EP501 HW5 Equations

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Problem 1 Equations

Following the method outlined in Section 8.4.1 of the textbook, the general form of the equation we are working with is:

$$\bar{y}'' + P(x)\bar{y}' + Q(x)\bar{y} = F(x) \ , \tag{1}$$

where

$$\begin{aligned} P(x) &= \frac{1}{\epsilon}\frac{d\epsilon}{dx} \\ Q(x) &= 0 \\ F(x) &= 0 \\ y &= \Phi \end{aligned}$$

Substituting second-order centered difference expressions for the derivatives as given by Equations 8.43 and 8.44 in the textbook and rearranging gives a version of Equation 8.46:

$$\left(1 - \frac{\Delta x}{2}P_i\right)\Phi_{i-1} - 2\Phi_i + \left(1 + \frac{\Delta x}{2}P_i\right)\Phi_{i+1} = 0 \tag{2}$$

P_i can be described by a second-order FDA at the interior points and a first-order FDA at the boundaries. Assuming that the computational grid begins at index 1 at the left boundary and ends at index N at the right boundary,

$$\begin{aligned} P(x) &= \frac{\epsilon_{i+1} - \epsilon_{i-1}}{2\epsilon_i \Delta x} \qquad \text{(interior)} \\ P(x) &= \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 \Delta x} \qquad \text{(left boundary)} \\ P(x) &= \frac{\epsilon_N - \epsilon_{N-1}}{\epsilon_N \Delta x} \qquad \text{(right boundary)} \end{aligned}$$

Substituting the interior $P(x)$ function into (2) gives the FDE for this system:

$$\left(1 - \frac{\epsilon_{i+1} - \epsilon_{i-1}}{4\epsilon_i}\right)\Phi_{i-1} - 2\Phi_i + \left(1 + \frac{\epsilon_{i+1} - \epsilon_{i-1}}{4\epsilon_i}\right)\Phi_{i+1} = 0 \tag{3}$$

Given our two initial conditions and the same index numbering scheme, the two resulting FDEs for the boundaries are

$$1000 = \frac{\Phi_2 - \Phi_1}{\Delta x} \qquad (\text{for } x = -a) \tag{4}$$

$$100 = \Phi_N \qquad (\text{for } x = a) \tag{5}$$

The resulting system of equations, expressed in matrix format, is

$$\begin{bmatrix} -1 & 1 & & & & & 0 \\ \alpha & -2 & \beta & & & & \\ & \alpha & -2 & \beta & & & \\ & & & \ddots & & & \\ & & & & \alpha & -2 & \beta \\ 0 & & & & & & 1 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \vdots \\ \Phi_{N-1} \\ \Phi_N \end{bmatrix} = \begin{bmatrix} 1000\Delta x \\ 0 \\ 0 \\ \vdots \\ 0 \\ 100 \end{bmatrix} \tag{6}$$

where blank elements are zero, and α and β are given by

$$\begin{aligned} \alpha &= 1 - \frac{\epsilon_{i+1} - \epsilon_{i-1}}{4\epsilon_i} \\ \beta &= 1 + \frac{\epsilon_{i+1} - \epsilon_{i-1}}{4\epsilon_i} \end{aligned}$$

Since the dielectric function varies rapidly at the edges, the derivative of Φ can be better approximated by a second-order FDA at $x = -a$:

$$\frac{d\Phi(-a)}{dx} = \frac{-\Phi_3 + 4\Phi_2 - 3\Phi_1}{2\Delta x} \tag{7}$$