SHIFTS BETWEEN TWO PERIODIC TIME SERIES

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ABSTRACT. This paper is concerned with examining rules of thumb and intuitions for finding lags/leads between two time series f, g indexed by a discrete ordered set $\{0, 1, \ldots, m-1\} = [0, m)$. The context motivating this paper is when f, g represent measurements of a wave with period at most m at two distinct points of the phase space and one is interested in interpolating the lagging measurement having the data from the leading one, with lag/lead modeled by a circular permutation of $[0, m)$. Since the cyclic group Z/mZ with generator σ acts freely and transitively on $[0, m)$ by the 'circular shift' $\sigma(t) = t + 1 \mod m$ hence Z/mZ can be identified with $[0, m)$ using $\sigma^t \in$Z/mZ$\rightarrow \sigma^t(0) = t \in [0, m)$. We equipp $[0, m)$ with a Z/mZ invariant measure giving each of its elements an equal weight. The vector space of complex valued functions $Map([0, m), C)$ carries the left Z/mZ group action $f \rightarrow f^s$ given by $f^s(t) = f(s^{-1}t)$ as well as the standard, invariant hermitian product $< f, g >= \sum_{t\in[0,m)} f(t)\overline{g(t)}$ with the corresponding norm $||$. Given a pair $g(t), f(t) \in Map([0, m), C)$ we define the square of the distance function $d : [0, m) \rightarrow R^+$ given by $d(f, g)(s) = |g - f^s|^2$ and the crosscorrelation function $\psi(f, g)(s) = Re < g, f^s >$. We define the set of shifts $\mathbf{S}(f, g)$ of f from g as $\mathbf{S}(f, g) = \arg \min_{t\in T_m} (d) = \arg \max_{t\in T_m} (\psi)$. Thus, a shift $s \in \mathbf{S}(f, g)$ is the circular transformation of g for which f approximates g^s better than for any other circular transfomation. In section 1 we establish some basic rules of thumb for how $\mathbf{S}(f, g)$ behaves under perturbations of f, g. In sections 2 and 3 we show, using the properties of the convolution operator, that $\mathbf{S}(f, g)$ is invariant under refining the mesh $[0, m)$ to $[0, ml)$ by adding an equal number l of points between each consecutive pair $\{k, k+1\}$ using the customary 'as of the last available measurement' rule. In Sections 4 we write an equation relating the crosscorelation function $\psi(f^*, g^*)$ to $\psi(f, g)$ where f^*, g^* denote restrictions of f, g to a subinterval $[0, m') \subset [0, m)$ for $m' < m$. From section 6 on, we consider the case of f, g periodic, nonnegative and containing sufficiently many fundamental periods $\hat{f}, \hat{g}$ of length $l << m$, which is reasonable if f, g approximate signals emited by a high frequency cyclical process. We show, under mild technical assumptions, that $\mathbf{S}(f, g)$ can be computed in terms of $\mathbf{S}(\hat{f}, \hat{g})$ and l. In particular, the result does not depend explicitely on the size of data windows m, m' and is entirely stable if $\mathbf{S}(\hat{f}, \hat{g}) = \{0\}$ or $\mathbf{S}(f, g) = \{0\}$ or $\mathbf{S}(f^*, g^*) = \{0\}$. This is of interest for the lags/leads between two 'nearly periodic' signals $\hat{f}, \hat{g}$ captured with time series f, g and sampled as f^*, g^*. In the appendix, we proof a well known fact that DFT can be used to O(mlog(m)) compute $\mathbf{S}(f, g)$.

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1. Shifts via energy and cross-correlation functions

Given a natural number $m > 1$, define $[0, m) = \{0, 1, \ldots, m-1\}$ and write T_m for the abelian group Z/mZ. Let σ_m denote the transformation of $[0, m)$ given by $\sigma_m(t) = t + 1 \mod m$.

It is easy to see that σ_m is the circular permutation $(0, 1, \ldots, m-1) \to (1, 2, \ldots, m-1, 0)$ and that $\sigma_m^m = id$. This defines the transitive action of the group of integers Z on $[0, m)$ where each element $s \in Z$ acts via $t \to \sigma_m^s(t) = t + s \mod m$. The correspondence $t \to t(0)$ identifies T_m with $[0, m)$ and σ_m^s with s. To simplify the notation, we will write $[0, m) = T_m$. Let τ denotes the involution of $[0, m)$ given by $t \to m - t$. Since $m - t = -t \mod m$ we can equivalently define τ as $\tau(t) = -t$ on the abelian group T_m.

Let $\mathbb{C}([0, m))$ denote the vector space of functions on $[0, m)$ with values in the field of complex numbers $\mathbb{C}$. Then T_m and $Z/2Z$ act on $\mathbb{C}([0, m))$ via $\sigma_m^s(f)(t) = f(\sigma_m^{-s}(t)) = f(t - s \mod m)$, $\tau(f)(t) = f(-t)$ for $f \in \mathbb{C}(T_m)$. To simplify notation we will denote $\sigma_m^s(f)$ as f^s. We have $\tau(f^s)(t) = f^s(-t) = f(-t - s \mod m)$ and $\tau(f)^s(t) = \tau(f)(t - s) = f(s - t)$ from which we get formulae $\tau(f^s) = \tau(f)^{-s}$.

Let $\delta_k \in \mathbb{C}([0, m))$ denotes the Dirac delta function on $[0, m)$ equal to 1 at k and 0 elswehere. Clearly δ_i, $0 \le i \le m$ is a basis of $\mathbb{C}([0, m))$ and $\delta_k^s = \delta_{k+s \mod m}$, $\tau(\delta_k) = \delta_{-k} = \delta_{m-k}$.

Explicitely, given vector $f = (f_0, f_1, \ldots, f_{m-1}) = \sum f_i \delta_i$, we have $f^s = (f_{m-s \mod m}, f_{m-s+1 \mod m}, \ldots, f_0, f_1, \ldots, f_{m-s-1}) = \sum f_i \delta_{i+s \mod m}$ and $\tau(f) = (f_0, f_{m-1}, f_{m-2}, \ldots, f_1) = \sum f_i \delta_{m-i \mod m}$.

Consider the T_m invariant norm $\|f\| = \sqrt{< f, f >}$ where $< f, g >= \sum_{t \in T_m} f(t)\overline{g(t)}$ is the T_m invariant hermitian inner product. Given $f, g \in \mathbb{C}([0, m))$ consider the energy function $d(f, g) : T_m \to \mathbb{R}^+$ given by $d(f, g)(t) = ||f - g^t||^2$. Consider the circular crosscorrelation [?] of f, g given as function $\psi(f, g) : T_m \to \mathbb{R}$ given by $\psi(f, g)(t) = Re < f, g^t >$ where $Re(z)$ denotes the real part of a complex number z. Since $\|\|$ is T_m invariant, it is easy to see, using the expansion $\|f - g^t\|^2 =< f - g^t, f - g^t >= ||f||^2 + ||g||^2 - 2Re < f, g^t >$, that the minimums of $d()$ coincide with the maximums of $\psi()$.

Definition 1.1. We define the *leads* of f *over* g (equivalently, *lags* of g *from* f) as the set $\mathbf{S}(f, g) = \arg \min\limits_{t \in T_m} (d(f, g)(t)) = \arg \max\limits_{t \in T_m} (\psi(f, g)(t))$. Let $m = \max\limits_{t \in T_m} (\psi(f, g)(t))$. Define $gap(f, g) = \min\limits_{t \in (T_m - \mathbf{S}(f,g))} (m - \psi(f, g)(t))$ when $\psi(f, g)(t)$ is not a constant function and as $gap(f, g) = 0$ otherwise.

Thus the set $\mathbf{S}(f, g)$ of shifts (lags of f from g or leads of g over f) consists of $s \in T_m$ for which g^t is the best approximation of f in $\|\|$ norm or, equivalently, for which the circular crosscorrelation function $\psi(f, g)$ attains maximum. Clearly, for any pair $f, g \in \mathbb{C}(T_m)$, $\emptyset \ne \mathbf{S}(f, g) \subset T_m$.

The following theorems lists the most elementary properties of $\mathbf{S}(f, g)$.

Theorem 1.2. *For any $f, g, h \in \mathbb{C}([0, m])$:*

1. $\mathbf{S}(f, \delta_0)$ *(resp.* $\mathbf{S}(f, -\delta_0)$ *is the set of maximums (resp. minimums) of* f
2. $\mathbf{S}(f, f) = T_m$ *if and only if* f *is a constant function*
3. *for any* $c \in \mathbb{C}$: $\mathbf{S}(c, f) = T_m$,
4. *for any* $t \in T_m$ *we have* $\mathbf{S}(f, g^t) = \mathbf{S}(f, g)^t = t + \mathbf{S}(f, g) \mod m$
5. *for any* $t \in T_m$ *we have* $\mathbf{S}(f, g) = \mathbf{S}(f^t, g^t)$

(6) $\mathbf{S}(g,f) = \tau(\mathbf{S}(f,g)) = m - \mathbf{S}(f,g) \mod m = -\mathbf{S}(f,g) \mod m$
(7) *for any* $c \in \mathbb{C}$*:* $\mathbf{S}(g+c,f) = \mathbf{S}(g,f)$
(8) *for* $r \in \mathbb{R}$*,* $r > 0$*:* $\mathbf{S}(rg,f) = \mathbf{S}(g,f)$
(9) *for* $\theta \in \mathbb{C}([0,m))$*, with* $|\theta(t)|^2$ *a positive constant:* $\mathbf{S}(\theta g, \theta f) = \mathbf{S}(g,f)$

Proof. We leave it as excercise to the reader. □

Given a set S write $\#S$ for the cardinality of S.

Theorem 1.3. *Given* $f \in \mathbb{C}([0,m))$ *:*

(1) *For any* $g \in \mathbb{C}([0,m))$ *and for any* $\epsilon > 0$ *there exist* $h_1, h_2 \in \mathbb{C}([0,m))$ *with* $\|h_1\|^2, \|h_2\|^2 < \epsilon$ *such that* $\#\mathbf{S}(f+h_1, g+h_2) = 1$
(2) *For any* $g \in \mathbb{C}([0,m))$ *there exists* $\epsilon > 0$ *so that for any* $h \in \mathbb{C}([0,m))$ *with* $\|h\|^2 < \epsilon$ *we have* $\mathbf{S}(g+h,f)) \subset \mathbf{S}(g,f)$*.*
(3) *For any* $h, g \in \mathbb{C}([0,m))$ *there exists* $\epsilon > 0$ *so that for any* c *with* $c > \epsilon$ *we have* $\mathbf{S}(cg+h,f)) \subset \mathbf{S}(g,f)$*.*

Proof. We leave it as excercise to the reader. □

With $\|h\|^2$ interpreted as the total energy of h the first item above asserts that for generic f, g, the set $\mathbf{S}(f,g)$ consists of a unique element. Further, if $\#\mathbf{S}(f,g) = 1$ then $\mathbf{S}(f,g)$ it is stable under perturbations of either argument by any function h of sufficeintly small energy . The third assertion gives a prescription for increasing the energy of g so that the stability holds for any given h. Together, this can be used to compute $\mathbf{S}(f,g)$ when g gets sufficiently spiked at $k \in [0,m)$.

Corollary 1.4. For any pair $f, g \in \mathbb{R}([0,m))$ and $k \in [0,m)$ there exists $\epsilon_1 > 0$ (respectively, $\epsilon_2 < 0$) so that for any $r_1 > \epsilon_1$ (resp. $r_2 < \epsilon_2$) for each $s \in \mathbf{S}(f, g + r_1\delta_k)$, $s + k$ is a maximum (resp. for each $s \in \mathbf{S}(f, g + r_2\delta_k)$, $s + k$ is a minimum) of f.

Proof. We leave it as excercise to the reader. □

Thus, if f is a generic function and g is 'sufficiently spiked' up (resp. down) at k then $\mathbf{S}(f,g)$ is contained in the set of shifts from k to $\arg\max_{t \in T_m}(f)$ (resp. $\arg\min_{t \in T_m}(f)$) .

For the reminder of this section we will restrict to the subset $\mathbb{R}([0,m)) \subset \mathbb{C}([0,m))$ of real valued functions on $[0,m)$ and write $\mathbb{R}([0,m))^+$, $\mathbb{R}([0,m))^{++}$ (respectively $\mathbb{R}([0,m))^-$, $\mathbb{R}([0,m))^{--}$) for the subset of real valued, non-decreasing, strictly increasing (respectively, non increasing, strictly decreasing) functions.

Theorem 1.5. (1) *If* $f, g \in \mathbb{R}([0,m))^{++}$ *or* $f, g \in \mathbb{R}([0,m))^{--}$ *then* $\mathbf{S} = \{0\}$*.*
(2) *If* $f \in \mathbb{R}([0,m))^{++}$ *and* $g \in \mathbb{R}([0,m))^{--}$ *then* $0 \notin \mathbf{S}$*.*

Proof. For (1), denote $f = (f_0, f_1, \ldots, f_{m-1}), g = (g_0, g_1, \ldots, g_{m-1})$. We will consider the case $f_i < f_{i+1}, g_i < g_{i+1}$. Using induction on m we will show that the crosscorrelation function $\psi(s) =< f, s(g) >$ has a maximum at $s = 0$. From Theorem 1 (6) , we loose no generality by setting $f_0 = 0$. First consider the case $m = 2$. Then $< f, g >= f_1 g_1 < f_1 g_0 =< f, 1(g) >$. Recall that $1(g)$ denotes the circular 1-shift of g. Our inductive hypothesis is that the theorem holds for $m = n$. For $m = n+1$, if $s = 1$ then, since for $i > 0$ we have $f_i g_{i-1} < f_i g_i$ hence $< f, 1(g) >= f_1 g_0 + \ldots + f_n g_{n-1} < f_1 g_1 + \ldots f_n g_n =< f, g >$. For $s = k$, $1 < k \le n$ we have $< f, k(g) >= f_1 g_{n-k+2} + \ldots + f_{k-1} g_n + f_k g_0 + \ldots + f_n g_{n-k} <$

$f_1g_0+\ldots+f_{n-k+1}g_{n-k}+f_{n-k+2}g_{n-k+2}+\ldots+f_ng_n < f_1g_1+\ldots+f_{n-k+1}g_{n-k+1}+f_{n-k+2}g_{n-k+2}+\ldots+f_ng_n =< f,g >$ where the first inequality is the inductive hypothesis and the second follows from $f_ig_{i-1} < f_ig_i$.

For (2), we lose no generality by assuming that $f_0 = 0$, $f_i < f_{i+1}$ and $g_i > g_{i+1}$ in which case we have $< f, 1(g) >= f_1g_0+\ldots+f_{m-1}g_{m-2} > f_1g_1+\ldots+f_{m-1}g_{m-1} =< f,g >$.

□

Closer examination of this proof shows that the theorem holds under weaker assumption that $f \in \mathbb{R}([0,m))^+$, f is not constant and g is strictly monotonic. We leave this as an excercise to the reader.

Together, both theorems give a rule of thumb for computing $\mathbf{S}(f,g)$ when $m = 2$.

Corollary 1.6. If $m = 2$ then:

$$\mathbf{S} = \begin{cases} \{0,1\} \text{ if and only if at least one of } f,g \text{ is constant} \\ \{0\} \text{ if and only if } f,g \text{ are both strictly increasing or both strictly decreasing} \\ \{1\} \text{ if and only if } f \text{ is strictly increasing and } g \text{ strictly decreasing or vice versa} \end{cases}.$$

Suppose that for a function $f \in \mathbb{R}([0,m))$ there exists $t \in T_m$ for which f^t is monotonic. We could call such functions 'shift-monotonic'. Notice that if f is shift-monotonic then, unless it is a constant function, it is either shift-decreasing or shift-increasing and the t for which f^t is monotonic is unique. We notice that the theorem above has a natural reformulation for a shift monotonic pair f,g, since knowing $\mathbf{S}(f^t,g^{t'})$ (with $f^t,g^{t'}$ monotonic) determines $\mathbf{S}(f,g)$, via identity given by Theorem 1.2 as: $\mathbf{S}(f,g) = \mathbf{S}(f^t,g^{t'})^{t-t'} = t-t'+\mathbf{S}(f^t,g^{t'}) \mod m$.

For $m = 3$ we have the following.

Lemma 1.7. *Any $f \in \mathbb{R}([0,3))$ is shift-monotonic.*

Proof. We leave it as excercise to the reader. □

Thus, to compute $\mathbf{S}(f,g)$ for $m = 3$, one can assume that f is monotonic and g is shift-monotonic, in which case one can write down a a set of rules similiar to Corollary 1.6. We leave details to the reader.

2. Shifts via convolution

Let $*$ denotes the circular convolution operator [**?**] on $\mathbb{C}(T_m)$.

Theorem 2.1. *The operator $*$ is commutative, associative and :*

(1) $\overline{f*g} = \overline{f}*\overline{g}$
(2) $f^s = \delta_s * f$
(3) $\tau(f*g) = \tau(f)*\tau(g)$
(4) $(f*g)^s = (f^s)*g = f*(g^s) = (f^s)*g$

.

Proof. For 3), use: $\tau(f*g)(t) = (f*g)(-t) = \sum_{t'} f_{-t-t'}g_s = \sum_{t'} \tau(f)_{t+t'}\tau(g)_{-t'} = \tau(f)*\tau(g)(t)$. For 4), use $(f*g)^s = \delta_s*(f*g) = (\delta_s*f)*g = (f^s)*g$ and thus $(f*g)^s = (g*f)^s = f*(g^s)$. We leave the details to the reader. □

With this, the material in Section 1 can be refrased in the language of convolutions and Dirac functions. The well known [?] result below gives a formulae for the crosscorrelation function $\psi(t) =< f, t(g) >$ in such language. Two applications of this are given in the later sections of this paper. Recall that $\overline{z}$ denotes the complex conjugate of z .

Theorem 2.2. *For the crosscorrelation function* $\psi(t) =< f, t(g) >$, *we have :* $\psi = f * \tau(\overline{g})$.

Proof. $(f * \tau(\overline{g}))(t) = \sum_{t'} f_{t'} \overline{\tau(g)}_{-t'+t} = \sum f_{t'} \overline{g}_{t'-t} =< f, t(g) >$ □

Corollary 2.3. Given $f, f', g, g' \in \mathbb{C}(T_m)$ the crosscorrelation function $\psi(f * f', g * g')(t) =< f * f', (g * g')^t >$ can be written as $\psi(f * f', g * g') = (f' * \tau(\overline{g'})) * (f * \tau(\overline{g})) = \psi(f, g) * \psi(f', g')$.

Proof. We have: $(f * f') * \tau(\overline{(g * g')}) = (f * f') * (\tau(\overline{g}) * \tau(\overline{g'})) = f * \tau(\overline{g}) * f' * \tau(\overline{g'}) = (f * \tau(\overline{g})) * (f' * \tau(\overline{g'}))$. □

Definition 2.4. For $f, g \in \mathbb{C}(T_m)$ define $f \hat{*} g = f * \tau(\overline{g})$.

With this we can summarize the results of this secition by writing out the following identities.

Corollary 2.5. (1) $\delta_s \hat{*} f = \tau(\overline{f})^s = \tau(\overline{f^{-s}}) = \tau(\delta_{-s} * \overline{f})$
(2) $(f \hat{*} g)^s = f^s \hat{*} g = f * \tau(\overline{g})^s = f * \tau(\overline{g^{-s}}) = f \hat{*} g^{-s}$
(3) $(f * f') \hat{*} (g * g') = (f \hat{*} g) * (f' \hat{*} g')$
(4) $(f \hat{*} f') * (g \hat{*} g') = (f * g) \hat{*} (f' * g')$
(5) $\psi(f, g) = f \hat{*} g$
(6) $\psi(f^s, g) = \psi(f, g)^s$, $\psi(f, g^s) = \psi(f, g)^{-s}$, $\psi(f,^s g^s) = \psi(f, g)$
(7) $\psi(f * f', g * g') = \psi(f, g) * \psi(f', g')$

Definition 2.6. Given $S \subset T_m$ let δ_S denotes the characteristic function of S: $\delta_S(s) = 1$ for $s \in S$ and $\delta_S(t) = 1$ for $t \notin S$.

We can extend the definitions of $*$, $\hat{*}$ to sets: for $S, S' \subset T_m$ define $S * S' = supp(\delta_S * \delta_S)$, $S \hat{*} S' = supp(\delta_S \hat{*} \delta_S)$ where $supp(f) = \{s : f(s) \neq 0\}$. Clearly $S * S' = \cup_{s \in S, s' \in S'} \{s + s'\}$, $S \hat{*} S' = \cup_{s \in S, s' \in S'} \{s - s'\}$.

3. Mesh refinements

Suppose that $m = m_1 m_2$ with $m_1, m_2 > 1$. Then, by means of the canonical injection (a homomorphism of abelian groups) $T_{m_1} = [0, m_1) \xrightarrow{i} [0, m) = T_m$ given by $i(k) = m_2 k$, T_m can be considered as the m_2-subdivision of T_{m_1}.

Given $f \in \mathbb{C}(T_{m_1})$ consider function $i_*(f) \in \mathbb{C}(T_m)$ given by $i_*(f)(t) = f(t)$ if $t \in T_{m_1}$ and $i_*(f)(t) = 0$ otherwise. Clearly for any pair of $f, g \in \mathbb{C}(T_{m_1})$ we have $m_2 S(f, g) = S(i_*(f), i_*(g))$ or equivalently, $S(i_*(f), i_*(g)) = i_*(S(f, g)$. Going forward, to simplify notation, we will continue writing f for $i_*(f)$. This section is concerned with methods for finding a function $f' \in \mathbb{C}(T_m)$ for which $S(f * f', g * f') = S(f, g)$. In particular, we show that $f' = 1_{T_{m_2}}$, equal to 1 on $T_{m_2} = \{0, 1, \ldots, m_2 - 1\} \subset T_m$ and 0 otherwise, has this property. In fact, f' is a generic example of such function. Observe that if f is a time series initially defined on mesh T_{m_1} then $f * 1_{T_{m_2}}$ is obtained by first extending f to $i_* f$ as *null* on $T_m - T_{m_1}$ and then filling the missing values with the 'most recent value' - the usual choice for time series with gaps. For example, for a time series f representing quotes of a price , $f * 1_{T_{m_2}}$ represents a refinement signal where the new intermediate price values are the most recent quoted prices.

Theorem 3.1. *Let $f' = 1_{T_{m_2}}$. Then for any $f, g \in \mathbb{C}(T_{m_1})$ we have $S(f * f', g * f') = S(f, g)$.*

Proof. Write $p' = f' \hat{*} f'$. From Corollary 2.3, it suffices to show that for any crosscorrelation function $\psi = f \hat{*} g$ the maximum of p coincide with maximums of $p' * p$. It is easy to see that p is a function with support in $i_*(T_{m_1}) \subset T_m$. By direct calculation, p' is a nonnegative integer valued function with support $T_{m_2} \cup -T_{m_2} = \{-m_2+1, \ldots, 0, \ldots, m_2-1\}$ given by formulae $p'(u) = m_2 - |u|$. We leave to the reader to prove as an excercse that if p'' is an arbitrary nonnegative function with support $T_{m_2} \cup -T_{m_2}$, for which $p'(0) \le p''(0)$ and $p'(s) \ge p''(s)$ for $s \neq 0$ then for any function p with support $i_*(T_{m_1})$ the maximums of p coincide with maximums of $p'' * p$.

□

One can show that the class of functions f' for which the theorem holds is given by the requirement that the function $\psi^{'} = f' \hat{*} f'$, after multiplying by a positive constant, has a spike at $t = 0$ at least as sharp as the crosscorrelation function $1_{T_{m_2}} \hat{*} 1_{T_{m_2}}$ from the proof above. This is why we refer to $1_{T_{m_2}}$ as generic.

4. Changing the size of the sample

Given m, and $a, b \in [0, m)$ write $r = \#[a, b) = b - a$, $[a, b) \subset [0, m)$. For $f \in \mathbb{C}([0, nl))$ let $f|_{[b,a)}$ denote the restriction of f to $[b, a) = [0, nl) - [a, b)$.

Write $m' = m - r$. For $f \in \mathbb{C}([0, m))$ let $f^* \in \mathbb{C}([0, m'))$ denotes the restriction of f to to $[0, m')$. We will write a general equation relating the crosscorrelation functions $\psi(f^*, g^*)$ and $\psi(f, g)$.

Theorem 4.1. *For $s \in T_{m'}$ and $f, g \in \mathbb{C}([0, m))$ we have:*

$$(g^*)^s = ((\delta_{[0,m'-s)} g)^s)^* + ((\delta_{[m'-s,m')} g)^{s+r})^* \tag{4.1}$$

$$\psi(f^*, g^*)(s) = \psi(f, \delta_{[0,m'-1-s]} g)(s) + \psi(f, \delta_{[m'-s,m'-1]} g)(s + r) \tag{4.2}$$

Proof. For a fixed t, $0 \le t \le m' - 1$ and $g = \delta_t$, the left hand side of 4.1 can be interpreted as an equation of motion $\delta_t \to \delta_t^s$ on $[0, m')$ with the right hand side an equation of the coresponding motion in $[0, m)$. Since both sides of equation 4.1 are linear with respect to g and $g = \sum_{t=0}^{t=m'-1} g_t \delta_t$ hence it suffices to check it on basis $g = \delta_t$, $0 \le t \le m' - 1$ of $\mathbb{C}(T_{m'})$. We leave this as an excercise to the reader. Equation 4.2 follows from

$$< f^*, (g^*)^s >=< f, (\delta_{[0,m'-1-s]} g)^s > + < f, (\delta_{[m'-s,m'-1]} g)^{s+r} > \tag{4.3}$$

which is a consequence of 4.1.

□

Definition 4.2. Given $a, b \in [0, m)$ let $\sigma_{a,b} \in T_m$ denotes the unique circular permutation for which $\sigma(a) = b$. Let $T_{a,b} = \{0, 1, \ldots, \sigma_{a,b}\}$ Denote $[a, b] = a + T_{a,b}$, $[a, b) = [a, b] - \{b\}$, $(a, b] = [a, b] - \{a\}$. Then $[a, b] \subset [0, m)$ will be called interval (from a to b).

With this definition an interval is always an ordered, proper and, if closed , a nonempty subset of $[0, m)$. Moreover:

(1) For $s \in T_m$: $[a, b]^s = [a + s, b + s]$

(2) $\#[a, b) = \begin{cases} b - a \text{ if } a < b \\ 0 \text{ if } a = b \\ m - \#[b, a) = m + (b - a) \text{ if } a > b \end{cases}$

(3) $[a, b]^\tau = [-b, -a]$

If we view $[0, m)$ as a discrete subset of the unit circle (with clockwise orientation), then a pair of its points a, b determines two intervals $[a, b)$, $[b, a)$ realized as the two complementary arcs of the circle.

Given $f \in \mathbb{C}([0, m))$ and an interval $i = [a, b) \subset [0, m)$ define $i^*(f)$ to be the restriction of $f \in \mathbb{C}([0, m))$ to the interval $[b, a) = [0, m) - [a, b)$ and set $r = \#[a, b)$. Theorem 4.1 generalizes, with the same proof, to the following.

Theorem 4.3. *For $s \in T_{b,a}$ and $f, g \in \mathbb{C}([0, m))$ we have:*

$$i^*(f)^s = i^*(\delta_{[b,a-s)} g)^s) + i^*(\delta_{[a-s,a)} g)^{s+r}) \tag{4.4}$$

$$\psi(i^*(f), i^*(g))(s) = \psi(f, \delta_{[b,a-s)} g)(s) + \psi(f, \delta_{[a-s,a)} g)(s + r) \tag{4.5}$$

5. Periodic extensions

We have seen that for a pair $f, g \in \mathbb{C}([0, m))$ the set $\mathbf{S}(f, g)$ can be sensitive to small perturbations of either argument and/or restriction to subintervals of T_m. This section defines a notion of f, g being 'sufficiently l-periodic' so that $\mathbf{S}(f, g)$ remains almost stable under localizations and/or sampling. In fact, will see in the next sections, that the computation of the shortest shift $s \in \mathbf{S}(f, g)$ for f, g in such class is independent of the size of the data sample and invariant under certain perturbations as long as the samples contain sufficiently many periods.

For a pair of positive integers m, l denote $[m/l] = floor(m/l) = max\{k \in \mathbb{Z} : 0 \leq m - kl\}$. Define $\phi : T_m \to T_l$, $\phi(t) = t \mod l$ and the induced homomorphism of $\mathbb{C}$ vector spaces $\phi^* : \mathbb{C}(T_l) \to \mathbb{C}(T_m)$. We have $m = [m/l]l + k$ with $k = m \mod l$. For a set of positive integers $L = \{l_0, l_1, \ldots, l_n\}$ denote (L) for the greatest common divisor. Recall that given m, L, the elements $\{\phi(l_0), \phi(l_1), \ldots, \phi(l_n)\} \subset T_m$ generate subgroup T_L of T_m given by equation $T_L = \{t \in T_m : t = 0 \mod (L \cup m)\}$. Recall that for $f \in \mathbb{C}(T_m)$, the integer l is a period of f (equivalently, f is l-periodic) if $f = f^l$. It follows that if L is the set of all periods of f then each period is divisible by the fundamental period $l = min((L \cup m))$ and that f is determined by its restriction $\hat{f}$ (also called fundamental period of f) to $T_l = [0, l) \subset [0, m) = T_m$.

Definition 5.1. Given a positive integer l, $f \in \mathbb{C}([0, m))$ is an l-periodic extension with $[m/l]$ periods and the reminder $k = \phi(m) = m \mod l$ when there exists $\mathbf{f} \in \mathbb{C}([0, l))$ such that $f = \phi^*(\mathbf{f})$. We will call l an extended period of f. Any smallest extended period of f will be called the extended fundamental period. Let $P_l(m) \subset \mathbb{C}([0, m))$ denotes the $\mathbb{C}^*$-algebra of l-periodic extensions on T_m.

Lemma 5.2. *The following are equivalent:*

(1) *$f \in \mathbb{C}(T_m)$ is l-periodic extension with $[m/l]$ periods and the reminder k*
(2) *f can be extended to l-periodic function on T_{m+l-k}*

(3) *restriction of f to any interval $[a,b] \subset T_m, 0 \le a \le b < m$ with $\#[a,b]$ divisible by l is l-periodic*
(4) *f is a restriction of l-periodic $\mathbf{f} \in \mathbb{C}([0,[m/l]+k))$ to a sub-interval of length m*

Proof. We leave this as excercise to the reader. $\square$

It follows that if l is the extended fundamental period, then any extended period l' is divisible by l. In addition, if l divides m, then any l-periodic extension $f \in \mathbb{C}(T_m)$ is l-periodic with m/l of l-periods.

Example 5.3. Given a natural number m let $\nu_m = e^{2\pi i/m}$ denote the primitive m'th root of unity. The set of characters $\chi_m^k \in \mathbb{C}(T_m)$, $\chi_m^k(t) = \frac{1}{\sqrt[2]{m}}\nu_m^{tk}$, $0 \le k < m$ is an orthonormal basis of $\mathbb{C}(T_m)$ with respect to the inner product $<>$, $\mathbb{C}(T_m)$ is a $\mathbb{C}^*$ algebra generated by χ_m^1. Since $P_l(m)$ is closed under pointwise addition and multiplication, it is a $\mathbb{C}^*$-sub-algebra of $\mathbb{C}(T_m)$. If $m = nl$ then it is easy to see that element χ_m^n generates $P_l(m)$, in addition, $\{\chi_m^{0n}, \chi_m^{1n}, \dots \chi_m^{(l-1)n}\} = \{\chi_m^{nk} : k \in [0,l)\}$ is an orthonrmal basis over $\mathbb{C}$ of $P_l(m)$. For an arbitrary m let $m' = nl$ with $m \le m'$. Let i denotes the natural inclusion $[0,m) \subset [0,m']$. It follows that $P_l(m)$ is generated, as $\mathbb{C}^*$-algebra, by the element $i^*(\chi_{nl}^1)^n$ and $\{i^*(\chi_{nl}^1)^{nk} : k \in [0,l)\}$ is a basis (not orthonormal, unless $l|m$) over $\mathbb{C}$ of $P_l(m)$. Of course, while χ_m^k are characters (group homomorphisms from T_m to $\mathbb{C}^*$), $i^*(\chi_{nl}^1)^{nk}$ are not, unless $l|m$.

6. Localization of periodic signals

Given $\hat{f} \in \mathbb{C}([0,l))$ let $f \in \mathbf{P}_l(ql)$ be the l-periodic extension of $\hat{f}$. For an arbitrary $g \in \mathbb{C}([0,ql))$ the crosscorelation fuction $\psi(f,g)$ is l-periodic and thus so is the function $\delta_{\mathbf{S}(f,g)}$. This is obvious for $g = \delta_i$, $i \in [0,ql)$, and the general case follws from $g = \sum g_i\delta_i$ and bilinearity of ψ. If, in addition, $g \in \mathbf{P}_l(ql)$ is also an l-periodic extensions of $\hat{g} \in \mathbb{C}([0,l))$ then $\psi(f,g)$ can be directly related to (the periodic extension of) $\psi(\hat{f},\hat{g})$ (see equation 6.1 below). The lemma below considers a slightly more general setting which will be needed in the next section. Let $[a,b] \subset [0,ql)$ be an arbitrary interval and denote $n = [\#[a,b)/l]$, $k = \#[a,b)$ mod l so that $\#[a,b) = nl + k$. Denote: $\rho(\hat{f},\hat{g},a,k) = \psi(\delta_{[0,k)} \cdot \hat{f}^{(-a)},\hat{g})$, Notice that since $\delta_{[0,0)} \cong 0$, hence $\rho(\hat{f},\hat{g},a,0) \cong 0$.

Lemma 6.1. *Suppose $f,g \in \mathbf{P}_l(ql)$. Then: $\psi(\delta^s_{[a,b)}f,g) = n\cdot\psi(\hat{f},\hat{g}) + \rho(\hat{f}^{-s},\hat{g}^{-s},a,k)^a$, with $\rho(\hat{f}^{-s},\hat{g}^{-s},a,k)^a(t)$ l-periodic in t, s and a .*

Proof. We have $\psi(\delta^s_{[a,b)}f,g) = \psi((\delta_{[a,b]}f^{-s})^s,g) = \psi(\delta_{[a,b]}f^{-s},g^{-s})$. Since $\hat{f}^{-s} = \hat{f^{-s}}$ we get $\psi(\hat{f},\hat{g}) = \psi(\hat{f}^{-s},\hat{g}^{-s}) = \psi(\hat{f^{-s}},\hat{g^{-s}})$. Thus, it suffices to prove the lemma for $s = 0$. Similarly, from $\psi(\delta_{[a,b)}f,g) = \psi(\delta^a_{[0,nl+k)}f,g) = \psi((\delta_{[0,nl+k)}f^{-a})^a,g) = \psi((\delta_{[0,nl+k)}f^{-a}),g^{-a})$, it suffices to prove the lemma for $a = 0$. By bilinearity, we have $\psi(\delta_{[0,b)}f,g) = \psi(\delta_{[0,k)}f,g) + \psi(\delta_{[k,nl+k)}f,g) = \rho(\hat{f},\hat{g},0,k) + \psi(\delta^k_{[0,nl)}f,g)$. We leave to the reader to check that if $l|\#[a,b)$ then $\psi(\delta^s_{[a,b))}f,g)$ does not depend on s. Then the identity $\psi(\delta_{[0,nl))}f,g)(s) = n\cdot\psi(\hat{f},\hat{g})(s \mod l)$ completes the proof. $\square$

The expression above can be interpreted as a resonance of ψ. For the reminder of this section we will use the special case of the lemme when $k = 0$, $q = n$ and

$[a,b) = [0, nl)$, in which case the lemma simplifies to

$$\psi(f,g) = n \cdot \psi(\hat{f}, \hat{g}) \tag{6.1}$$

. We see that as n increases, unless $\psi(f,g)$ is a constant function, the extrema of $\psi(f,g)$ become more pronounced. More precisely; if $\psi(\hat{f},\hat{g})$ is not a constant function and $g(\hat{f},\hat{g})$ denotes the *gap* between $\hat{f},\hat{g}$ as defined in Definition 1 then:

$$g(f,g) = ng(\hat{f},\hat{g}) \tag{6.2}$$

Of course, for the l-period $\hat{\delta}_{\mathbf{S}(f,g)}$ of $\delta_{\mathbf{S}(f,g)}$ we have $\hat{\delta}_{\mathbf{S}(f,g)} = \delta_{\mathbf{S}(\hat{f},\hat{g})}$, $\#\mathbf{S}(f,g) = n \cdot \#\mathbf{S}(\hat{f},\hat{g})$.

Let $f'_n, g'_n \in \mathbb{C}([0,nl))$ is a sequence of pairs of functions for whch $\psi(f,g'_n)(t)$, $\psi(f'_n,g)(t)$, $\psi(f'_n,g'_n)(t)$ are all uniformly bounded for all n,t. This holds if, for example, there exists k, $Z \subset \mathbb{C}([0,kl))$ for which functions f'_n, g'_n are identically zero outside of Z for $n > k$. Lets describe the relationship between the maximums of $\psi(f,g)$ and the maximums of $\psi(f+f',g+g')$, for large n. Of course, the correspondence $f \to f+f'$ is modeling a small perturbation of f. We have :

$$\psi(f'+f,g'+g) = n \cdot \psi(\hat{f},\hat{g}) + \psi(f,g') + \psi(f',g) + \psi(f',g') \tag{6.3}$$

. Denote $\psi'' = \psi(f',g')$ and $\psi' = \psi(f',g) + \psi(f,g')$. We can rewrite:

$$\psi(f'+f,g'+g) = n \cdot \psi(\hat{f},\hat{g}) + \psi' + \psi'' \tag{6.4}$$

. Since $|\psi'(t)|, |\psi''(t)|$ are uniformly bounded for all n,t , hence, for sufficiently large n,

$$\arg\max_{t \in T_{nl}} (n\psi(\hat{f},\hat{g}) + \psi' + \psi'') = \arg\max_{t \in \mathbf{S}(f,g)} (\psi' + \psi'') \tag{6.5}$$

. To see this, notice that when $\psi(\hat{f},\hat{g})$ is a constant function then the equation above is a tautology and otherwise it follows from the experession 5.2. In addition, if $\#\mathbf{S}(\hat{f},\hat{g}) = 1$ then, since ψ' is l-periodic and thus constant on $\mathbf{S}(f,g)$ hence we have

$$\arg\max_{t \in \mathbf{S}(f,g)} (\psi' + \psi'') = \arg\max_{t \in \mathbf{S}(f,g)} (\psi'') \tag{6.6}$$

. We have proved the following theorem.

Theorem 6.2. *Let $f,g \in \mathbf{P}_l(ln)$ denote the l-periodic extensions of $\hat{f},\hat{g} \in \mathbb{C}([0,l))$. Suppose that for $f',g' \in \mathbb{C}([0,nl))$ the functions $\psi(f,g'), \psi(f',g), \psi(f',g')$ are uniformly bounded with respect to n,t.*

1. *There exists $N1 > 0$ such that for $n > N1$: $\mathbf{S}(f+f',g+g') = arg \max_{t \in \mathbf{S}(f,g)} (\psi' + \psi(f',g'))$.*
2. *If $\#\mathbf{S}(\hat{f},\hat{g}) = 1$ then $arg \max_{t \in \mathbf{S}(f,g)} (\psi' + \psi(f',g')) = arg \max_{t \in \mathbf{S}(f,g)} (\psi(f',g'))$*

We saw that the assumption $\#\mathbf{S}(\hat{f},\hat{g}) = 1$ can be weakened by requiring that ψ' is constant on $\mathbf{S}(f,g)$. One can also obtain a versions of the second assertion in our theorem with assumption that $\arg\max_{t \in \mathbf{S}(f,g)} (\psi') \subset \arg\max_{t \in \mathbf{S}(f,g)} (\psi'')$ or $\arg\max_{t \in \mathbf{S}(f,g)} (\psi'') \subset \arg\max_{t \in \mathbf{S}(f,g)} (\psi')$. In general, it is possible to construct examples where neither $\arg\max_{t\in\mathbf{S}(f,g)}(\psi'+\psi'') \subset \arg\max_{t\in\mathbf{S}(f,g)}(\psi(f',g'))$ nor $\arg\max_{t\in\mathbf{S}(f,g)}(\psi(f',g')) \subset \arg\max_{t\in\mathbf{S}(f,g)}(\psi'+\psi'')$.

For the reminder of this section we will discuss one specific class of functions f', g' localizing f, g, in which case the assertion 2) of the theorem above can be strengthened.

Definition 6.3. Given $Z \subset [0,m)$ let δ_Z denotes the characteristic function of Z: $\delta_Z(s) = 1$ for $s \in Z$ and $\delta_Z(t) = 0$ for $t \notin Z$. For $f \in \mathbb{C}([0,m))$ the product $\delta_Z f$ will be called localization of f to Z.

Denote $\theta_Z = 1 - \delta_Z$. Thus θ_Z is the characteristic function of the complement Z^o of Z in T_m and $\theta_Z f$ is the localization of f away from Z. We have

$$\psi(\theta_Z f, \theta_Z g) = \psi(f,g) - \psi(\delta_Z f, g) - \psi(f, \delta_Z g) + \psi(\delta_Z f, \delta_Z g) \tag{6.7}$$

tying up the crosscorelations functions of f, g and their localizations.

We start with the obvious application of the theorem.

Corollary 6.4. With notation of the theorem, suppose that $\mathbf{S}(\hat{f},\hat{g}) = \{s\}$ and for $Z \subset [0,nl)$ $\psi(\delta_Z f, \delta_Z g)(t) = 0$ for $t = s \mod l$. Then $\mathbf{S}(\theta_Z f, \theta_Z g) = \mathbf{S}(f,g)$.

We have seen in section 2 that *supp*$(\psi(\delta_Z f, \delta_Z g)) \subset Z \circledast Z$. In particular if $Z = \{s\}$ then *supp*$(\psi(\delta_Z f, \delta_Z g)) \subset \{0\}$. When $f(s) \neq 0$ and $g(s) \neq 0$ then *supp*$(\psi(\delta_Z f, \delta_Z g)) = \{0\}$ and corrolary 6.3 gives us the following result (which helped the authors to construct interesting examples , i.e. example below) .

Corollary 6.5. With notation of the theorem suppose that $Z = \{s\}$and $\mathbf{S}(\hat{f},\hat{g}) = \{p\}$ and that $f(s) \neq 0, g(s) \neq 0$. Then $\mathbf{S}(f,g) = \mathbf{p} = \{p, p+l, \ldots, p+(n-1)l\}$ and, if $p = 0$ then :

(1) if $f(s)g(s) < 0$ then $0 \notin \mathbf{S}(\theta_Z f, \theta_Z g)$ (thus $\mathbf{S}(\theta_Z f, \theta_Z g) \neq \mathbf{S}(f,g)$)
(2) if $f(s)g(s) > 0$ then $\{0\} = \mathbf{S}(\theta_Z f, \theta_Z g)$ (thus $\mathbf{S}(\theta_Z f, \theta_Z g) = \mathbf{S}(f,g)$)

while if $p \neq 0$ then:

(1) $\mathbf{S}(\theta_Z f, \theta_Z g) = \mathbf{S}(f,g)$.

In particular, we see that if $\mathbf{S}(\hat{f},\hat{g}) = \{0\}$ then $f(s)g(s) < 0$ implies a change: $0 \notin \mathbf{S}(\theta_Z f, \theta_Z g)$ while $fs)g(s) > 0$ imples conservation: $0 \in \mathbf{S}(\theta_Z f, \theta_Z g)$. We close this section by extending this result to the case when Z is an interval.

Lemma 6.6. *Suppose $h \in \mathbb{R}([0,nl))$ and $h(s) \geq 0$ for all $s \in [a,b] \subset [0,nl)$. For an arbitrary $p \in [0,l)$ consider sequence $\mathbf{p} = \{p, p+l, \ldots, p+l(n-1)\} = \{t \in T_{nl} : t = p \mod l\}$. Let ψ^* denotes the restriction of $\psi(\delta_{[a,b]}h, \delta_{[a,b]})$ to $\mathbf{p}$.*

(1) *$\psi(\delta_{[a,b]}h, \delta_{[a,b]})$ has maximum at $s = 0$*
(2) *If $nl > 2\#[a,b]$ then $\psi^*(s) \leq min(\psi^*(p), \psi^*(p+l(n-1)))$ for $s \notin \{p, p+l(n-1)\}$. In particular, ψ^* has maximum at at least one of $\{p, p+(n-1)l\}$.*

Proof. The first assertion follows from $(\forall s \in [0,nl)([a,b] \cap [a,b]^s \subset [a,b]))$. The second assertion follows from $(\forall s \in \mathbf{p}([a,b] \cap [a,b]^s \subset [a,b]^p \vee [a,b] \cap [a,b]^s \subset [a,b]^{p-l}))$ and $-l = (n-1)l$. □

Theorem 6.7. *Suppose $\hat{f}, \hat{g} \in \mathbb{C}([0,l))$ with $0 \leq Re(\hat{f}(t) \cdot \overline{\hat{g}(t)})$ for all t and $\mathbf{S}(\hat{f},\hat{g}) = \{p\}$. Let $f, g \in \mathbb{C}([0,nl))$ are l-periodic extensions of $\hat{f}, \hat{g}$. Suppose $Z = [a,b] \subset T_{lk}$. Then, for sufficeintly large n, $n \geq k$ we have:*

(1) *if* $p = 0$ *then* $p \in \mathbf{S}(\theta_Z f, \theta_Z g)$
(2) *if* $p \neq 0$ *then* $\mathbf{S}(\theta_Z f, \theta_Z g) \cap \{p, p + (n-1)l\} \neq \emptyset$.

Proof. Consider subset $\mathbf{p} = \{p, p+l, \ldots, p+(n-1)l\} = \{t \in T_{ln} : t = p \mod l\} \subset T_{ln}$. Denote $\psi = \psi(\delta_Z f, \delta_Z g)|_{\mathbf{p}}$. From Theorem 6.1, it suffices to show that when $p = 0$ then ψ has a maximum at 0 and if $p \neq 0$ then ψ has a maximum at one of $p, p + l(n-1)$. We have $\psi(s) = Re < \delta_{[a,b]} f, \delta_{[a+s,b+s]} g^s >$. Since g is l-periodic we have $Re < \delta_{[a,b]} f, \delta_{[a+s,b+s]} g^s >= Re < \delta_{[a,b]} f, \delta_{[a+s,b+s]} g >= Re < \delta_{[a,b]} f \overline{g}, \delta_{[a+s,b+s]} >$. Lemma 6.6 completes the argument. □

If $0 < Re(\hat{f} \cdot \overline{\hat{g}})$ then we see that when $p = 0$ then ψ has a unique maximum , thus $0 = \mathbf{S}(\theta_Z f, \theta_Z g)$. If $p \neq 0$ then it is still possible state necessary conditions for ψ to have a unique maximum in which case $\mathbf{S}(\theta_Z f, \theta_Z g)$ equals to exactly one of $\{p, p + (n-1)l\} \neq \emptyset$. We leave the details to the reader.

Corollary 6.8. With notation of the theorem, if $\hat{f}, \hat{g} \in \mathbb{C}([0, l))$ are real and both either nonnegative or nonpositive then:
(1) if $p = 0$ then $p \in \mathbf{S}(\theta_Z f, \theta_Z g)$
(2) if $p \neq 0$ then $\mathbf{S}(\theta_Z f, \theta_Z g) \cap \{p, p - l\} \neq \emptyset$.

Proof. We have $(n-1)l = -l \mod nl$. □

If we consider $\mathbf{S}(f, g) \subset T_{nl} = [0, nl)$ as an ordered set then the corollary above asserts that $\mathbf{S}(\theta_Z f, \theta_Z g)$ contains either the smallest or the largest element from $\mathbf{S}(f, g)$. If we view this corollary as a result on localization to the subdomain $[0, nl) - Z$ then Theorem 7.13 from the next section is it's analog for the restriction to this subdomain.

Example 6.9. We show that the requirement of Corollary 6.8, that $\hat{f}, \hat{g} \in \mathbb{R}([0, l))$ are both either nonnegative or nonpositive, is essential (we have pointed this out already, for the special case $\mathbf{S}(\hat{f}, \hat{g}) = \{0\}$, $Z = [s, s]$ a one point interval, in the discussion following Corollary 6.3). Take $l = 3$ and let $\hat{f}, \hat{g} \in \mathbb{R}([0, 3))$ are given by $\hat{f} = \{2, 0.5, 0\}$, $\hat{g} = (-2, -1, 0)$. Then $\psi(\hat{f}, \hat{g})(t) = \{-4.5, -1, -2\}$. It folows that $\mathbf{S}(\hat{f}, \hat{g}) = \{1\}$. For $n > 1$ and for the periodic extensions $f, g \in \mathbb{R}(T_{3n})$ of $\hat{f}, \hat{g}$ we have: $\psi(f, g)(t) = \begin{cases} -4.5n \text{ if } t = 0 \mod 3 \\ -n \text{ if } t = 1 \mod 3 \\ -2n \text{ if } t = 2 \mod 3 \end{cases}$. Thus $\mathbf{S}(f, g) = \{1, 4, \ldots, 1 + 3(n-1)\} = \{s : s = 1 \mod 3\}$. Take $a = 0, b = 2$ so that $[a, b] = \{0, 1, 2\}$. Then $\psi(\delta_{[0,2]} f, g)(t) = \psi(f, \delta_{[0,2]}, g)(-t)$, $\psi(f, \delta_{[0,2]} g)(t) = \psi(\hat{f}, \hat{g})(t \mod 3)$ and $\psi(\delta_{[0,2]} f, \delta_{[0,2]} g)(t) = \begin{cases} -2 \text{ if } t = -1 \\ -4.5 \text{ if } t = 0 \\ -1 \text{ if } t = 1 \\ 0 \text{ otherwise} \end{cases}$. Explicitely, $\psi(f, \delta_{[0,2]} g)(t) = \begin{cases} -4.5 \text{ if } t = 0 \mod 3 \\ -1 \text{ if } t = 1 \mod 3 \\ -2 \text{ if } t = 2 \mod 3 \end{cases}$,
$\psi(\delta_{[0,2]} f, g)(t) = \begin{cases} -4.5 \text{ if } t = 0 \mod 3 \\ -2 \text{ if } t = 1 \mod 3 \\ -1 \text{ if } t = 2 \mod 3 \end{cases}$.

From $\psi(\theta_{[0,2]} f, \theta_{[0,2]} g) = \psi(f, g) - \psi(f, \delta_{[0,2]} g) - \psi(\delta_{[0,2]} f, g)(t) + \psi(\delta_{[0,2]} f, \delta_{[0,2]} g)(t)$, it follows that for $n > 1$, $\mathbf{S}(\theta_{[0,2]} f, \theta_{[0,2]} g) = \{4, \ldots, , 1 + 3(n-1)\}$ and thus $min(\mathbf{S}(\theta_{[a,b]} f, \theta_{[a,b]} g)) = 4$.

7. Sampling periodic signals

Given $\hat{f}, \hat{g} \in \mathbb{C}([0,l))$ let $f, g \in \mathbb{C}([0,nl))$ are l-periodic extensions of $\hat{f}, \hat{g}$. To simplify notation, we will surpress the dependence of f, g on n. Suppose for $n \geq k$, one selects intervals $[a_n, b_n) \subset T_{nl}$ with $\#[a_n, b_n)$ uniformly bounded by a constant C for all n. Again, to simplify the notation, we will surpress the dependance of a_n, b_n on n. Following the notation from Section 4, let f^*, g^* denote restrictions of $f, g,$ to the complement $[b, a)$ of $[a, b)$. Clearly the set of all such f^*, g^* is $P_l(\#[b, a))$ of section 5. In this section, we will write $\psi(f^*, g^*)$ as the sum of three functions with decreasing, increasing and periodic pattern respectively. Using this, for $\hat{f}, \hat{g}$ real-valued, of the same sign, and meeting an additional mild assumption, we obtain Theorem 7.13 - a stabilty result similiar to Theorem 6.7. We willl first address the special case $a_n = nl + k$, $b_n = (n+1)l$, and, at the end of the section, we will present an argument reducing the general case to it. We combine two facts: Lemma 6.1 implying that $|\psi(\delta_{[a,b]} f, g)(t) - [\#[a,b]/l] \cdot \psi(\hat{f}, \hat{g})(t)| < C$ where C is a constant depeneding on $\hat{f}, \hat{g}$, universal for all values of: a, b, n, t and Theorem 4.1 expressing $\psi(f^*, g^*)$ as a sum of two waves.

Using Lemma 6.1, with $r_1 = (m' - s) \mod l$, $r_2 = s \mod l$, we have:

(7.1)
$$\psi(f, \delta_{[0,m'-1-s]} g)(s) = \begin{cases} ((m'-s)/l)\psi(\hat{f}, \hat{g})(s) \text{ if } r_1 = 0 \mod l \\ [(m'-s)/l]\psi(\hat{f}, \hat{g})(s) + \psi(\hat{f}, \delta_{[0,r_1-1]}\hat{g})(s) \text{ if } r_1 \neq 0 \mod l \end{cases}$$

(7.2)
$$\psi(f, \delta_{[m'-s,m'-1]} g)(s+r) = \begin{cases} (s/l)\psi(\hat{f}, \hat{g})(s+r) \text{ if } r_2 = 0 \mod l \\ [s/l]\psi(\hat{f}, \hat{g})(s+r) + \psi(\hat{f}, \delta_{[r_1,r_1+r_2-1]}\hat{g})(s+r) \text{ if } r_2 \neq 0 \mod l \end{cases}$$

Let $\theta_{a,b} = 1 - \delta_{a,b}$ denotes the indicator function given by $\theta_{a,b} = 1$ if $a \neq b$ and $\theta_{a,b} = 0$ if $a = b$. Let $\phi : [0, nl+k) \to [0, l)$ denotes the reduction $\mod l$. For $p \in [0, l)$, denote

$$\lambda(p) = \begin{cases} n \text{ if } 0 \leq p < k \text{ and } k > 0 \\ n-1 \text{ if } k \leq p < l \text{ and } k > 0 \\ n-1 \text{ if } k = 0 \end{cases} \tag{7.3}$$

Theorem 7.1. *Denote:* $r_1(s) = \phi(m'-s)$, $r_2(s) = \phi(s)$, $\rho_1(s) = \psi(\hat{f}, \delta_{[0,r_1)}\hat{g})(s)$, $\rho_2(s) = \psi(\hat{f}, \delta_{[r_1,r_1+r_2)}\hat{g})(s+r)$, $\rho(s) = \rho_1(s) + \rho_2(s)$. *Then* ρ_1, ρ_2 , ρ *are* l*-periodc and:*

$$\psi(f^*, g^*)(s) = [(m'-s)/l]\psi(\hat{f}, \hat{g})(s) + [s/l]\psi(\hat{f}, \hat{g})(s+r) + \rho(s) \tag{7.4}$$

$$[(m'-s)/l] + [s/l] = \lambda(\phi(s)) + \delta_k(\phi(s)) \tag{7.5}$$

(7.6)
$$\rho(s) - \rho(k+s) = \psi(\hat{f}, \delta_{[0,k-s)}\hat{g})(s) - \psi(\hat{f}, \delta_{[-s,k)}\hat{g})(s) + \psi(\hat{f}, \delta_{[k-s,k)}\hat{g})(s+r) - \psi(\hat{f}, \delta_{[0,-s)}\hat{g})(s+k)$$

$$\rho(0) - \rho(k) = 0 \tag{7.7}$$

Proof. The first equation follows directly from (4.2), (7.1) and (7.2). Since both sides of (7.5) are invariant under $s \to s+k$ it suffices to verify it for $s \in [0, l)$ which we leave it as an excercise to the reader. The third formulae is a straightforward consequence of the definitions (also see the columns: 4-8 rows: 1-5 in the table below). The forth formulae is a consequence of symmetry in (7.6). ☐

The theorem can be expressed as the following table.

s	$[(m'-s)/l]\psi(\hat{f},\hat{g})$	$[s/l]\psi(\hat{f},\hat{g})$	$r_1(s)$	$\theta_{0,r_1}\cdot\rho_1(s)$	$r_2(s)$	$\theta_{0,r_2}\cdot\rho_2(s)$
0	$n\psi_0$	$0\psi_r$	k	$\psi([0,k-1])_0$	0	0
1	$n\psi_1$	$1\psi_{r+1}$	$k-1$	$\psi([0,k-2])_1$	1	$\psi([k-1,k-2])_{r+1}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
k-1	$n\psi_{k-1}$	$0\psi_{l-1}$	1	$\psi([0,0])_{k-1}$	$k-1$	$\psi([1,k-1])_{l-1}$
k	$n\psi_k$	$0\psi_0$	0	0	k	$\psi([0,k-1])_0$
k+1	$(n-1)\psi_{k+1}$	$0\psi_1$	$l-1$	$\psi([0,l-2])_{k+1}$	$k+1$	$\psi([l-1,l-1+k])_1$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
l-1	$(n-1)\psi_{l-1}$	$0\psi_{r-1}$	$k+1$	$\psi([0,k])_{l-1}$	$l-1$	$\psi([k+1,k+l-1])_{r-1}$
l	$(n-1)\psi_0$	$1\psi_r$	k	$\psi([0,k-1])_k$	0	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
l+k-1	$(n-1)\psi_{k-1}$	$1\psi_{l-1}$	1	$\psi([0,0])_{k-1}$	$k-1$	$\psi([1,k-1])_{l-1}$
l+k	$(n-1)\psi_k$	$1\psi_0$	0	0	k	$\psi([0,k-1]_0$
l+k+1	$(n-2)\psi_{k+1}$	$1\psi_1$	$l-1$	$\psi([0,l-1])_{k+1}$	$k+1$	$\psi([l-1,l-1+k])_1$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
2l-1	$(n-2)\psi_{l-1}$	$1\psi_{r-1}$	$k+1$	$\psi([0,k])_{l-1}$	$l-1$	$\psi([k+1,k+l-1])_{r-1}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
(n-1)l	$1\psi_0$	$(n-1)\psi_r$	k	$\psi([0,k-1])_0$	0	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
(n-1)l+k-1	$1\psi_{k-1}$	$(n-1)\psi_{l-1}$	1	$\psi([0,0])_{k-1}$	$k-1$	$\psi([1,k-1])_{l-1}$
(n-1)l+k	$0\psi_k$	$(n-1)\psi_0$	0	0	k	$\psi([0,k-1])_0$
(n-1)l+k+1	$0\psi_{k+1}$	$(n-1)\psi_1$	$l-1$	$\psi([0,l-2])_{k+1}$	$k+1$	$\psi([l-1,l-1+k])_1$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
(n-1)l-1	$0\psi_{l-1}$	$(n-1)\psi_{r-1}$	$k+1$	$\psi([0,k])_{l-1}$	$l-1$	$\psi([k+1,k+l-1])_{r-1}$
nl	$0\psi_0$	$n\psi_r$	k	$\psi([0,k-1])_0$	0	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
nl+k-1	$0\psi_{k-1}$	$n\psi_{l-1}$	1	$\psi([0,0])_{k-1}$	$k-1$	$\psi([1,k-1])_{l-1}$

Using (7.5) we can rewrite (7.4) as:

$$\psi(f^*,g^*)(s)=(\lambda(\phi(s))+\delta_k(\phi(s))-[s/l])\psi(\hat{f},\hat{g})(s)+[s/l]\psi(\hat{f},\hat{g})(s+r)+\rho(s) \tag{7.8}$$

We notice that the amplitude of each of the three terms in the above equations are, respectively, decreasing , increasing and periodic. We will give a geometrical interpretation of these two equations by examining the fibers of $\phi:[0,nl+k)\to[0,l)$ given by reduction $\mod l$. For each $p\in[0,l)$, the fiber $\phi_p=\phi^{-1}(p)$ is an ordered set $\phi_p=\{p,l+p,\dots,\lambda(p)l+p\}$ of cardinality:

$$\#\phi_p=\lambda(p)+1 \tag{7.9}$$

, isomorphic as an ordered set to $\{0,1,\dots,\lambda(p)\}$. Thus we have an isomorphism $\Theta:\coprod_{p\in[0,l)}\{p\}\times\{0,1,\dots,\lambda(p)\}\cong[0,m')$ given by $\Theta(p,k)=kl+p$. Under this isomorphism ϕ corresponds to the projection on the first coordinate. To simplify the notation we will write $\lambda=\lambda(p)$. If we arrange the elements of $[0,nl+k)$ into a partial matrix

$$\begin{pmatrix} nl & nl+1 & \dots & nl+k-1 & & & \\ (n-1)l & (n-1)l+1 & \dots & (n-1)l+k-1 & (n-1)l+k & \dots & nl-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ l & l+1 & \dots & l+k-1 & l+k & \dots & 2l-1 \\ 0 & 1 & \dots & k-1 & k & \dots & l-1 \end{pmatrix}$$

, then ϕ is the vertical projection onto the bottom row with the fibers ϕ_p represented by columns whose height (depth?) is given by the equation (7.9). The function $s \to [s/l]$ being the height h_s of s. With this notation we can represent any function on $[0, nl+k)$ as such partial matrix: by listing values at each cell. In particular, the matrix above represents the identity function.

We can also rewrite (7.8) as:
(7.10)

$$\psi(f^*, g^*)(h, s) = h(\psi(\hat{f}, \hat{g})(\phi_{s+r}) - \psi(\hat{f}, \hat{g})(\phi_s)) + (\#\phi_s - 1 + \delta_{\phi_k}(s))\psi(\hat{f}, \hat{g})(\phi_s) + \rho(\phi_s)$$

, with all the terms, except for height $h_s = [s/l]$, being constant on every fiber ϕ_s. In essence, the reminder of this section is concerned with solving the optimization problem for function: $h_s(\psi(\hat{f}, \hat{g})(s+r) - \psi(\hat{f}, \hat{g})(s)) + (\lambda(s) + \delta_{\phi_k}(s))\psi(\hat{f}, \hat{g})(s) + \rho(s)$ subject to constrains $s \in [0, l), 0 \le h_s \le n + \delta_{[0,k)}(s), 0 << n$. We find that the matrix language makes it easier to do this.

Definition 7.2. Define sets $\Lambda(n,k)_0 = \{0\} \times [0, l)$, $\Lambda(n,k)_1 = \{1\} \times [0, l)$, $\Lambda(n,k) = \Lambda(n,k)_0 \coprod \Lambda(n,k)_1$ and $\Lambda : \Lambda(n,k) \to [0, 2n+k)$ as $\Lambda(\epsilon, s) = \begin{cases} s \text{ if } \epsilon = 0 \\ (n-1)l + k + s \text{ if } \epsilon = 1 \end{cases}$.

We notice that $\Lambda(n,k)$ carries the natural lexicographic ordering and that Λ is the map of ordered sets: Λ_0 parametrizing the bottom row:

$$\Lambda(\Lambda(n,k)_0) = \begin{pmatrix} 0 & 1 & \dots & k-1 & k & \dots & l-1 \end{pmatrix}$$

, Λ_1 the top ridge

$$\Lambda(\Lambda(n,k)_1) = \begin{pmatrix} nl & nl+1 & \dots & nl+k-1 & & & \\ & & & & (n-1)l+k & \dots & nl-1 \end{pmatrix}$$

of the matrix above, each consisting of l elements.

Example 7.3. Let's consider the case $k = 0$, $m' = nl$. Since $\lambda(p) = n-1$, $\epsilon(p) = 0$, hence:

$$id_{[0,nl)}(s) = \begin{pmatrix} (n-1)l & (n-1)l+1 & \dots & (n-1)l+k-1 & (n-1)l+k & \dots & nl-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ l & l+1 & \dots & l+k-1 & l+k & \dots & 2l-1 \\ 0 & 1 & \dots & k-1 & k & \dots & l-1 \end{pmatrix}$$

,

$$[(nl-s)/l] = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n-1 & n-2 & \dots & n-2 & n-2 & \dots & n-2 \\ n & n-1 & \dots & n-1 & n-1 & \dots & n-1 \end{pmatrix}$$

,

$$[s/l] = \begin{pmatrix} n-1 & n-1 & \dots & n-1 & n-1 & \dots & n-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

.

Thus, in agreement with (7.5):

$$[(nl-s)/l] + [s/l] = \begin{pmatrix} n & n-1 & \dots & n-1 & n-1 & \dots & n-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n & n-1 & \dots & n-1 & n-1 & \dots & n-1 \\ n & n-1 & \dots & n-1 & n-1 & \dots & n-1 \end{pmatrix}$$

.

Since ψ_s is l-periodic hence ψ is constant on each fiber of ϕ. Since $r = 0$ hence

$$\tilde{\psi}(n,0)_s = \begin{pmatrix} n\psi_0 & (n-1)\psi_1 & \dots & (n-1)\psi_{k-1} & (n-1)\psi_k & \dots & (n-1)\psi_{l-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n\psi_0 & (n-1)\psi_1 & \dots & (n-1)\psi_{k-1} & (n-1)\psi_k & \dots & (n-1)\psi_{l-1} \\ n\psi_0 & (n-1)\psi_1 & \dots & (n-1)\psi_{k-1} & (n-1)\psi_k & \dots & (n-1)\psi_{l-1} \end{pmatrix}$$

Since $\phi(nl-s) + \phi(s) = \begin{cases} 0 \text{ if } k = 0 \mod l \\ l \text{ otherwise} \end{cases}$, hence

$$\rho(s) = \begin{pmatrix} 0 & \psi_1 & \dots & \psi_{k-1} & \psi_k & \dots & \psi_{l-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \psi_1 & \dots & \psi_{k-1} & \psi_k & \dots & \psi_{l-1} \\ 0 & \psi_1 & \dots & \psi_{k-1} & \psi_k & \dots & \psi_{l-1} \end{pmatrix}$$

Adding these two matricies we get $\psi(f^*, g^*)(s) = \tilde{\psi}(n,0)_s + \rho(s) = n\psi_{\phi(s)}$, getting back the resonance equation (6.1) from the previous section.

Example 7.4. Let's consider the case $m' = nl + k$ with $k > 0$.

Then

$$[(nl+k-s)/l]\psi_s = \begin{pmatrix} 0 & 0 & \dots & 0 & & & \\ 1\psi_0 & 1\psi_1 & \dots & 1\psi_{k-1} & 1\psi_k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (n-1)\psi_0 & (n-1)\psi_1 & \dots & (n-1)\psi_{k-1} & (n-1)\psi_k & \dots & (n-2)\psi_{l-1} \\ n\psi_0 & n\psi_1 & \dots & n\psi_{k-1} & n\psi_k & \dots & (n-1)\psi_{l-1} \end{pmatrix}$$

and , since $\psi_{s+k+r} = \psi_{s+l} = \psi_s$

$$[s/l]\psi_{s+r} = \begin{pmatrix} n\psi_r & n\psi_{1+r} & \dots & n\psi_{k-1+r} & & & \\ (n-1)\psi_r & (n-1)\psi_{1+r} & \dots & (n-1)\psi_{k-1+r} & (n-1)\psi_0 & \dots & (n-1)\psi_{r-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1\psi_r & 1\psi_{1+r} & \dots & 1\psi_{k-1+r} & 1\psi_0 & \dots & 1\psi_{r-1} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

. We see that

$$[(nl+k-s)/l] + [s/l] = \begin{cases} n \text{ if } s \mod k \in [0,k] \\ n-1 \text{ if } s \mod k \in (k,l) \end{cases}$$

again, in agreement with (7.5). If we write:

$$\rho(s) = \begin{pmatrix} \rho_0 & \rho_1 & \dots & \rho_{k-1} & & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \rho_0 & \rho_1 & \dots & \rho_{k-1} & \rho_k & \dots & \rho_{l-1} \\ \rho_0 & \rho_1 & \dots & \rho_{k-1} & \rho_k & \dots & \rho_{l-1} \end{pmatrix}$$

then the resonance formulae (7.4) is the sum of these 3 matricies.

For a function f on $[0, nl+k)$ let $\Lambda^*(f)$ denotes the pullback (reparametrisation) of the restriction of f to $Im(\Lambda)$. The following lemma lists identities which are the direct consequence of identities from Theorem 7.1 and the definitions. We leave the proofs as an excercise for the reader.

Lemma 7.5. *Denote* $\tilde{\psi}(n,k)(s) = [(nl+k-s)/l]\psi_s + [s/l]\psi_{s+r}$. *If* $n, k > 0$ *then, for* $s \in [0, l)$, $q \in [0, \lambda(p)]$ *and* $\epsilon \in [0, 1]$*:*

$$\tilde{\psi}(n,k)(ql+s) = (\lambda(s)+\delta_k(s)-q)\psi_s + q\psi_{s+r} = (\lambda(s)-q)\psi_s + q\psi_{s+r} + \psi_k\delta_k(s) \tag{7.11}$$

$$\Lambda(\epsilon, s) = \epsilon((n-1)l+k) + s = \begin{cases} s \ \textit{if} \ \epsilon = 0 \\ s - l \mod (nl+k) \ \textit{if} \ \epsilon = 0 \end{cases} \tag{7.12}$$

$$\Lambda^*(\rho)(\epsilon, s) = \rho(\epsilon k + s) \tag{7.13}$$

$$\Lambda^*(\tilde{\psi}(n,k))(\epsilon, s) = \begin{cases} (\lambda(s) + \delta_k(s))\psi_s \ \textit{if} \ \epsilon = 0 \\ \lambda(\phi(k+s))\psi_s + \delta_0(s)\psi_k \ \textit{if} \ \epsilon = 1 \end{cases} \tag{7.14}$$

$$\Lambda^*(\psi(f^*, g^*))(\epsilon, s) = \Lambda^*(\tilde{\psi}(n,k))(\epsilon, s) + \rho(\epsilon k + s) \tag{7.15}$$

Example 7.6. Consider the case $l = 2$ and $\tilde{f} = \tilde{g} = \delta_0$. It is easy to see that ψ is given by $\psi_0 = 1, \psi_1 = 0$ and $f = g$ is the parity function (a special case of a 'Dirac comb') given as 0 for odd and 1 for even arguments. First consider the case of $k = 0$. It is easy to see that $\rho(1) = \rho(0) = 0$ and we get

$$\psi(f^*, g^*) = \begin{pmatrix} n & 0 \\ \dots & \dots \\ n & 0 \\ n & 0 \end{pmatrix}$$

Consider the case of $k = 1$. Then $m' = 2n+1$ and the parity functions $f^* = g^*$ are now defined on $[0, 2n+1)$. We have $\rho_1(0) = 1, \rho_2(0) = 0$ thus $\rho(0) = 1$. By (7.7), $\rho(1) = \rho(0) = 1$. It follows that

$$\psi(f^*, g^*) = \rho + \tilde{\psi}(n,1) = \begin{pmatrix} 1 & \\ 1 & 1 \\ \dots & \dots \\ 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \\ 1 & n-1 \\ \dots & \dots \\ n-1 & 1 \\ n & 0 \end{pmatrix} = \begin{pmatrix} 1 & \\ 2 & n \\ \dots & \dots \\ n & 2 \\ n+1 & 1 \end{pmatrix}$$

While these two expressions differ, in both cases the maximum of $\psi(f^*, g^*)$ is attained at 0 , in the left bottom corner, thus remaining where the maximum 1 of of $\psi(\hat{f}, \hat{g})$ is in the first place. Notice that the other point, at which $\psi(f^*, g^*)$ is close to maximum, is $2(n-1) + 1 = 2n + k - 2 = -l \mod m'$ in the right upper corner.

For the reminder of this section, we will assume that ψ_s is nonnegative, in which case the relationship between $\psi(f^*, g^*)$ and $\psi(f, g)$ can be expressed as conservation laws for the lags. The first law, while allowing for a perturbation of of f^*, g^*, identifies the fibers of ϕ, while the second law identifies the elemets in these fibers at which $\psi(f^*, g^*)$ certainly attains maximum.

We begin with finding the maximums of the restriction $\psi(f^*, g^*)_p$ of $\psi(f^*, g^*)$ to a fiber ϕ_p. Since, from (7.8) we have $\psi(f^*, g^*) = \tilde{\psi}(n, k) + \rho$ with $\tilde{\psi}(n, k)$ defined in Lemma 7.5 and ρ constant on each fiber ϕ_p, hence $S_p = \arg\max_{s\in\phi_p}(\psi(f^*, g^*)) = \arg\max_{s\in\phi_p}(\tilde{\psi}(n, k))$. The following asserts that for each p either both $\tilde{\psi}(n, k)_p$, $\psi(f^*, g^*)_p$ are constant functions or they attain maximum at exactly one point which must be either the first or at the last element of ϕ_p (equivalently, their maximum values are at the ridges $\Lambda(\Lambda(n, k)_*) \subset [0, nl + k)$ from Definition 7.2).

Lemma 7.7. *Suppose $0 \le \psi_t$ for all t. Then, for sufficeintly large n :*

$$S_p = \begin{cases} \{p\} \in \Lambda(\Lambda(n,k)_0) \ \text{if } \psi_p > \psi_{p+r} \\ \{\lambda l + p\} \in \Lambda(\Lambda(n,k)_1) \ \text{if } \psi_p < \psi_{p+r} \\ \phi_p \ \text{if } \psi_p = \psi_{p+r} \end{cases} \tag{7.16}$$

and for $s \in S_p$ we have:

$$\tilde{\psi}(n,k)(s) = \begin{cases} \lambda(p)\psi_p + \delta_k(p)\psi_k \ \text{if } \psi_p \ge \psi_{p+r} \\ \lambda(p)\psi_{p+r} + \delta_k(p)\psi_k \ \text{if } \psi_p < \psi_{p+r} \end{cases} \tag{7.17}$$

Proof. From (7.14), $\psi(f^*, g^*)_p(q) = (\lambda(p) + \delta_k(p) - q)\psi_s + q\psi_{p+r}$. Checking from definitions shows that $(\lambda(\phi(s)) + \delta_k(s) - q)\psi_s + q\psi_{s+r} = (n - q)\psi_s + q\psi_{s+r} + C$ where C is a constant which does not depend on n, q, hence (7.16). The formulaes (7.17) are evaluations of (7.11) at $q \in \{p, \lambda(p)n + p\}$. $\square$

Denote $M = \max(\psi_s)$. The next result states that $\psi(f^*, g^*)$ attains global maximum only at fibers ϕ_p for which either $\psi_p = M$ or $\psi_{p+r} = M$.

Theorem 7.8. *Suppose that for a sequence of $\alpha_n \in \mathbb{R}(T_{nl+k})$ there exists a constant A such that $|\alpha_n(t)| < A$ for all $t \in T_{nl+k}$ and all n. Suppose $0 < k$, $0 \le \psi_t$ for all t and at least for one t $0 < \psi_t$. There exists N such that for $n > N$, for any $s \in arg\max_{T_{m'}}(\psi(f^*, g^*) + \alpha_n)$ and $p = \phi(s)$ we have $\max(\psi_p, \psi_{p+r}) = M$. Moreover:*

1. *if $\psi_p = M$ then $p \in arg\max_{T_{m'}}(\tilde{\psi}(n, k))$*
2. *if $\psi_{p+r} = M$ then $p + \epsilon \in arg\max_{T_{m'}}(\tilde{\psi}(n, k))$ with $p + \epsilon \in [(n-1)l + k, nl + k)$ and $\epsilon = 0 \mod l$*

Proof. If we show that $\max(\psi_p, \psi_{p+r}) = M$ then the rest of the theorem follws directly from Lemma 7.7. Since, for at least one t $0 < \psi_t$ hence $0 < M$. Suppose $\max(\psi_p, \psi_{p+r}) < M$. Then there exists $q \in [0, l)$ such that $max(\psi_q, \psi_{q+r}) = M$ and $\max(\psi_p, \psi_{p+r}) = M - \epsilon$ with $0 < \epsilon < M$. From (7.4), $\max_{\phi_p}(\psi(f^*, g^*) + \alpha_n) = \max_{\phi_p}(\tilde{\psi}(n, k) + \alpha_n + \rho) \le \lambda(p)(M - \epsilon) + A + B + \psi_k \le n(M-\epsilon) + A + B + \psi_k$ where $A = \max_{\phi_p}(\alpha_n)$ and $B = \max_{\phi_p}(\rho) = \rho(p)$. Since we have either $\psi_q = M$ or $\psi_{q+r} = M$ hence, by Lemma 7.7 , we have $\max_{\phi_q}(\psi(f^*, g^*)+\alpha_n) = \max_{\phi_q}(\tilde{\psi}(n,k)+\alpha_n+\rho) \geq \lambda(q)M+A'+B' \geq (n-1)M+A'+B'$ where $A' = \min_{\phi_q}(\alpha_n)$ and $B' = \min_{\phi_q}(\rho) = \rho(q)$.

Now since the quantity $((n-1)M + B' + A') - (n(M-\epsilon) + B + A + \psi_k)) = -M+B'+A'-A-B-\psi_k+n\epsilon$ is positive for large n hence so is $\max_{\phi_q}(\psi(f^*, g^*)+\alpha_n) - \max_{\phi_p}(\psi(f^*, g^*)+\alpha_n)$ - a contradiction. QED $\square$

Corollary 7.9. If $n > N$ and $p = \phi(s)$ for $s \in \arg\max_{T_{m'}}(\psi(f^*, g^*)+\alpha_n))$ then either $p \in \mathbf{S}(\hat{f}, \hat{g})$ or $p + r \mod l \in \mathbf{S}(\hat{f}, \hat{g})$.

Since the equation $p + r = s \mod l$ is equivalent to $p = s + k \mod l$ hence the assertion of the corollary can be written as

(7.18)
$$\phi(\arg\max_{[0,2n+k)}(\psi(f^*, g^*) + \alpha_n)) \subset \mathbf{S}(\hat{f}, \hat{g}) \cup (k + \mathbf{S}(\hat{f}, \hat{g})) = \mathbf{S}(\hat{f}, \hat{g}) \cup (\mathbf{S}(\hat{f}, \hat{g}) - r)$$

Example 7.10. In Example 11 we had $\mathbf{S}(\hat{f}, \hat{g}) = \{0\}$ with $\arg\max_{T_{m'}}(\psi(f^*, g^*)+\alpha_n)) \mod 2 = \{0\}$. Adding to $\psi(f^*, g^*)$ the disturbance $\alpha = 2\delta_{2n}$ gives $\arg\max_{T_{m'}}(\psi(f^*, g^*)+\alpha_n)) \mod 2 = \{1\}$, disturbance $\alpha = \delta_{2n}$ gives $\arg\max_{T_{m'}}(\psi(f^*, g^*)+\alpha_n)) \mod 2 = \{0, 1\}$. This shows that Corollary 7.9 can not be strenghtened without additional assumptions.

However when the function ψ is sufficiently generic and the disturbance α is (nearly) zero then the result above can be strengthen.

Definition 7.11. We define S_p to be of type I if it consist of one point, and ψ to be of type I if for all $p \in \arg\max_{s\in T_m}(\hat{\psi}(n,k)(s))$ S_p is of type I, otherwise we will say that ψ is of type II.

Notice that the condition $\psi_t \neq \psi_{t+r}$ for $t \in [0, l), M = max(\psi_t, \psi_{t+r})$ implies that ψ is of type I . In particular, the condition $\#\mathbf{S}(\hat{f}, \hat{g}) = 1$ considered in the previous section implies that ψ is of type I .

Corollary 7.12. If $0 \leq \psi_s$ for all s and ψ is of type I then for sufficiently large n: $\arg\max_{[0,2n+k)}(\psi(f^*, g^*)) = \arg\max_{(\epsilon,s)\in\{0,1\}\times\mathbf{S}(\hat{f},\hat{g})}(\lambda(\phi(\epsilon k+s))\psi_s+\psi_k\delta_{k(1-\epsilon)}(s)+\rho(\epsilon k+s))$.

Proof. If $s \in \arg\max_{[0,2n+k)}(\psi(f^*, g^*))$ and $p = \phi(s)$ then, by the theorem , either $\psi_p \in M$ or $\psi_{p+r} \in M$. If $\psi_p \in M$ then since ϕ is of type I , by Lemma 7.7 , s is the first point of ϕ_p and thus $s \in \Lambda(n,k)_0$ while if $\psi_{p+r} \in M$ then s is the last point of ϕ_p and thus $s \in \Lambda(n,k)_1$. This shows that $\arg\max_{[0,2n+k)}(\psi(f^*, g^*)) = \arg\max_{\Lambda(n,k)_*}(\Lambda^*(\psi(f^*, g^*)))$. From (7.14) we have $\Lambda^*(\psi(f^*, g^*)) = \lambda(\phi(\epsilon k+s))\psi_s + \delta_{k(1-\epsilon)}(s)\psi_k + \rho(\epsilon k+s)$. Since $\lambda(\phi(\epsilon k+s))\psi_s + \delta_{k(1-\epsilon)}(s)\psi_k + \rho(\epsilon k+s) = (n-1)\psi_s+C(k,s)$ with $C(k,s)$ not dependending on n hence, the left hand side, for large

n, can be a maximum only if $\psi_s = M$. Thus $\arg \max_{\Lambda(n,k)_*} (\Lambda^*(\psi(f^*, g^*))(\epsilon, s))) = \arg \max_{(\epsilon,s)\in\Lambda(n,k)_*:\psi_s=M} (\lambda(\phi(\epsilon k + s))\psi_s + \delta_{k(1-\epsilon)}(s)\psi_k + \rho(\epsilon k + s))$. □

We can now write down the main result of this section, paralell to the Theorem 6.5 from the previous section.

Theorem 7.13. *Suppose $0 < k$, $0 \le \psi_t$ for all t, and that $\mathbf{S}(\hat{f}, \hat{g}) = \{s\}$. Then for sufficciently large n:*

1. *If $s = 0$ then $\mathbf{S}(f^*, g^*) = \{0\}$.*
2. *If $s \neq 0$ then $\mathbf{S}(f^*, g^*) \subset \{s, s - l\}$.*

Proof. When $\mathbf{S}(\hat{f}, \hat{g}) = \{s\}$ then ψ is of Type I and our optimization problem reduces to finding maximum of $\gamma(\epsilon) = \lambda(\phi(\epsilon k + s))\psi_s + \delta_{k(1-\epsilon)}\psi_k + \rho(\epsilon k + s)$. Define $\Delta = \gamma(0) - \gamma(1)$. From (6.11) we have $\mathbf{S}(f^*, g^*) = \begin{cases} \{s\} \text{ if } \Delta > 0 \\ \{s - l\} \text{ if } \Delta < 0 \\ \{s, s - l\} \text{ if } \Delta = 0 \end{cases}$.

If $s = 0$ we have $\gamma(\epsilon) = \begin{cases} \lambda(0)\psi_0 + \rho(0) \text{ if } \epsilon = 0 \\ \lambda(k)\psi_0 + \psi_k + \rho(k) \text{ if } \epsilon = 1 \end{cases}$ which, using (6.10), gives $\Delta = (\lambda(0) - \lambda(k))\psi_0 - \psi_k = \psi_0 - \psi_k > 0$. Finally, $-l = nl + k - l \mod (nl + k) = (n - l)l + k \mod (nl + k)$. □

Observe that the above result agrees with Corollary 7.5 after reduction $\mod l$. It can be viewed as a conservation law for the lag between f, g given the lag between $\hat{f}, \hat{g}$ using an arbitrary data window, in addition, it does not depend, at least explicitely, on size of the data window

Example 13

If $0 < k$ then for $s \in (0, l)$ we have:

$$\lambda(\phi(k + s)) = \begin{cases} n - 1 \text{ if } 0 \le s < r \\ n \text{ if } r \le s < l \end{cases}.$$

$$\gamma(0) - \gamma(1) = \begin{cases} M - \psi_k \text{ if } s = 0 \\ \begin{cases} M + \psi_k + \rho(k) - \rho(2k) \text{ if } k < r \\ -M + \psi_k + \rho(k) - \rho(2k) \text{ if } r <= k \end{cases} \text{ if } s = k \\ \begin{cases} M + \rho(s) - \rho(k + s) \text{ if } s < min(k, r) \\ -M + \rho(s) - \rho(k + s) \text{ if } max(k, r) <= s \\ \rho(s) - \rho(k + s) \text{ otherwise} \end{cases} \text{ if } s \notin \{0, k\} \end{cases}$$

Thus, as long as $|\rho(s) - \rho(k + s)| \le M$ and $s \neq k$, we have : $\gamma(0) - \gamma(1) > 0$ for $0 \le s < min(k, r)$, $\gamma(0) - \gamma(1) < 0$ if $max(k, r) < s$.

Example 14

Take $l = 3$. We will show that the second assertion in Theorem 9 is best possible: if $s \neq 0$, then, with suitable choice of f and g, $\mathbf{S}(f^*, g^*)$ can be any one of: $\{s, s - l\}, \{s\}, \{s - l\}$. With matrix notation, for any $x, y, z \ge 0$ define:

$$\hat{f} = \begin{pmatrix} x & y & z \end{pmatrix}$$

$$\hat{g} = \hat{f}^{(-1)} = \begin{pmatrix} y & z & x \end{pmatrix}$$

Thus, for generic x, y, z, we will have $\mathbf{S}(\hat{f}, \hat{g}) = \{1\}$, and:

$$f = \begin{pmatrix} x & y & z & \dots & x & y & z & x \end{pmatrix}$$

$$g = \begin{pmatrix} y & z & x & \dots & y & z & x & y \end{pmatrix}$$

$$g^1 = \begin{pmatrix} y & y & z & x & \dots & y & z & x \end{pmatrix}$$

$$g^{(-2)} = \begin{pmatrix} x & y & z & \dots & x & y & y & z \end{pmatrix}$$

$$\psi(f,g)(1) = \psi(f, g^1) =$$
$$= \begin{pmatrix} x & y & z & \dots & x & y & z & x \end{pmatrix} \begin{pmatrix} y & y & z & x & \dots & y & z & x \end{pmatrix}^T =$$
$$= xy + n\psi(\hat{f}, \hat{g}) = xy + n(x^2 + y^2 + z^2)$$

$$\psi(f,g)(-2) = \psi(f, g^{(-2)}) =$$
$$= \begin{pmatrix} x & y & z & \dots & x & y & z & x \end{pmatrix} \begin{pmatrix} x & y & z & \dots & x & y & y & z \end{pmatrix}^T =$$
$$= (n-1)\psi(\hat{f}, \hat{g}) + x^2 + y^2 + zy + xz = (n-1)(x^2 + y^2 + z^2) + x^2 + y^2 + zy + xz$$

.

Thus $\psi(f,g)(1) - \psi(f,g)(-2) = z^2 + xy - zy + xz = z^2 + z(x-y) + xy$. Setting $z = 1, y = x + 2$ we get a family of functions f_x parametrised by $x > 0$ for which $\Delta(x) = \psi(f_x, g_x)(1) - \psi(f_x, g_x)(-2) = -1 + x(x+2)$.

Since the quantity $\Delta(x)$ is independent of n, hence, from Theorem 9, for n large : $\mathbf{S}(f_x, g_x) = \begin{cases} \{1\} \text{ if } \Delta(x) > 0 \\ \{-2\} \text{ if } \Delta(x) < 0 \\ \{1, -2\} \text{ if } \Delta(x) = 0 \end{cases}$

Since the quantity $\Delta(x)$, with a suitable choice of x can be positive ,negative or zero hence any of the three alternatives above can take place, showing the the assertion 2) of Theorem 9 is the best possible.

We will close this section with an argument generalizing all the results obtained for the interval $[0, nk + l)$ to an arbitrary interval $[a, b)$ of length $nk + l$. Given $f \in \mathbb{C}(T_{nl})$ and an interval $i = [a, b) \subset T_{nl}$ define $i^*(f)$ to be the restriction of $f \in \mathbb{C}(T_{nl})$ to the interval $[b, a) = [0, nl) - [a, b)$ .By naturality, for any $\sigma \in T_{nl}$, we have

$$(i^*(f))^\sigma = (i^\sigma)^*(f^\sigma) \tag{7.19}$$

. Select $\sigma_b \in T_{nl}$ so that $b + \sigma_b + 1 = 0 \mod nl$. Let $r = \#[b, a)$, then $i^{\sigma_b} = [a + \sigma_b, b + \sigma_b) = [nl - r, nl)$ and thus $(i^{\sigma_b})^*(f) = f^*$. From the equation above we get $i^*(f^{-\sigma_b}) = (f^*)^{-\sigma_b}$ or, equivalently:

$$i^*(f) = ((f^{\sigma_b})^*)^{-\sigma_b} \tag{7.20}$$

From this we get $\psi(i^*(f), i^*(g)) = \psi(((f^{\sigma_b})^*)^{-\sigma_b}, ((g^{\sigma_b})^*)^{-\sigma_b}) = \psi((f^{\sigma_b})^*, (g^{\sigma_b})^*)$. Thus all the results of this section for $\psi(f^*, g^*)$ remain valid after substitution of $f, \hat{f}, g, \hat{g}$ by $f^{\sigma_b}, \hat{f}^{\sigma_b}, g^{\sigma_b}, \hat{g}^{\sigma_b}$. Moreover, since for any s, $\psi(\hat{f}^s, \hat{g}^s) = \psi(\hat{f}, \hat{g})$ hence the values ψ_s are unchanged after such substitution.

Thus, given intervals $i_n = [a_n, b_n) \subset T_{nl}$, with $a_n, b_n \in T_{nl}$ such that $\#[a_n, b_n) < C$ for a constant C independent of n, the results of this section remain valid for f^*, g^* replaced by $i_n^*(f), i_n^*(g)$ and $f, \hat{f}, g, \hat{g}$ by $f^{\sigma_b}, \hat{f}^{\sigma_b}, g^{\sigma_b}, \hat{g}^{\sigma_b}$

8. Appendix 1: Shifts via Discrete Fourier Transform DFT

The results of this section are well know in the case when T_m is replaced by the unit circle or the real line [**?**] and are presented here for completeness only.

Given $g \in \mathbb{C}(T_m)$, denote $g_i = g(\nu_m^i), G_i =< g, \chi_i >, G_i^{-1} =< g, \chi_{n-i} >$ and $F(g)(i) = G_i, F^{-1}(g)(i) = G_i^{-1}$. Then the operator $\mathrm{F} : \mathbb{C}(T_m) \to \mathbb{C}(T_m)$ given by $g \to F(g)$ is the well known Discrete Fourier Transform DFT with inverse $\frac{1}{m}F^{-1}$ [**?**]. For any f, g we have:

(1) $\chi_t \tau = \overline{\chi_t} = \chi_{-t}$
(2) $\mathrm{F}(f\tau) = \mathrm{F}(f)\tau = \mathrm{F}^{-1}(f)$
(3) $\mathrm{F}(\overline{f})\tau = \overline{\mathrm{F}}(f)$ and $\mathrm{F}^{-1}(\overline{f})\tau = \overline{\mathrm{F}^{-1}}(f)$
(4) $\mathrm{F}^{-1}(f\tau) = \mathrm{F}^{-1}(f)\tau = \mathrm{F}(f)$
(5) $\mathrm{F}^{-1}(\mathrm{F}(f)) = \mathrm{F}(\mathrm{F}^{-1}(f)) = mf$
(6) $\mathrm{F}(fg) = \mathrm{F}(f) * \mathrm{F}(g)$ and $\mathrm{F}^{-1}(fg) = \mathrm{F}^{-1}(f) * \mathrm{F}^{-1}(g)$
(7) $\mathrm{F}(f * g) = \mathrm{F}(f)\mathrm{F}(g)$ and $\mathrm{F}^{-1}(f * g) = \mathrm{F}^{-1}(f)\mathrm{F}^{-1}(g)$
(8) $F(\delta_t) = \chi_t$

Theorem 7

$p : T_m \to \mathbb{R}$ given by $p(t) =< f, t(g) >$ can be written as $p = m^{-2}\mathrm{F}^{-1}(\mathrm{F}(f)\overline{\mathrm{F}(g)})$

Proof.

From properties of DFT we have: $\mathrm{F}^{-1}(\mathrm{F}(f)\overline{\mathrm{F}}(f)) = \mathrm{F}^{-1}((\mathrm{F}(f)) * \mathrm{F}^{-1}(\overline{\mathrm{F}}(f)) = m^2 f * \overline{g}\tau$. Theorem 3 completes the proof. QED

Theorem 4 gives us a new method for computing $S(f, g)$.

Corollary 3

$S(f, g) = \arg\max_{t \in T_m}(\psi)$ where $\psi = Re(\mathrm{F}^{-1}(\mathrm{F}(f)\overline{\mathrm{F}(g)}))$.

Corollary 4

For $m = 2^n$, $S(f, g)$ is $O(mlog(m))$ computable.

Proof. It is well known that the Fast Fourier Transform algorithm FFT [**?**] computing F, F^{-1} is $O(mlog(m))$. Neither the expression for ψ nor search for $\arg\max_{t \in T_m}(\psi)$ increase the complexity. QED

Note that the computation of $S(f, g)$ using the definition given in Section 1 is $O(m^2)$.

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