EP 501 Midterm exam: Chapters 1,3-5

October 31, 2019

Instructions:

- Answer all questions
- You may use the course textbook during this exam
- You may log into your computer and use Matlab.
- You may use *your own* course notes for this exam
- You may use my course notes.
- You may access the course Canvas site during this exam: `https://erau.instructure.com/courses/101937`
- You may visit the course repository: `https://github.com/mattzett/EP501`
- You *may not* use an internet browser to access search capabilities and internet references.

1. Use the Taylor Series method (§5.4 in the textbook) to develop derivatives for a *nonuniform* (unequally spaced) grid consisting of the three points x_{i-1}, x_i, x_{i+1} and function values at those points f_{i-1}, f_i, f_{i+1} defined by:

$$
\begin{aligned}
x_{i-1} &= x_i - \Delta x_b && (1)\\
x_{i+1} &= x_i + \Delta x_f && (2)\\
(\Delta x_b &\neq \Delta x_f) && (3)\\
f_{i-1} &= f(x_i - \Delta x_b) && (4)\\
f_{i+1} &= f(x_i + \Delta x_f) && (5)
\end{aligned}
$$

   (a) Develop a *centered* difference approximation for the first derivative with respect to x at the i^{th} grid point, i.e. derive an approximation for:

$$f'(x_i) = \left[\frac{df}{dx}\right]_i \tag{6}$$

   (b) Show that the truncation error for your finite difference formula is $\mathcal{O}(\Delta x_f - \Delta x_b)$

   (c) Obtain an approximate second derivative:

$$f''(x_i) = \left[\frac{d^2 f}{dx^2}\right]_i \tag{7}$$

   by iteratively applying your first derivative formula derived from part (a), e.g.:

$$\left[\frac{d^2 f}{dx^2}\right]_i \approx \frac{\left[\frac{df}{dx}\right]_{i+1/2} - \left[\frac{df}{dx}\right]_{i-1/2}}{x_{i+1/2} - x_{i-1/2}} \tag{8}$$

(This page intentionally left blank for additional space for question 1)

2. Derive and explain matrix condition numbers.

   (a) Use a method similar to that presented in §1.6.3.2 to show that the condition number relates variations in the solution vector $\underline{b}$ to variations in the matrix $\underline{\underline{A}}$ via the formula:

$$\frac{||\delta \underline{b}||}{||\underline{b}||} \leq \mathcal{C}(\underline{\underline{A}}) \frac{||\delta \underline{\underline{A}}||}{||\underline{\underline{A}}||} \tag{9}$$

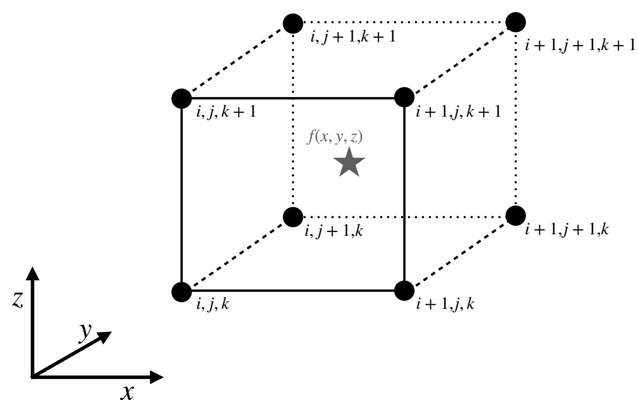
   where $\mathcal{C}(\underline{\underline{A}})$ is the condition number of the matrix $\underline{\underline{A}}$.

   (b) Provide a brief (several sentence) explanation of the meaning of the condition number $\mathcal{C}(\underline{\underline{A}})$. Include a discussion of what large and small condition numbers mean and why they are undesirable.

3. *Trilinear* interpolation involves approximation of an underlying three dimensional function using a polynomial of the form:

$$f(x, y, z) \approx a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 xz + a_8 xyz \tag{10}$$

for the region $x_i \le x \le x_{i+1}, y_j \le x \le y_{j+1}, z_k \le x \le z_{k+1}$. Set up a system of equations that can be solved for the coefficients $\underline{a} \equiv a_\ell$ using the value of the function f at the eight points defining the vertices of this cube-shaped region, shown in the diagram below. Express your system of equations in matrix form.



4. Suppose we wish to perform a least squares fit (§4.10) to a set of measurements y_i sampled at independent variable locations x_i using the functional form:

$$y(x) = ax^2 + bx^5 \tag{11}$$

Derive a system of equations can can be solved to determine the coefficients a, b. Express your system in matrix form.