approximation error.

Local Truncation Error

IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

Definition of LTE

The difference method

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \cdots, N-1,
\end{aligned}
$$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \cdots, N-1$, where y_i and y_{i+1} denote the solution at t_i and t_{i+1}, respectively.

Local Truncation Error

Example

Euler's method has local truncation error at the ith step

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \cdots, N - 1$$

- This error is a local error because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step.
- As such, it depends on the differential equation, the step size, and the particular step in the approximation.

Local Truncation Error

LTE in Euler's Method (Cont'd)

Earlier, we have seen that, for Euler's method:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i),$$

so that the LTE is

$$\tau_{i+1}(h) = \frac{h}{2}y''(\xi_i), \text{ for some } \xi_i \text{ in } (t_i, t_{i+1})$$

When $y''(t)$ is known to be bounded by a constant M on $[a, b]$, this implies

$$|\tau_{i+1}(h)| \leq \frac{h}{2}M$$

so the local truncation error in Euler's method is $O(h)$.

Outline

Higher-Order Taylor Methods

Motivation

- One way to select difference-equation methods for solving ordinary differential equations is in such a manner that their local truncation errors are $O(h^p)$ for as large a value of p as possible while keeping the number and complexity of calculations of the methods within a reasonable bound.
- Euler's method was derived by using Taylor's Theorem with $n = 1$ to approximate the solution of the differential equation.
- Can we extend this technique of derivation to larger values of n in order to find methods for improving the convergence properties of difference methods?

Higher-Order Taylor Methods

Assumption

The solution $y(t)$ to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has $(n + 1)$ continuous derivatives.

Taylor Expansion about t_i

If we expand the solution, $y(t)$, in terms of its nth Taylor polynomial about t_i and evaluate at t_{i+1}, we obtain

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

for some ξ_i in (t_i, t_{i+1}).

Higher-Order Taylor Methods

Derivation (Cont'd)

Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), \ y''(t) = f'(t, y(t)), \cdots, \ y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

Substituting these results into

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

gives

$$\begin{aligned} y(t_{i+1}) = \quad & y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots \\ & + \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

Higher-Order Taylor Methods

Derivation (Cont'd)

The difference-equation method corresponding to

$$y(t_{i+1}) = \quad y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots$$
$$+ \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))$$

is obtained by deleting the remainder term involving ξ_i.

Higher-Order Taylor Methods

Taylor's Method of order n

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \cdots, N-1$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i, w_i)$$

Higher-Order Taylor Methods

Example

Apply Taylor's method of orders
- 2 and
- 4

with $N = 10$ to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

Higher-Order Taylor Methods

Order 2 Method: Summary of Numerical Results

	Taylor Order 2	Error
t_i	w_i	$\|y(t_i) - w_i\|$
0.0	0.500000	0
0.2	0.830000	0.000701
0.4	1.215800	0.001712
0.6	1.652076	0.003135
0.8	2.132333	0.005103
⋮	⋮	⋮
1.6	4.306146	0.022663
1.8	4.846299	0.031122
2.0	5.347684	0.042212

Higher-Order Taylor Methods

Order 4 Method: Summary of Numerical Results

	Taylor Order 4	Error
t_i	w_i	$\|y(t_i) - w_i\|$
0.0	0.500000	0
0.2	0.829300	0.000001
0.4	1.214091	0.000003
0.6	1.648947	0.000006
⋮	⋮	⋮
1.4	3.732432	0.000032
1.6	4.283529	0.000045
1.8	4.815238	0.000062
2.0	5.305555	0.000083

Higher-Order Taylor Methods

Theorem

If Taylor's method of order n is used to approximate the solution to

$$y'(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

with step size h and if $y \in C^{n+1}[a, b]$, then the local truncation error is $O(h^n)$.

Higher-Order Taylor Methods

Proof (1/2)

When deriving Taylor Method, we obtained the expression

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

and this can be rewritten in the form

$$\begin{aligned} y_{i+1} \quad &-y_i - hf(t_i, y_i) - \frac{h^2}{2}f'(t_i, y_i) - \cdots - \frac{h^n}{n!}f^{(n-1)}(t_i, y_i) \\ &= \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

for some ξ_i in (t_i, t_{i+1}).

Higher-Order Taylor Methods

$$y_{i+1} \quad -y_i - hf(t_i, y_i) - \frac{h^2}{2}f'(t_i, y_i) - \cdots - \frac{h^n}{n!}f^{(n-1)}(t_i, y_i)$$
$$= \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))$$

Proof (2/2)

So the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i))$$

for each $i = 0, 1, \cdots, N - 1$. Since $y \in C^{n+1}[a, b]$, we have
$y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i(h) = O(h^n)$, for each
$i = 1, \cdots, N$.

Outline

Runge-Kutta Methods

Taylor Methods vs. Runge-Kutta Methods

- Taylor methods have the desirable property of high-order local truncation error.
- But the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$.
- This is a complicated and time-consuming procedure for most problems, so the Taylor methods are seldom used in practice.
- Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t, y)$.

Runge-Kutta Methods

Taylor Theorem in 2 Variables (1/2)

Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n + 1$ are continuous on $D = \{(t, y) | a \leq t \leq b, c \leq y \leq d\}$, and let $(t_0, y_0) \in D$. For every $(t, y) \in D$, there exists ξ between t and t_0 and μ between y and y_0 with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

The function $P_n(t, y)$ is called the *n*th Taylor polynomial in two variables for the function f about (t_0, y_0), and $R_n(t, y)$ is the remainder term associated with $P_n(t, y)$.

Runge-Kutta Methods

Taylor Theorem in 2 Variables (2/2)

$$P_n(t, y) = f(t_0, y_0) + \left[(t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right]$$

$$+ \left[\frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right]$$

$$+ \cdots + \left[\frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} (t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \right]$$

$$R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t - t_0)^{n+1-j}(y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu)$$

Basic Structure of RK2 Methods

Our starting point is to assume that the numerical method has the following structure:

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + h \cdot a_1 f(t_i + \alpha_1, w_i + \beta_1)
\end{aligned}
$$

for $i = 0, 1, \cdots, N - 1$, where a_1, α_1, and β_1 are parameters to be determined to ensure a local truncation error of $O(h^2)$.

Runge-Kutta Methods of Order Two

Method of Derivation

The first step is to determine values for a_1, α_1, and β_1 with the property that

$$a_1 f(t + \alpha_1, y + \beta_1) \approx T^{(2)}(t, y) = f(t, y) + \frac{h}{2} f'(t, y),$$

with error no greater than $O(h^2)$, which is the same as the order of the truncation error for the Taylor method of order two. Since

$$f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t) \ \text{ and } y'(t) = f(t, y),$$

we have

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y) \cdot f(t, y)$$

Runge-Kutta Methods of Order Two

Method of Derivation (Cont'd)

Expanding $f(t + \alpha_1, y + \beta_1)$ in its Taylor polynomial of degree one about (t, y) gives

$$a_1 f(t + \alpha_1, y + \beta_1) = a_1 f(t, y) + a_1 \alpha_1 \frac{\partial f}{\partial t}(t, y)$$

$$+ a_1 \beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 R_1(t + \alpha_1, y + \beta_1)$$

where

$$R_1(t + \alpha_1, y + \beta_1) = \frac{\alpha_1^2}{2} \frac{\partial^2 f}{\partial t^2}(\xi, \mu) + \alpha_1 \beta_1 \frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) + \frac{\beta_1^2}{2} \frac{\partial^2 f}{\partial y^2}(\xi, \mu)$$

for some ξ between t and $t + \alpha_1$ and μ between y and $y + \beta_1$.

Runge-Kutta Methods of Order Two

Method of Derivation (Cont'd)

Matching the coefficients of f and its derivatives in

$$a_1 f(t + \alpha_1, y + \beta_1) = a_1 f(t, y) + a_1 \alpha_1 \frac{\partial f}{\partial t}(t, y)$$

$$+ a_1 \beta_1 \frac{\partial f}{\partial y}(t, y) + a_1 R_1(t + \alpha_1, y + \beta_1)$$

and

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y) + \frac{h}{2} \frac{\partial f}{\partial y}(t, y) \cdot f(t, y)$$

gives the three equations

$$a_1 = 1 \qquad a_1 \alpha_1 = \frac{h}{2} \qquad a_1 \beta_1 = \frac{h}{2} f(t, y)$$

Runge-Kutta Methods of Order Two

$$a_1 = 1 \qquad a_1\alpha_1 = \frac{h}{2} \qquad a_1\beta_1 = \frac{h}{2}f(t,y)$$

Method of Derivation (Cont'd)

The parameters a_1, α_1, and β_1 are therefore

$$a_1 = 1 \qquad \alpha_1 = \frac{h}{2} \qquad \beta_1 = \frac{h}{2}f(t,y)$$

so that

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}t(f,y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right)$$

Runge-Kutta Methods of Order Two

Method of Derivation (Cont'd)

Earlier, we saw that

$$R_1(t + \alpha_1, y + \beta_1) = \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi, \mu) + \alpha_1\beta_1\frac{\partial^2 f}{\partial t\partial y}(\xi, \mu) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi, \mu)$$

which leads to

$$R_1\left(t + \frac{h}{2}, y + \frac{h}{2}t(f, y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi, \mu) + \frac{h^2}{4}f(t, y)\frac{\partial^2 f}{\partial t\partial y}(\xi, \mu)$$

$$+\frac{h^2}{8}(f(t, y))^2\frac{\partial^2 f}{\partial y^2}(\xi, \mu).$$

which is $O(h^2)$ if all the second-order partial derivatives of f are bounded.

Runge-Kutta Methods of Order Two

The difference-equation method resulting from replacing $T^{(2)}(t, y)$ in Taylor's method of order two by $f(t + (h/2), y + (h/2)f(t, y))$ is specific Runge-Kutta method known as the Midpoint Method.

The Midpoint Method

$$
\begin{aligned}
w_0 &= \alpha \\
w_{i+1} &= w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right)
\end{aligned}
$$

for each $i = 0, 1, \cdots, N - 1$.

Higher Order Runge-Kutta Methods

The Heun Method of Order 3

The term $T^{(3)}(t, y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$$

involving 4 parameters, but the algebra involved in the determination of $\alpha_1, \delta_1, \alpha_2,$ and δ_2 is quite involved. The most common $O(h^3)$ method is that of Huen, given by

$$w_0 = \alpha$$
$$w_{i+1} = w_i +$$

$$\frac{h}{4} \left(f(t_i, w_i) + 3f \left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f \left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i) \right) \right) \right)$$

Higher Order Runge-Kutta Methods

Runge-Kutta Order 4 Method

$$
\begin{aligned}
w_0 &= \alpha \\
k_1 &= hf(t_i, w_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, w_i + k_3) \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

for each $i = 0, 1, \cdots, N-1$. This method has local truncation error $O(h^4)$, provided the solution $y(t)$ has five continuous derivatives.

Higher Order Runge-Kutta Methods

Example

- For the problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

Euler's method with $h = 0.025$, the Midpoint method with $h = 0.05$, and the Runge-Kutta 4th-order method with $h = 0.1$ are compared at the common mesh points of these method 0.1, 0.2, 0.3, 0.4, and 0.5.

- Each of these techniques requires 20 function evaluations to determine the values (listed in the following table) to approximate $y(0.5)$.

Higher Order Runge-Kutta Methods

t_i	Exact	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta Order Four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
0.3	1.0150706	1.0089334	1.0147254	1.0150701
0.4	1.2140877	1.2056345	1.2136079	1.2140869
0.5	1.4256394	1.4147264	1.4250141	1.4256384

In this example, the fourth-order method is clearly superior.

Assignment

- Reading assignment: Chap. 5.1-5.4