

EP 501 Homework 4: Differentiation, Integration, and Multidimensional functions

November 21, 2019

Instructions:

- Submit all source code and publish Matlab results in .pdf form via Canvas. Please zip all contents of your solution into single file and then submit in a single zip file.
- Discussing the assignment with others is fine, but you must not copy anyone's code.
- I must be able to run your code and produce all results by executing a single top-level Matlab script, e.g. `assignment1.m` or similar.
- You may use any of the example codes from our course repository: <https://github.com/mattzett/EP501/>.
- Do not copy verbatim any other codes (i.e. any source codes other than from our course repository). You may use other examples as a reference but you must write your own programs (except for those I give you).

Purpose of this assignment:

- Deal with numerical differentiation to solve complex problems.
- Develop good coding and documentation practices, such that your programs are easily understood by others.
- Exercise good judgement in numerical problem setup.
- Demonstrate higher reasoning to synthesize a problem and devise a basic algorithm to solve it.

1. Vector derivatives and multidimensional plotting:

- (a) Plot the two components of the vector magnetic field defined by the piecewise function:

$$\mathbf{B}(x, y) = \begin{cases} \frac{\mu_0 I}{2\pi a^2} \sqrt{x^2 + y^2} \left(-\frac{y}{\sqrt{x^2 + y^2}} \hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2 + y^2}} \hat{\mathbf{e}}_y \right) & \left(\sqrt{x^2 + y^2} < a \right) \\ \frac{\mu_0 I}{2\pi \sqrt{x^2 + y^2}} \left(-\frac{y}{\sqrt{x^2 + y^2}} \hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2 + y^2}} \hat{\mathbf{e}}_y \right) & \left(\sqrt{x^2 + y^2} \geq a \right) \end{cases} \quad (1)$$

Assume the parameters in this equation have the values:

$$\begin{aligned} I &= 10 & (\text{A}) \\ \mu_0 &= 4\pi \times 10^{-7} & (\text{H/m}) \\ a &= 0.005 & (\text{m}) \end{aligned}$$

Use an image plot (`imagesc`, `pcolor`) for each component and have your plot show the region $-3a \leq x \leq 3a, -3a \leq y \leq 3a$. Make sure you add a colorbar and axis labels to your plot. You will need to define a range and resolution in x and y , and create a meshgrid from that. Be sure to use a resolution fine enough to resolve important variations in this function.

- (b) Make a quiver plot of the magnetic field \mathbf{B} ; add labels, etc.
(c) Compute the numerical curl of \mathbf{B} , i.e. $\nabla \times \mathbf{B}$. Use centered differences on the interior grid points and first-order derivatives on the edges. Plot your result using `imagesc`, or `pcolor`.
(d) Compute $\nabla \times \mathbf{B}$ analytically. Plot the alongside your numerical approximation. Demonstrate that they are suitably similar.
(e) Compute and plot the scalar field:

$$\Phi(x, y, z) = \begin{cases} \frac{Q}{4\pi\epsilon_0 a} - \frac{Q}{8\pi\epsilon_0 a^3} (x^2 + y^2 + z^2 - a^2) & \left(\sqrt{x^2 + y^2 + z^2} < a \right) \\ \frac{Q}{4\pi\epsilon_0 \sqrt{x^2 + y^2 + z^2}} & \left(\sqrt{x^2 + y^2 + z^2} \geq a \right) \end{cases} \quad (2)$$

Use the parameters:

$$\begin{aligned} Q &= 1 & (\text{C}) \\ a &= 1 & (\text{m}) \\ \epsilon_0 &= 8.854 \times 10^{-12} & (\text{F/m}) \end{aligned}$$

and plot this function in the region $-3a \leq x \leq 3a, -3a \leq y \leq 3a$ in the $z = 0$ plane. Be sure to use a resolution fine enough to resolve variations in this function (aside from those associated with the singularity).

- (e) Write a function to numerically compute the Laplacian of a scalar field, i.e. $\nabla^2 \Phi$. Plot your result and include appropriate labels and colorbars.

2. Integration in multiple dimensions.

- (a) Numerically compute the electrostatic energy in the region $R \equiv -3a \leq x \leq 3a, -3a \leq y \leq 3a, -3a \leq z \leq 3a$, defined by the integral:

$$W_E = -\frac{1}{2} \iiint_R (\epsilon_0 \nabla^2 \Phi) \Phi \, dx dy dz \quad (3)$$

using a iterated trapezoidal or multi-dimensional trapezoidal method.

- (b) Compute and plot the parametric path

$$\mathbf{r}(\phi) \equiv x(\phi)\hat{\mathbf{e}}_x + y(\phi)\hat{\mathbf{e}}_y = r_0 \cos \phi \hat{\mathbf{e}}_x + r_0 \sin \phi \hat{\mathbf{e}}_y \quad (0 \leq \phi \leq 2\pi) \quad (4)$$

in the x, y plane on the same axis as your magnetic field components from the previous problem (plot the path in the figure for each component). Take $r_0 = 2a$. You will need to define a grid in ϕ to do this.

- (c) Plot the two components of the magnetic field $\mathbf{B}(x(\phi), y(\phi))$ at the x, y points along \mathbf{r} (I recommend `yyaxis` for this) and visually compare against your image plots of the magnetic field.
- (d) Numerically compute the tangent vector to the path \mathbf{r} by performing the derivative: $d\mathbf{r}/d\phi$. Compare your numerical results against the analytical derivative (e.g. plot the two) and refine your grid in ϕ (if necessary) such that you get visually acceptable results - i.e. such that the path appears circular.
- (e) Numerically compute the magnetic field integrated around the path \mathbf{r} , i.e.:

$$\frac{1}{\mu_0} \int_{\mathbf{r}} \mathbf{B} \cdot d\boldsymbol{\ell} \quad (5)$$

where the differential path length is given by:

$$d\boldsymbol{\ell} = \frac{d\mathbf{r}}{d\phi} d\phi \quad (6)$$