PAUL YUSKA

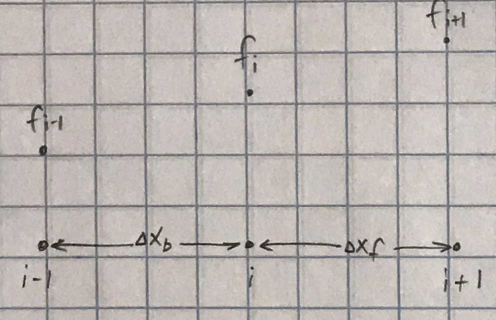
1)

$x_{i-1} = x_i - \Delta x_b$

$x_{i+1} = x_i + \Delta x_f \qquad \Delta x_b \neq \Delta x_f$

$f_{i-1} = f(x_i - \Delta x_b)$

$f_{i+1} = f(x_i + \Delta x_f)$



Taylor Series definition : $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

for $a = x_i$ and $x = x_{i+1}, x_{i-1}$, the Taylor series for f_{i+1} & f_{i-1} become, respectively:

$$f_{i+1} = f_i + f_i'\underbrace{(x_i + \Delta x_f - x_i)}_{\Delta x_f} + \frac{1}{2}f''(\Delta x_f)^2 + \ldots$$

$$f_{i-1} = f_i + f_i'\underbrace{(x_i - \Delta x_b - x_i)}_{-\Delta x_b} + \frac{1}{2}f_i''(-\Delta x_b)^2 + \ldots = f_i - f_i'\Delta x_b + \frac{1}{2}f_i''\Delta x_b^2 + \ldots$$

A centered-difference approximation for the first derivative can be found by $(f_{i+1} - f_{i-1})$, truncating the higher order terms, and solving for f_i':

$$f_{i+1} - f_{i-1} = f_i - f_i + f_i'\Delta x_f + f_i'\Delta x_b + \frac{1}{2}f_i''(\Delta x_f)^2 - \frac{1}{2}f_i''(\Delta x_b)^2 + \ldots$$

$$f_{i+1} - f_{i-1} = f_i'(\Delta x_f + \Delta x_b) + \frac{1}{2}f_i''\left[(\Delta x_f)^2 - (\Delta x_b)^2\right]$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{\Delta x_f + \Delta x_b} - \frac{1}{2}f_i''\frac{\left[(\Delta x_f)^2 - (\Delta x_b)^2\right]}{(\Delta x_f + \Delta x_b)}$$

a. $\boxed{f_i' = \dfrac{f_{i+1} - f_{i-1}}{\Delta x_f + \Delta x_b}}$

b. by inspection of the truncated 2^{nd}-order term, the order of the error is : CONT. $\longrightarrow$

1) CONT. PAUL YUSKA

$$\frac{\Delta x_f^2 - \Delta x_b^2}{\Delta x_f + \Delta x_b} = \frac{\Delta x_f^2}{\Delta x_f + \Delta x_b} - \frac{\Delta x_b^2}{\Delta x_f + \Delta x_b} \approx \boxed{\Delta x_f - \Delta x_b}$$

APPROX.

Hand-wavey, I know.
I'm still missing something here.

c. define $\left[\frac{df}{dx}\right]_{i+1/2} = f'_{i+\frac{1}{2}}$ as $\frac{f'_i + f'_{i+1}}{2}$ ← halfway between grid pts., regardless of spacing

where f'_{i+1} is found by solving the Taylor series about f_{i+1} in a similar matter as above

for $a = x_{i+1}$ and $x = x_i, x_{i+2}$, and $x_{i+2} = x_{i+1} + \Delta x_{f2}$

$$f'_{i+1} = \frac{f_{i+2} - f_i}{\Delta x_f + \Delta x_{f2}} ; \quad f'_i = \frac{f_{i+1} - f_{i-1}}{\Delta x_f + \Delta x_b}$$

define $\left[\frac{df}{dx}\right]_{i-\frac{1}{2}}$ similarly, finding f'_{i-1} in the same manner as f'_{i+1}:

$$f'_{i-1} = \frac{f_i - f_{i-2}}{\Delta x_b + \Delta x_{b2}} ; \quad f'_i = $$

define $x_{i\pm\frac{1}{2}}$ as $\frac{x_i + x_{i+1}}{2} = \frac{x_i + (x_i + \Delta x_f)}{2} = x_i + \frac{1}{2}\Delta x_f$ and

$$\frac{x_{i-1} + x_i}{2} = \longrightarrow = x_i - \frac{1}{2}\Delta x_b$$

then, f''_i can be approximated as:

$$f''_i \approx \left[\frac{f'_{i+1} + f'_i}{2} - \frac{f'_i + f'_{i-1}}{2}\right] \frac{1}{\frac{1}{2}(\Delta x_f + \Delta x_b)}$$

$$f''_i \approx \left[\frac{\left(\frac{f_{i+2} - f_i}{\Delta x_f + \Delta x_{f2}} + \frac{f_{i+1} - f_{i-1}}{\Delta x_f + \Delta x_b} \right)}{\Delta x_f + \Delta x_b} - \frac{\left(\frac{f_{i+1} - f_{i-1}}{\Delta x_f + \Delta x_b} + \frac{f_i - f_{i-2}}{\Delta x_b + \Delta x_{b2}} \right)}{\Delta x_f + \Delta x_b} \right]$$

2) a. Start w/ $\underline{\underline{A}}\,\underline{x} = \underline{b}$ PAUL YUSKA

Perturb the system: $\underline{\underline{A}} \to \underline{\underline{A}} + \delta\underline{\underline{A}}$, $\underline{b} \to \underline{b} + \delta\underline{b}$

$$(\underline{\underline{A}} + \delta\underline{\underline{A}})\underline{x} = \underline{b} + \delta\underline{b}$$

$$\underline{\underline{A}}\,\underline{x} + \delta\underline{\underline{A}}\,\underline{x} = \underline{b} + \delta\underline{b}$$ subtract unperturbed system to get

$$\delta\underline{\underline{A}}\,\underline{x} = \delta\underline{b} \longrightarrow \underbrace{\underline{\underline{A}}\,\underline{\underline{A}}^{-1}}_{\underline{\underline{I}}}\delta\underline{\underline{A}}\,\underline{x} = \delta\underline{b}$$

$$\|\underline{\underline{A}}\|\,\|\underline{\underline{A}}^{-1}\| \geq \|\underline{\underline{I}}\| = \|\underline{\underline{A}}\,\underline{\underline{A}}^{-1}\|$$

$$\underbrace{\|\underline{\underline{A}}\|\,\|\underline{\underline{A}}^{-1}\|}_{C(A)}\,\|\delta\underline{\underline{A}}\|\,\|\underline{x}\| \geq \|\delta\underline{b}\| \longrightarrow C(A)\,\|\delta\underline{\underline{A}}\|\,\|\underline{x}\| \geq \|\delta\underline{b}\|$$

$$C(A)\,\frac{\|\delta\underline{\underline{A}}\|}{\|\underline{\underline{A}}\|}\,\cancelto{1}{\frac{\|\underline{x}\|}{\|\underline{x}\|}} \geq \frac{\|\delta\underline{b}\|}{\|\underline{b}\|}$$

b. The condition # is a measure of sensitivity of changes in the solution of a system to perturbation of the input. High condition numbers ($C \gg 1$) are bad. C on the order of 1 is desirable.

PAUL YUSKA

4) $y(x) = ax^2 + bx^5$

$$S(a,b) = \sum_{i=1}^{N} (y_i - ax^2 - bx^5)^2$$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{N} 2(y_i - ax^2 - bx^5)(2x) = \sum_{i=1}^{N} 4(y_i x - ax^3 - bx^6) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{N} 2(y_i - ax^2 - bx^5)(5x^4) = \sum_{i=1}^{N} 10(y_i x^4 - ax^6 - bx^9) = 0$$

$$\begin{bmatrix} \sum_{i=1}^{N} x^3 & \sum_{i=1}^{N} x^6 \\ \sum_{i=1}^{N} x^6 & \sum_{i=1}^{N} x^9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} y_i x \\ \sum_{i=1}^{N} y_i x^4 \end{bmatrix}$$