In[1]:=
```
(* Clear content *)
ClearAll
```
Out[1]= ClearAll

In[2]:=
```
(* Define the wavelenghts and corresponding summand *)
lambdai[i_] = λ + (i - 1) * δ
lambdaj[i_, j_] = λ + (j - 1) * δ
summand[i_, j_] = FullSimplify[(lambdai[i] * lambdaj[i, j] / (lambdai[i] - lambdaj[i, j])) ^ 2]
```
Out[2]= $(-1 + i)\,\delta + \lambda$

Out[3]= $(-1 + j)\,\delta + \lambda$

Out[4]= $$\frac{((-1 + i)\,\delta + \lambda)^2\,((-1 + j)\,\delta + \lambda)^2}{(i - j)^2\,\delta^2}$$

In[5]:=
```
(* Alternative, simplified form of the summand *)
summandFull[i_, j_] = 1 / (i - j) ^ 2 * (1 + Δ * (i - 1)) ^ 2 * (1 + Δ * (j - 1)) ^ 2
```
Out[5]= $$\frac{(1 + (-1 + i)\,\Delta)^2\,(1 + (-1 + j)\,\Delta)^2}{(i - j)^2}$$

In[6]:=
```
(* Calculate the inner sum over i. NN
 is the total number of wavelengths consider *)
summedInner[i_] = FullSimplify[Sum[summandFull[i, j], {j, i + 1, NN}]]
```
Out[6]= $$\frac{1}{6}\,(1 + (-1 + i)\,\Delta)^2\,\big(6\,\Delta\,(2\,\mathrm{EulerGamma} + (2\,\mathrm{EulerGamma}\,(-1 + i) - i + NN)\,\Delta) + (\pi + (-1 + i)\,\pi\,\Delta)^2 + 12\,\Delta\,(1 + (-1 + i)\,\Delta)\,\mathrm{PolyGamma}[0,\,1 - i + NN] - 6\,(1 + (-1 + i)\,\Delta)^2\,\mathrm{PolyGamma}[1,\,1 - i + NN]\big)$$

In[7]:=
```
(* Calculate the outer sum over j *)
Sum[summedInner[i], {i, 1, NN}] - summedInner[NN]
```

Out[7]=

$$
\begin{aligned}
&-\frac{1}{6}(1+(-1+NN)\,\Delta)^2\left(-12\,\text{EulerGamma}\,\Delta\,(1+(-1+NN)\,\Delta)-\pi^2\,(1+(-1+NN)\,\Delta)^2+\right.\\
&\quad\left.6\,\Delta\,(2\,\text{EulerGamma}+2\,\text{EulerGamma}\,(-1+NN)\,\Delta)+(\pi+(-1+NN)\,\pi\,\Delta)^2\right)+\\
&\frac{1}{360}\left(-360\,\text{EulerGamma}+60\,NN\,\pi^2+720\,\text{EulerGamma}\,\Delta-720\,\text{EulerGamma}\,NN\,\Delta-\right.\\
&\quad 120\,NN\,\pi^2\,\Delta+120\,NN^2\,\pi^2\,\Delta-360\,\text{EulerGamma}\,\Delta^2+1080\,\text{EulerGamma}\,NN\,\Delta^2+360\,NN^2\,\Delta^2-\\
&\quad 1080\,\text{EulerGamma}\,NN^2\,\Delta^2+60\,NN\,\pi^2\,\Delta^2-180\,NN^2\,\pi^2\,\Delta^2+120\,NN^3\,\pi^2\,\Delta^2-360\,\text{EulerGamma}\,NN\,\Delta^3-\\
&\quad 360\,NN^2\,\Delta^3+1080\,\text{EulerGamma}\,NN^2\,\Delta^3+360\,NN^3\,\Delta^3-720\,\text{EulerGamma}\,NN^3\,\Delta^3+60\,NN^2\,\pi^2\,\Delta^3-\\
&\quad 120\,NN^3\,\pi^2\,\Delta^3+60\,NN^4\,\pi^2\,\Delta^3+12\,\text{EulerGamma}\,\Delta^4+57\,NN^2\,\Delta^4-180\,\text{EulerGamma}\,NN^2\,\Delta^4-\\
&\quad 186\,NN^3\,\Delta^4+360\,\text{EulerGamma}\,NN^3\,\Delta^4+117\,NN^4\,\Delta^4-180\,\text{EulerGamma}\,NN^4\,\Delta^4-\\
&\quad 2\,NN\,\pi^2\,\Delta^4+20\,NN^3\,\pi^2\,\Delta^4-30\,NN^4\,\pi^2\,\Delta^4+12\,NN^5\,\pi^2\,\Delta^4-360\,\text{PolyGamma}[0,\,1+NN]+\\
&\quad 720\,\Delta\,\text{PolyGamma}[0,\,1+NN]-720\,NN\,\Delta\,\text{PolyGamma}[0,\,1+NN]-360\,\Delta^2\,\text{PolyGamma}[0,\,1+NN]+\\
&\quad 1080\,NN\,\Delta^2\,\text{PolyGamma}[0,\,1+NN]-1080\,NN^2\,\Delta^2\,\text{PolyGamma}[0,\,1+NN]-\\
&\quad 360\,NN\,\Delta^3\,\text{PolyGamma}[0,\,1+NN]+1080\,NN^2\,\Delta^3\,\text{PolyGamma}[0,\,1+NN]-\\
&\quad 720\,NN^3\,\Delta^3\,\text{PolyGamma}[0,\,1+NN]+12\,\Delta^4\,\text{PolyGamma}[0,\,1+NN]-180\,NN^2\,\Delta^4\,\text{PolyGamma}[0,\,1+NN]+\\
&\quad 360\,NN^3\,\Delta^4\,\text{PolyGamma}[0,\,1+NN]-180\,NN^4\,\Delta^4\,\text{PolyGamma}[0,\,1+NN]-\\
&\quad 360\,NN\,\text{PolyGamma}[1,\,1+NN]+720\,NN\,\Delta\,\text{PolyGamma}[1,\,1+NN]-720\,NN^2\,\Delta\,\text{PolyGamma}[1,\,1+NN]-\\
&\quad 360\,NN\,\Delta^2\,\text{PolyGamma}[1,\,1+NN]+1080\,NN^2\,\Delta^2\,\text{PolyGamma}[1,\,1+NN]-\\
&\quad 720\,NN^3\,\Delta^2\,\text{PolyGamma}[1,\,1+NN]-360\,NN^2\,\Delta^3\,\text{PolyGamma}[1,\,1+NN]+\\
&\quad 720\,NN^3\,\Delta^3\,\text{PolyGamma}[1,\,1+NN]-360\,NN^4\,\Delta^3\,\text{PolyGamma}[1,\,1+NN]+\\
&\quad 12\,NN\,\Delta^4\,\text{PolyGamma}[1,\,1+NN]-120\,NN^3\,\Delta^4\,\text{PolyGamma}[1,\,1+NN]+\\
&\quad \left.180\,NN^4\,\Delta^4\,\text{PolyGamma}[1,\,1+NN]-72\,NN^5\,\Delta^4\,\text{PolyGamma}[1,\,1+NN]\right)
\end{aligned}
$$

In[8]:=
```
(* Simplify this expression *)
sumTot = FullSimplify[%]
```

Out[8]=

$$
\begin{aligned}
&\frac{1}{360}\Big(12\,\text{EulerGamma}\,\big(-30-60\,(-1+NN)\,\Delta-30\,(1+3\,(-1+NN)\,NN)\,\Delta^2-30\,(-1+NN)\,NN\,(-1+2\,NN)\,\Delta^3+\\
&\qquad\big(1-15\,(-1+NN)^2\,NN^2\big)\,\Delta^4\big)+NN\,\big(3\,NN\,\Delta^2\,\big(120+120\,(-1+NN)\,\Delta+(19+NN\,(-62+39\,NN))\,\Delta^2\big)+\\
&\qquad 2\,\pi^2\,\big(30+(-1+NN)\,\Delta\,\big(60+\Delta\,\big(-30+60\,NN+30\,(-1+NN)\,NN\,\Delta+\big(1+NN-9\,NN^2+6\,NN^3\big)\,\Delta^2\big)\big)\big)\big)-\\
&\quad 12\,\big(30+60\,(-1+NN)\,\Delta+30\,(1+3\,(-1+NN)\,NN)\,\Delta^2+30\,(-1+NN)\,NN\,(-1+2\,NN)\,\Delta^3+\\
&\qquad\big(-1+15\,(-1+NN)^2\,NN^2\big)\,\Delta^4\big)\,\text{PolyGamma}[0,\,1+NN]-\\
&\quad 12\,NN\,\big(30+(-1+NN)\,\Delta\,\big(60+\Delta\,\big(-30+60\,NN+30\,(-1+NN)\,NN\,\Delta+\big(1+NN-9\,NN^2+6\,NN^3\big)\,\Delta^2\big)\big)\big)\\
&\qquad\text{PolyGamma}[1,\,1+NN]\Big)
\end{aligned}
$$

In[9]:=
```
(* Use the ToMatlab extension to write this expression to Matlab *)
<< ToMatlab`
sumTot // ToMatlab
```

Out[10]=
```
(1/360).*(12.*EulerGamma.*((-30)+(-60).*((-1)+NN).*Δ+(-30).*(1+3.* ...
  ((-1)+NN).*NN).*Δ.^2+(-30).*((-1)+NN).*NN.*((-1)+2.*NN).*Δ.^3+(1+( ...
  -15).*((-1)+NN).^2.*NN.^2).*Δ.^4)+NN.*(3.*NN.*Δ.^2.*(120+120.*(( ...
  -1)+NN).*Δ+(19+NN.*((-62)+39.*NN)).*Δ.^2)+2.*pi.^2.*(30+((-1)+NN) ...
  .*Δ.*(60+Δ.*((-30)+60.*NN+30.*((-1)+NN).*NN.*Δ+(1+NN+(-9).*NN.^2+ ...
  6.*NN.^3).*Δ.^2))))+(-12).*(30+60.*((-1)+NN).*Δ+30.*(1+3.*((-1)+ ...
  NN).*NN).*Δ.^2+30.*((-1)+NN).*NN.*((-1)+2.*NN).*Δ.^3+((-1)+15.*(( ...
  -1)+NN).^2.*NN.^2).*Δ.^4).*PolyGamma(0,1+NN)+(-12).*NN.*(30+((-1)+ ...
  NN).*Δ.*(60+Δ.*((-30)+60.*NN+30.*((-1)+NN).*NN.*Δ+(1+NN+(-9).* ...
  NN.^2+6.*NN.^3).*Δ.^2))).*PolyGamma(1,1+NN));
```

In[11]:=
```
(* Change Δ to Range / (N-1) *)
variableChange = FullSimplify[sumTot /. {Δ → R / (NN - 1)}]
```

Out[11]=

$$\frac{1}{360}\Bigg(NN\Bigg(\frac{3\,NN\,R^2\left(120+R\left(120+\frac{(19+NN\,(-62+39\,NN))\,R}{(-1+NN)^2}\right)\right)}{(-1+NN)^2}+2\,\pi^2\left(30+R\left(60+\frac{R\left(-30+60\,NN+30\,NN\,R+\frac{(1+NN-9\,NN^2+6\,NN^3)\,R^2}{(-1+NN)^2}\right)}{-1+NN}\right)\right)\Bigg)+12\,\text{EulerGamma}\left(-30+R\left(-60+\frac{R\left(-30+R^2+30\,NN\,(5+R)-15\,NN^4\,(6+R\,(4+R))+30\,NN^3\,(9+R\,(5+R))-15\,NN^2\,(20+R\,(8+R))\right)}{(-1+NN)^4}\right)\right)-12\left(30+R\left(60+\frac{1}{(-1+NN)^4}R\left(30\,(-1+NN)^2\,(1+3\,(-1+NN)\,NN)+30\,(-1+NN)^2\,NN\,(-1+2\,NN)\,R+\left(-1+15\,(-1+NN)^2\,NN^2\right)R^2\right)\right)\right)\text{PolyGamma}[0,\,1+NN]-12\,NN\left(30+R\left(60+\frac{R\left(-30+60\,NN+30\,NN\,R+\frac{(1+NN-9\,NN^2+6\,NN^3)\,R^2}{(-1+NN)^2}\right)}{-1+NN}\right)\right)\text{PolyGamma}[1,\,1+NN]\Bigg)$$

In[12]:=
```
(* Now define the Sum as shown in the article (Eq. 30) *)
Sfunction = FullSimplify[variableChange / (NN ^ 2 * R ^ 2)]
```

Out[12]=

$$\frac{1}{360\,NN^2\,R^2}\left(NN\left(\frac{3\,NN\,R^2\left(120+R\left(120+\frac{(19+NN\,(-62+39\,NN))\,R}{(-1+NN)^2}\right)\right)}{(-1+NN)^2}+2\,\pi^2\left(30+R\left(60+\frac{R\left(-30+60\,NN+30\,NN\,R+\frac{(1+NN-9\,NN^2+6\,NN^3)\,R^2}{(-1+NN)^2}\right)}{-1+NN}\right)\right)+12\,\text{EulerGamma}\left(-30+R\left(-60+\frac{R\left(-30+R^2+30\,NN\,(5+R)-15\,NN^4\,(6+R\,(4+R))+30\,NN^3\,(9+R\,(5+R))-15\,NN^2\,(20+R\,(8+R))\right)}{(-1+NN)^4}\right)\right)\right)-12\left(30+R\left(60+\frac{R\left(30\,(-1+NN)^2\,(1+3\,(-1+NN)\,NN)+30\,(-1+NN)^2\,NN\,(-1+2\,NN)\,R+\left(-1+15\,(-1+NN)^2\,NN^2\right)R^2\right)}{(-1+NN)^4}\right)\right)\text{PolyGamma}[0,\,1+NN]-12\,NN\left(30+R\left(60+\frac{R\left(-30+60\,NN+30\,NN\,R+\frac{(1+NN-9\,NN^2+6\,NN^3)\,R^2}{(-1+NN)^2}\right)}{-1+NN}\right)\right)\text{PolyGamma}[1,\,1+NN]\right)$$

In[13]:=
```
(* To Matlab again *)
Sfunction // ToMatlab
```

Out[13]=
```
(1/360).*NN.^(-2).*R.^(-2).*(NN.*(3.*((-1)+NN).^(-2).*NN.*R.^2.*( ...
  120+R.*(120+((-1)+NN).^(-2).*(19+NN.*((-62)+39.*NN)).*R))+2.* ...
  pi.^2.*(30+R.*(60+((-1)+NN).^(-1).*R.*((-30)+60.*NN+30.*NN.*R+(( ...
  -1)+NN).^(-2).*(1+NN+(-9).*NN.^2+6.*NN.^3).*R.^2))))+12.* ...
  EulerGamma.*((-30)+R.*((-60)+((-1)+NN).^(-4).*R.*((-30)+R.^2+30.* ...
  NN.*(5+R)+(-15).*NN.^4.*(6+R.*(4+R))+30.*NN.^3.*(9+R.*(5+R))+(-15) ...
  .*NN.^2.*(20+R.*(8+R)))))+(-12).*(30+R.*(60+((-1)+NN).^(-4).*R.*( ...
  30.*((-1)+NN).^2.*(1+3.*((-1)+NN).*NN)+30.*((-1)+NN).^2.*NN.*((-1) ...
  +2.*NN).*R+((-1)+15.*((-1)+NN).^2.*NN.^2).*R.^2))).*PolyGamma(0,1+ ...
  NN)+(-12).*NN.*(30+R.*(60+((-1)+NN).^(-1).*R.*((-30)+60.*NN+30.* ...
  NN.*R+((-1)+NN).^(-2).*(1+NN+(-9).*NN.^2+6.*NN.^3).*R.^2))).* ...
  PolyGamma(1,1+NN));
```

In[16]:=
```
(* Asymptotic analysis as N is large *)
Series[Sfunction, {NN, ∞, 1}]
```

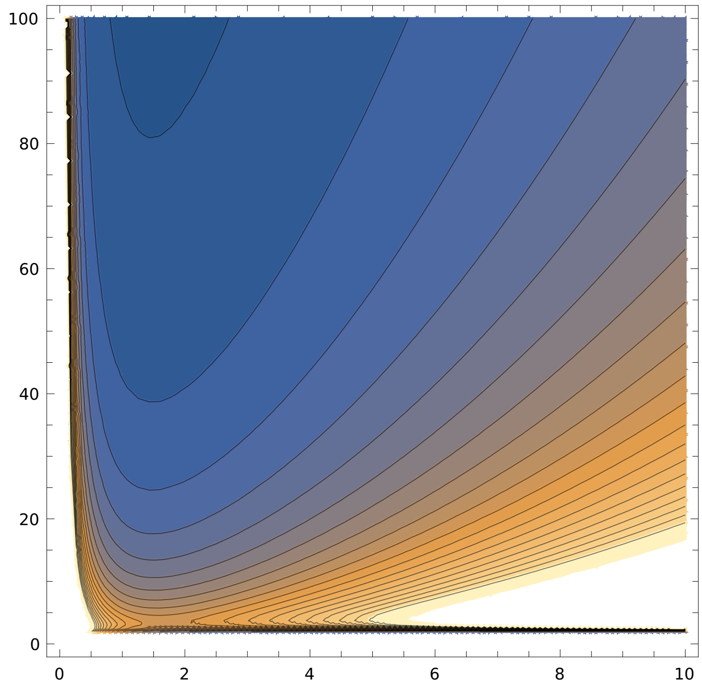
Out[16]=
$$\frac{\pi^2\left(5+10\,R+10\,R^2+5\,R^3+R^4\right)}{30\,R^2\,NN}+O\left[\frac{1}{NN}\right]^2$$

```
(* Same idea *)
infAsymptote = FullSimplify[Normal[Series[Sfunction, {NN, ∞, 1}]]]
```

Out[19]=
$$\frac{\pi^2\,(5+R\,(10+R\,(10+R\,(5+R))))}{30\,NN\,R^2}$$

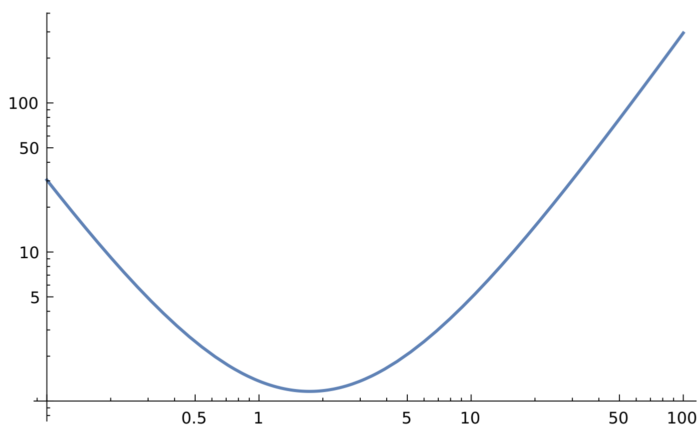
```
(* Contour plot of the Sum (Figure 4 in article) *)
ContourPlot[Sfunction, {R, 0.001, 10},
 {NN, 2, 100}, Contours → 20, PlotRange → Automatic]
```

Out[24]=



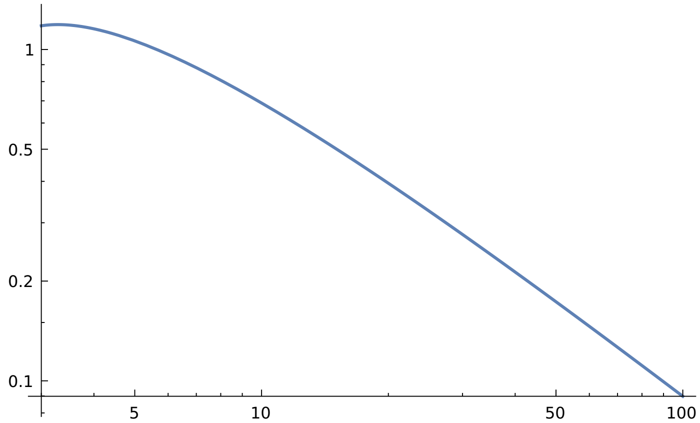

```
(* Slice at N = 3 *)
LogLogPlot[Sfunction /. {NN → 3}, {R, 0.1, 100}]
```

Out[25]=



```
(* Slice at R → 1.4587 (optimal R for large N) *)
LogLogPlot[Sfunction /. {R → 1.4587}, {NN, 3, 100}]
```

Out[26]=




```
(* Find the optimal R for N=3 *)
dR = FullSimplify[D[Sfunction /. {NN → 3}, R]]
```

Out[27]= $\frac{(-3 + R^2)(3 + R(3 + R))}{18 R^3}$

In[28]:= `N[Solve[dR == 0, R]]`

Out[28]= {{R → -1.73205}, {R → 1.73205}, {R → -1.5 - 0.866025 ⅈ}, {R → -1.5 + 0.866025 ⅈ}}

```
(* Find the optimal R for large N *)
(* First define the function of interest *)
fR[R_] = 1 / R ^ 2 * (R ^ 4 / 5 + R ^ 3 + 2 * R ^ 2 + 2 * R + 1)
```

Out[30]= $\frac{1 + 2 R + 2 R^2 + R^3 + \frac{R^4}{5}}{R^2}$

```
(* Then compute its derivative and solve the point for
 which the derivative is 0. Only a single positive real rool *)
N[Solve[D[fR[R], R] == 0]]
```

Out[31]= {{R → -1.81987}, {R → 1.45897}, {R → -1.06955 - 0.859767 ⅈ}, {R → -1.06955 + 0.859767 ⅈ}}