Supplementary Material for "A Bayesian Approach for Robust Longitudinal Envelope Models"

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A Proofs

This section presents the proofs. Section A.1 provides the proof of Proposition 2 and Section A.2 shows the symmetry of the proposal distribution for P.

The Weinstein–Aronszajn identity states that for any two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$,

$$|I_m + AB| = |I_n + BA|, \tag{1}$$

where I_m and I_n denote the identity matrices of sizes m and n, respectively. We will use this identity in several places in the following proofs.

A.1 The modes of distributions

For any idempotent matrix P, we can write $P = \Gamma\Gamma^T$, where Γ is an $m \times k$ column-orthogonal matrix satisfying $\Gamma^T\Gamma = I_k$. To find the mode of a distribution, it suffices to find P, or equivalently Γ that maximizes the corresponding density function.

Consider the density

$$f(P) \propto \mathrm{etr}(MP),$$

where $M \in \mathbb{R}^{m\times m}$ is symmetric. Notice that

$$\begin{aligned}\arg\max_P f(P) &= \arg\max_P \mathrm{tr}(MP) = \arg\max_\Gamma \mathrm{tr}(\Gamma^T M\Gamma)\\ &= \arg\max_\Gamma \sum_{i=1}^k \gamma_i^T M\gamma_i,\end{aligned}$$

where γ_i is the ith column of Γ. Let the eigenvalues of M be $\lambda_1 \geq \cdots \geq \lambda_m$ and the associated eigenvectors be ξ_i's. Due to Proposition (iv) in Section 1f.2 in Rao (1973),

$$\sum_{i=1}^k \gamma_i^T M\gamma_i \leq \sum_{i=1}^k \lambda_k,$$

where the maximum is achieved when $\gamma_i = \xi_i$ for $i = 1, \ldots, k$. Hence, the mode is $P_0 = U_1U_1^T$, where $U_1 = (\xi_1, \ldots, \xi_k)$.

Consider the density

$$f(P) \propto |I_m - P + M^{-1}P|^{-m/2},$$

where $M \in \mathbb{R}^{m\times m}$ is symmetric. Notice that

$$\begin{aligned}
\arg\max_P f(P) &= \arg\min_P |I_m - (I_m - M^{-1})P| \\
&= \arg\min_\Gamma |I_k - \Gamma^T(I_m - M^{-1})\Gamma| \\
&= \arg\min_\Gamma |\Gamma^T M^{-1}\Gamma|,
\end{aligned}$$

where the second equality holds because of (1). Let the eigenvalues of M be $\lambda_1 \geq \cdots \geq \lambda_m$ and the associated eigenvectors are ξ_i's. Then the eigenvalues of M^{-1} are $\lambda_1^{-1} \leq \cdots \leq \lambda_m^{-1}$ and the associated eigenvectors are ξ_i's. Due to Proposition (viii) in Section 1f.2 of Rao (1973),

$$|\Gamma^T M^{-1}\Gamma| \geq \lambda_1^{-1}\cdots\lambda_k^{-1},$$

where the minimum is achieved when $\Gamma = U_1 = (\xi_1, \ldots, \xi_k)$. Hence, the mode is $P_0 = U_1U_1^T$.

A.2 The symmetry of the proposal distribution for P

The proposal distribution for P has the density

$$f(P \mid P^{(s)}) = |W|^{-u/2}|I_r - P + W^{-1}P|^{-r/2},$$

where $W = \sigma^2 I_r + P^{(s)}$ and $\sigma^2 > 0$ is a tuning parameter. Note that

$$|W| = (\sigma^2+1)^u\sigma^{2(r-u)}, \qquad W^{-1} = \frac{1}{\sigma^2}I_r - \frac{1}{\sigma^2(\sigma^2+1)}P^{(s)}.$$

We can write $P = \Gamma\Gamma^T$ for a properly chosen column-orthogonal matrix $\Gamma \in \mathbb{R}^{r\times u}$. Therefore,

$$\begin{aligned}
f(P \mid P^{(s)}) &= |W|^{-u/2}|I_r - (I - W^{-1})\Gamma\Gamma^T|^{-r/2} \\
&= |W|^{-u/2}|I_u - \Gamma^T(I - W^{-1})\Gamma|^{-r/2} \\
&= (\sigma^2+1)^{-u^2/2}\sigma^{-(r-u)u}|\Gamma^T W^{-1}\Gamma|^{-r/2} \\
&= (\sigma^2+1)^{-u^2/2}\sigma^{-(r-u)u}\left|\frac{1}{\sigma^2}I_u - \frac{1}{\sigma^2(\sigma^2+1)}\Gamma^T P^{(s)}\Gamma\right|^{-r/2} \\
&= (\sigma^2+1)^{-u^2/2}\sigma^{u^2}\left|I_u - \frac{1}{(\sigma^2+1)}\Gamma^T P^{(s)}\Gamma\right|^{-r/2} \\
&= (\sigma^2+1)^{-u^2/2}\sigma^{u^2}\left|I_r - \frac{1}{(\sigma^2+1)}P^{(s)}P\right|^{-r/2} \\
&= (\sigma^2+1)^{-u^2/2}\sigma^{u^2}\left|I_r - \frac{1}{(\sigma^2+1)}PP^{(s)}\right|^{-r/2},
\end{aligned}$$

where the second, sixth, and seventh equalities hold because of (1). The last equality above implies that P and $P^{(s)}$ are symmetric and hence $f(P \mid P^{(s)}) = f(P^{(s)} \mid P)$. Therefore, the proposal distribution for P is symmetric.

B Additional Figures and Tables for the Simulation Studies

Table S1, corresponding to Figure 1 in Section 5.1, reports the medians and interquartile ranges (IQRs) of $D(\hat{\beta}, \beta)$ in 500 replicates, where the random errors follow a t-distribution.

Table S1: Medians and IQRs of $D(\hat{\beta}, \beta)$ in Figure 1

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	5.298 (0.661)	2.406 (0.160)	1.646 (0.095)
	LEM	1.385 (0.389)	0.979 (0.275)	0.725 (0.178)
	RoLEM	0.953 (0.178)	0.651 (0.118)	0.456 (0.083)
$J = 10$	RLMM	2.347 (0.212)	1.613 (0.123)	1.119 (0.065)
	LEM	0.942 (0.281)	0.701 (0.178)	0.495 (0.130)
	RoLEM	0.647 (0.116)	0.452 (0.073)	0.314 (0.054)

Table S2, corresponding to Figure 2 in Section 5.1, reports the medians and interquartile ranges of $D(\hat{\Sigma}, \Sigma)$ in 500 replicates, where random errors follow a t-distribution.

Table S2: Medians and IQRs of $D(\hat{\Sigma}, \Sigma)$ in Figure 2

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	31.388 (1.875)	31.564 (1.724)	31.380 (2.096)
	LEM	38.084 (21.761)	34.722 (15.165)	33.438 (10.926)
	RoLEM	31.388 (1.875)	31.563 (1.724)	8.180 (1.994)
$J = 10$	RLMM	31.342 (1.961)	31.457 (1.946)	31.434 (1.961)
	LEM	34.111 (19.764)	32.625 (13.436)	31.469 (10.489)
	RoLEM	31.341 (1.964)	31.457 (1.946)	7.887 (1.960)

Table S3, corresponding to Figure 3 in Section 5.1, reports the medians and interquartile ranges of the length of the 95% HPD intervals for β across all components and 500 replicates, where random errors follow a t-distribution.

Table S3: Medians and IQRs of the Length of HPD Intervals in Figure 3

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	0.727 (0.143)	0.370 (0.057)	0.256 (0.037)
	LEM	0.173 (0.070)	0.125 (0.048)	0.089 (0.034)
	RoLEM	0.148 (0.055)	0.101 (0.037)	0.070 (0.026)
$J = 10$	RLMM	0.363 (0.059)	0.251 (0.038)	0.176 (0.025)
	LEM	0.123 (0.048)	0.088 (0.033)	0.063 (0.024)
	RoLEM	0.100 (0.037)	0.070 (0.025)	0.049 (0.018)

Table S4, corresponding to Figure 4 in Section 5.1, reports the medians and interquartile ranges of the average empirical coverage probability of the 95% HPD intervals for β across all components, where the random errors follow a t-distribution.

Table S5, corresponding to Figure 5 in Section 5.2, reports the medians and interquartile ranges of $D(\hat{\beta}, \beta)$ across 500 replicates, where the random errors follow a normal distribution.

Table S4: Medians and IQRs of the Coverage Probability of HPD Intervals in Figure 4

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	0.914 (0.018)	0.942 (0.014)	0.944 (0.014)
	LEM	0.874 (0.018)	0.867 (0.021)	0.856 (0.020)
	RoLEM	0.948 (0.014)	0.946 (0.014)	0.944 (0.014)
$J = 10$	RLMM	0.944 (0.014)	0.946 (0.014)	0.946 (0.014)
	LEM	0.876 (0.018)	0.864 (0.022)	0.865 (0.018)
	RoLEM	0.948 (0.014)	0.946 (0.014)	0.948 (0.014)

Table S5: Medians and IQRs of $D(\hat{\beta}, \beta)$ in Figure 5

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	7.672 (0.780)	3.338 (0.180)	2.308 (0.112)
	LEM	1.245 (0.224)	0.857 (0.155)	0.595 (0.125)
	RoLEM	1.247 (0.222)	0.860 (0.157)	0.593 (0.122)
$J = 10$	RLMM	3.245 (0.173)	2.259 (0.125)	1.571 (0.084)
	LEM	0.831 (0.153)	0.587 (0.113)	0.409 (0.073)
	RoLEM	0.832 (0.151)	0.588 (0.112)	0.408 (0.072)

Figure S1 shows the medians and interquartile ranges of $D(\hat{\Sigma}, \Sigma)$ across 500 replicates, where the random errors follow a normal distribution. Table S6 reports the numerical values.

Table S6: Medians and IQRs of $D(\hat{\Sigma}, \Sigma)$ in Figure S1

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	31.572 (1.948)	31.489 (1.833)	31.451 (1.944)
	LEM	38.760 (3.410)	34.874 (2.690)	33.138 (2.383)
	RoLEM	35.715 (3.411)	33.772 (2.673)	32.540 (2.412)
$J = 10$	RLMM	31.486 (1.967)	31.524 (1.873)	31.531 (1.983)
	LEM	34.921 (2.908)	33.107 (2.241)	32.224 (2.340)
	RoLEM	33.257 (2.854)	32.423 (2.198)	31.860 (2.361)

Table S7, corresponding to Figure 6 in Section 5.2, reports the medians and interquartile ranges of the length of the 95% HPD intervals for β across all components and 500 replicates, where the random errors follow a normal distribution.

Figure S2 shows the medians and interquartile ranges of the average empirical coverage probability of the 95% HPD intervals for β across all components, where the random errors follow a normal distribution. Table S8 reports the numerical values.

Table S9, corresponding to Figure 7 in Section 5.2, reports the medians and interquartile ranges of $D(\hat{\beta}, \beta)$ across 500 replicates, where the random errors follow a mixture-normal distribution.

Figure S3 shows the medians and interquartile ranges of $D(\hat{\Sigma}, \Sigma)$ across 500 replicates, where the random errors follow a mixture-normal distribution. Table S10 reports the numerical values.

Table S11, corresponding to Figure 8 in Section 5.2, reports the medians and interquartile ranges of the length of the 95% HPD intervals for β across all components and 500 replicates, where the random errors follow a mixture-normal distribution.

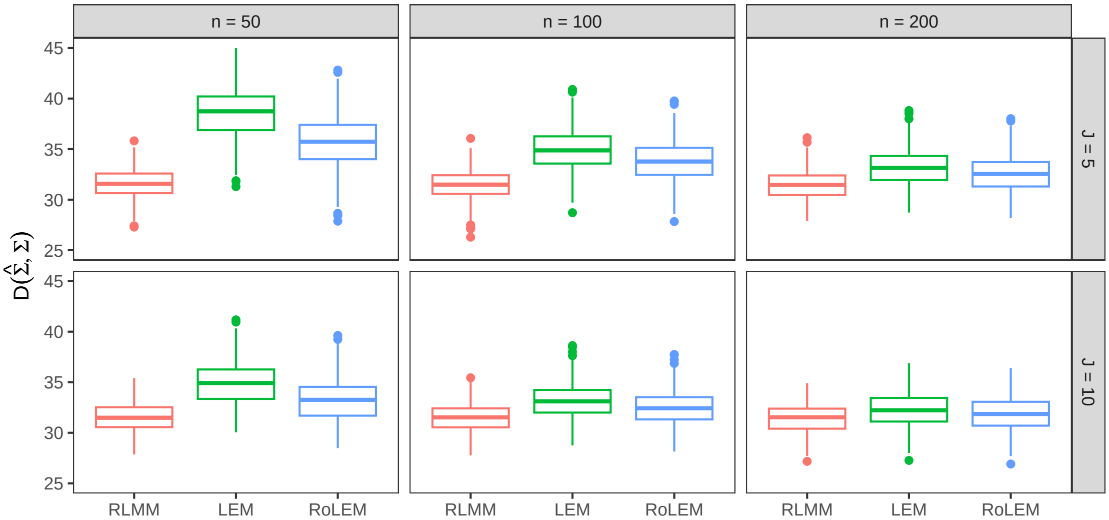

Figure S1: Performance of estimating Σ_ε using RLMM, LEM, or RoLEM when the random errors follow a normal distribution. Columns correspond to different sample sizes $n = 50$, 100, 200, and rows correspond to different time points $J = 5$ and 10.

Table S7: Medians and IQRs of the Length of HPD Intervals in Figure 6

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	0.910 (0.154)	0.513 (0.075)	0.359 (0.052)
	LEM	0.184 (0.068)	0.130 (0.049)	0.091 (0.034)
	RoLEM	0.184 (0.068)	0.130 (0.049)	0.091 (0.034)
$J = 10$	RLMM	0.503 (0.073)	0.352 (0.050)	0.247 (0.034)
	LEM	0.127 (0.047)	0.090 (0.034)	0.063 (0.024)
	RoLEM	0.127 (0.047)	0.090 (0.034)	0.063 (0.024)

Figure S4 shows the medians and interquartile ranges of the average empirical coverage probability of the 95% HPD intervals for β across all components, where the random errors follow a mixture-normal distribution. Table S12 reports the numerical values.

Table S13, corresponding to Figure 13 in Section 5.4, reports the medians and interquartile ranges of $D(\hat{\beta}, \beta)$ across 500 replicates, where random errors follow a t-distribution.

C Additional Figures and Tables for the OAI Analysis

This section includes additional figures and tables for the OAI analysis in Section 5.5.

Figure S5 displays the trace plots and autocorrelation plots for α_1, Ω_{11}, and $\Omega_{0,11}$, while Figure S6 presents the plots for ρ and ν.

Table S14 lists the posterior means of β for LEM, RLMM, and RoLEM, while Table S15 provides the corresponding 95% HPD intervals.

Figure S7 compares the length of the 95% HPD intervals for β using RLMM and RoLEM.

Figure S8 displays the posterior means of Σ_ε and their associated 95% HPD intervals obtained from LEM, RLMM, and RoLEM. Table S16 reports the posterior means of Σ_ε and Table S17

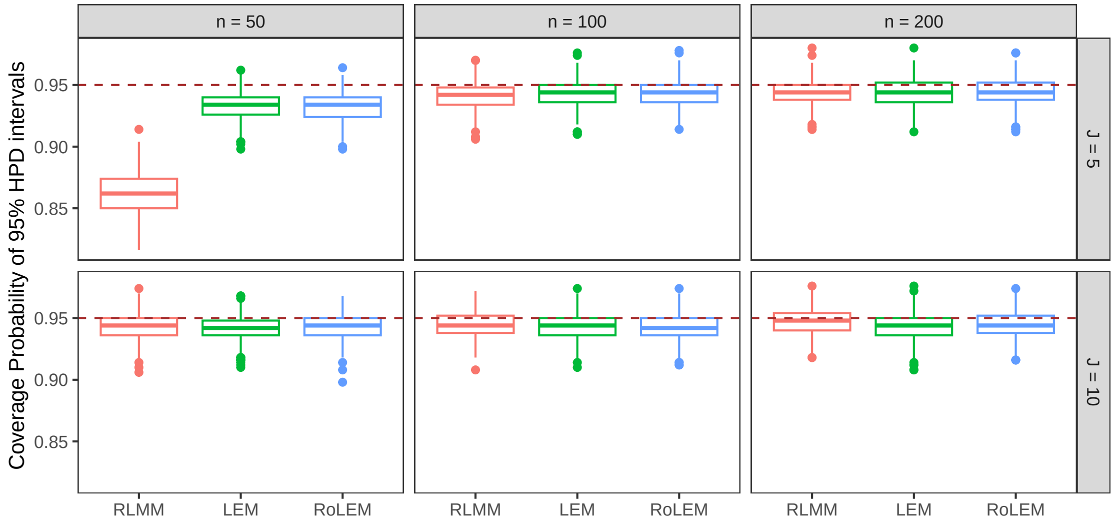

Figure S2: Empirical coverage probabilities of the 95% HPD intervals for β using RLMM, LEM, and RoLEM when the random errors follow a normal distribution. Columns correspond to different sample sizes $n = 50$, 100, 200, and rows correspond to different time points $J = 5$ and 10.

Table S8: Medians and IQRs of the Coverage Probability of HPD Intervals in Figure S2

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	0.862 (0.024)	0.942 (0.014)	0.944 (0.012)
	LEM	0.934 (0.014)	0.944 (0.014)	0.944 (0.016)
	RoLEM	0.934 (0.016)	0.944 (0.014)	0.944 (0.014)
$J = 10$	RLMM	0.944 (0.014)	0.944 (0.014)	0.948 (0.014)
	LEM	0.942 (0.012)	0.944 (0.014)	0.944 (0.014)
	RoLEM	0.944 (0.014)	0.942 (0.014)	0.944 (0.014)

reports the associated 95% HPD intervals.

References

Rao, C. R. (1973). *Linear Statistical Inference and its Applications* (2nd Edition ed.). Wiley New York.

Table S9: Medians and IQRs of $D(\hat{\beta}, \beta)$ in Figure 7

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	7.262 (0.737)	3.258 (0.192)	2.244 (0.120)
	LEM	1.265 (0.231)	0.867 (0.170)	0.613 (0.113)
	RoLEM	1.229 (0.236)	0.848 (0.165)	0.597 (0.112)
$J = 10$	RLMM	3.201 (0.192)	2.216 (0.127)	1.552 (0.084)
	LEM	0.864 (0.166)	0.595 (0.109)	0.416 (0.084)
	RoLEM	0.852 (0.160)	0.586 (0.109)	0.412 (0.084)

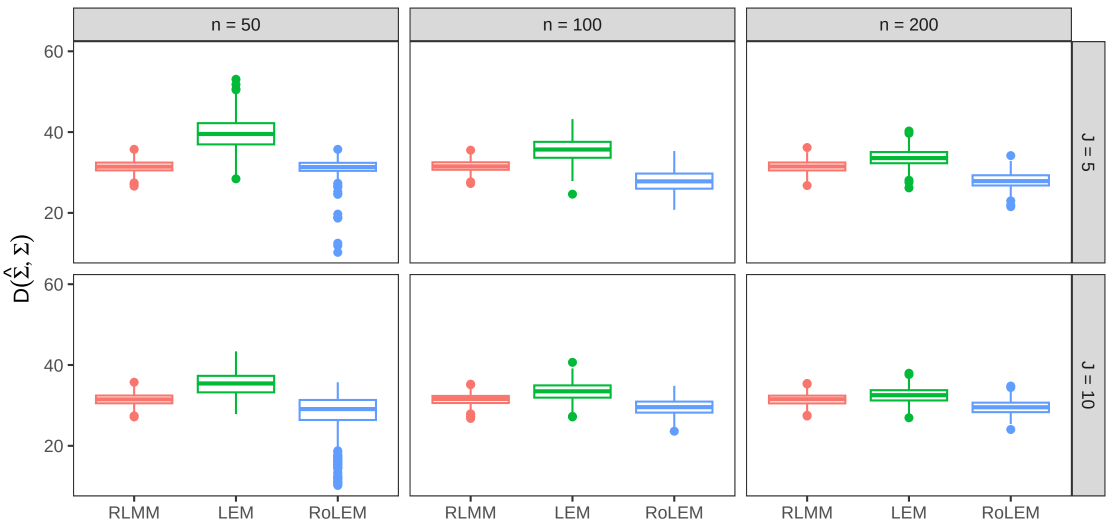

Figure S3: Performance of estimating Σ_ε using RLMM, LEM, or RoLEM when the random errors follow a mixture-normal distribution. Columns correspond to different sample sizes $n = 50$, 100, 200, and rows correspond to different time points $J = 5$ and 10.

Table S10: Medians and IQRs of $D(\hat{\Sigma}, \Sigma)$ in Figure S3

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	31.442 (1.931)	31.530 (1.848)	31.487 (1.959)
	LEM	39.530 (5.255)	35.686 (3.946)	33.566 (2.775)
	RoLEM	31.337 (1.986)	27.805 (3.747)	27.866 (2.530)
$J = 10$	RLMM	31.490 (1.917)	31.582 (1.757)	31.532 (1.917)
	LEM	35.429 (4.088)	33.488 (3.030)	32.528 (2.535)
	RoLEM	28.936 (5.810)	29.565 (2.719)	29.534 (2.348)

Table S11: Medians and IQRs of the Length of HPD Intervals in Figure 8

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	0.888 (0.159)	0.498 (0.079)	0.348 (0.052)
	LEM	0.183 (0.069)	0.128 (0.049)	0.091 (0.034)
	RoLEM	0.190 (0.069)	0.127 (0.048)	0.089 (0.033)
$J = 10$	RLMM	0.494 (0.079)	0.345 (0.052)	0.244 (0.034)
	LEM	0.127 (0.048)	0.089 (0.034)	0.063 (0.024)
	RoLEM	0.127 (0.048)	0.088 (0.033)	0.063 (0.024)

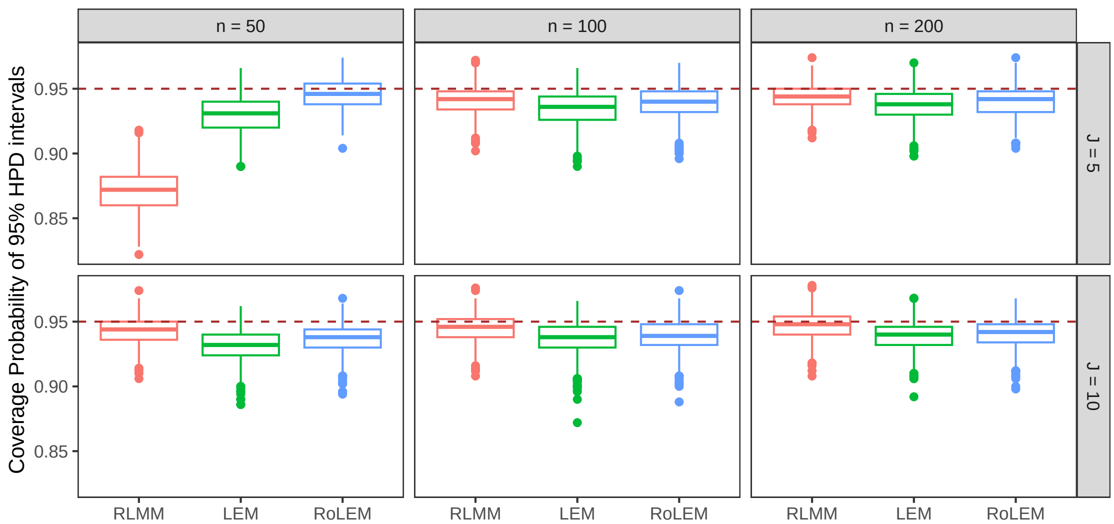

Figure S4: Empirical coverage probabilities of the 95% HPD intervals for β using RLMM, LEM, and RoLEM when the random errors follow a mixture-normal distribution. Columns correspond to different sample sizes $n = 50$, 100, 200, and rows correspond to different time points $J = 5$ and 10.

Table S12: Medians and IQRs of the Coverage Probability of HPD Intervals in Figure S4

	Method	$n = 50$	$n = 100$	$n = 200$
$J = 5$	RLMM	0.872 (0.022)	0.942 (0.014)	0.944 (0.012)
	LEM	0.931 (0.020)	0.936 (0.018)	0.938 (0.016)
	RoLEM	0.946 (0.016)	0.940 (0.016)	0.942 (0.016)
$J = 10$	RLMM	0.944 (0.014)	0.946 (0.014)	0.948 (0.014)
	LEM	0.932 (0.016)	0.938 (0.016)	0.940 (0.014)
	RoLEM	0.938 (0.014)	0.939 (0.016)	0.942 (0.014)

Table S13: Medians and IQRs of $D(\hat{\beta}, \beta)$ in Figure 13

	Hyperparameter	$n = 50$	$n = 100$	$n = 200$
$J = 5$	M_1	0.969 (0.181)	0.657 (0.123)	0.460 (0.070)
	M_2	0.817 (0.163)	0.549 (0.110)	0.391 (0.071)
	M_3	0.624 (0.141)	0.433 (0.101)	0.310 (0.069)
	M_4	0.397 (0.115)	0.273 (0.062)	0.208 (0.036)
$J = 10$	M_1	0.650 (0.122)	0.452 (0.082)	0.318 (0.056)
	M_2	0.546 (0.115)	0.383 (0.072)	0.276 (0.054)
	M_3	0.426 (0.098)	0.304 (0.061)	0.229 (0.049)
	M_4	0.268 (0.061)	0.208 (0.036)	0.174 (0.027)

Table S14: Posterior Means of β for LEM, RLMM, and RoLEM.

	Age	BMI	Diabetes	KL	Medication	Sex
			LEM			
CESD	−0.00	0.06	0.01	0.04	0.06	0.03
KOOS	0.13	−0.14	−0.05	−0.10	−0.18	−0.06
MCS	0.02	−0.02	−0.01	−0.02	−0.03	−0.01
NRS	−0.13	0.15	0.05	0.11	0.20	0.07
PASE	−0.33	0.02	0.05	−0.00	0.09	−0.05
PCS	0.00	−0.10	−0.02	−0.07	−0.11	−0.06
			RLMM			
CESD	−0.02	0.03	−0.02	0.04	−0.00	0.04
KOOS	0.14	−0.11	−0.10	−0.09	−0.21	−0.05
MCS	0.11	0.01	0.03	−0.02	0.02	−0.08
NRS	−0.10	0.10	0.06	0.07	0.16	0.04
PASE	−0.32	0.04	0.06	0.01	0.11	−0.06
PCS	−0.07	−0.20	−0.05	−0.02	−0.11	−0.02
			RoLEM			
CESD	−0.04	−0.00	0.00	0.00	0.00	−0.00
KOOS	0.09	−0.12	−0.07	−0.05	−0.16	−0.02
MCS	0.05	0.05	0.02	0.02	0.04	0.01
NRS	−0.08	0.14	0.08	0.06	0.18	0.02
PASE	−0.34	0.03	0.06	0.03	0.12	−0.03
PCS	−0.02	−0.11	−0.05	−0.04	−0.12	−0.03

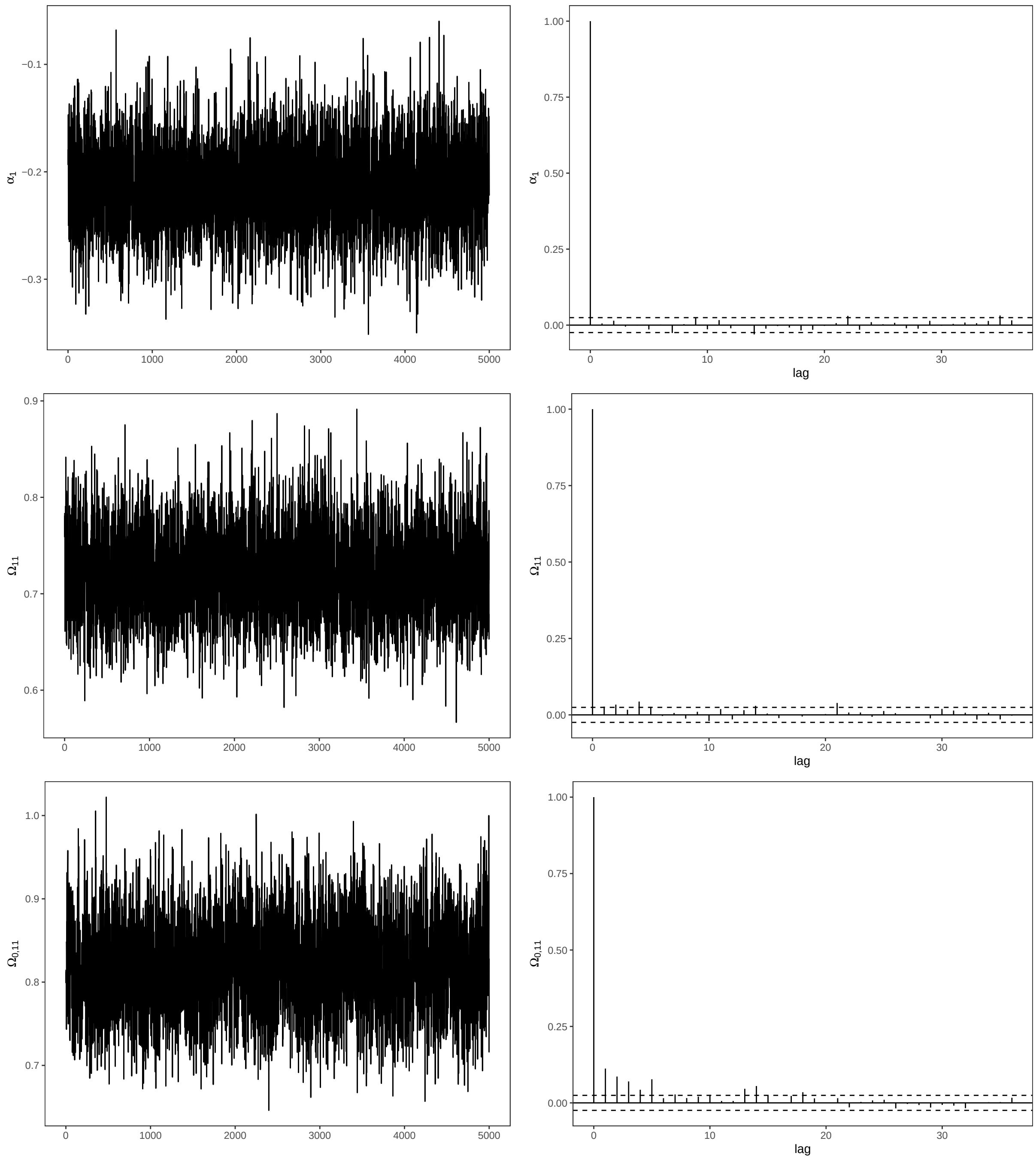

Figure S5: The trace plots and autocorrelation plots for the posterior draws of α_1, Ω_{11}, and $\Omega_{0,11}$.

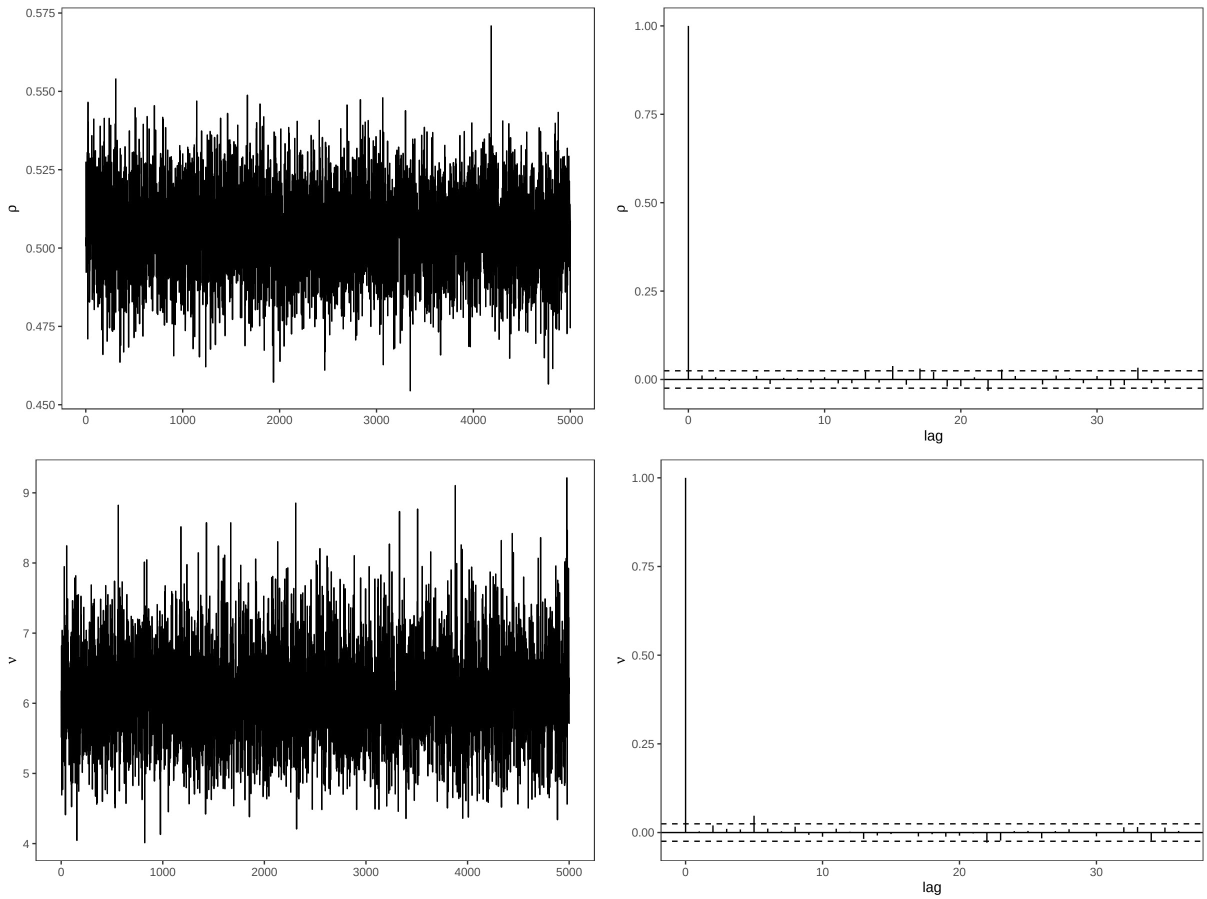

Figure S6: The trace plots and autocorrelation plots for the posterior draws of ρ and ν.

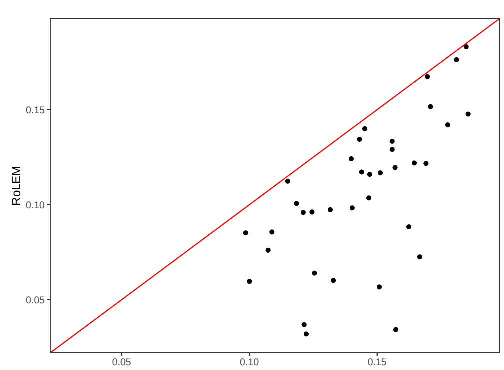

Figure S7: The comparison of the length of HPD intervals of β using RLMM and RoLEM.

Table S15: The 95% HPD Intervals of β for LEM, RLMM, RoLEM.

	Age	BMI	Diabetes	KL	Medication	Sex
			LEM			
CESD	$(-0.06, 0.05)$	$(0.02, 0.10)$	$(-0.01, 0.04)$	$(0.01, 0.07)$	$(0.03, 0.10)$	$(-0.00, 0.07)$
KOOS	$(0.06, 0.20)$	$(-0.20, -0.07)$	$(-0.10, 0.00)$	$(-0.15, -0.05)$	$(-0.23, -0.14)$	$(-0.12, 0.01)$
MCS	$(-0.01, 0.05)$	$(-0.05, 0.01)$	$(-0.02, 0.01)$	$(-0.04, 0.01)$	$(-0.06, 0.01)$	$(-0.03, 0.01)$
NRS	$(-0.21, -0.05)$	$(0.08, 0.22)$	$(-0.01, 0.11)$	$(0.05, 0.16)$	$(0.15, 0.24)$	$(-0.01, 0.14)$
PASE	$(-0.42, -0.25)$	$(-0.07, 0.10)$	$(-0.02, 0.12)$	$(-0.07, 0.06)$	$(0.04, 0.15)$	$(-0.14, 0.04)$
PCS	$(-0.07, 0.07)$	$(-0.15, -0.04)$	$(-0.06, 0.02)$	$(-0.11, -0.03)$	$(-0.15, -0.07)$	$(-0.11, -0.00)$
			RLMM			
CESD	$(-0.09, 0.05)$	$(-0.04, 0.11)$	$(-0.08, 0.04)$	$(-0.03, 0.10)$	$(-0.05, 0.05)$	$(-0.04, 0.12)$
KOOS	$(0.07, 0.21)$	$(-0.18, -0.03)$	$(-0.16, -0.04)$	$(-0.15, -0.03)$	$(-0.26, -0.16)$	$(-0.13, 0.02)$
MCS	$(0.03, 0.19)$	$(-0.07, 0.09)$	$(-0.03, 0.09)$	$(-0.09, 0.05)$	$(-0.03, 0.08)$	$(-0.16, 0.01)$
NRS	$(-0.18, -0.01)$	$(0.01, 0.19)$	$(-0.02, 0.13)$	$(-0.01, 0.14)$	$(0.10, 0.22)$	$(-0.06, 0.13)$
PASE	$(-0.41, -0.24)$	$(-0.05, 0.13)$	$(-0.01, 0.13)$	$(-0.06, 0.08)$	$(0.06, 0.17)$	$(-0.15, 0.03)$
PCS	$(-0.15, 0.01)$	$(-0.29, -0.12)$	$(-0.12, 0.02)$	$(-0.09, 0.04)$	$(-0.17, -0.06)$	$(-0.11, 0.06)$
			RoLEM			
CESD	$(-0.09, 0.01)$	$(-0.03, 0.02)$	$(-0.02, 0.02)$	$(-0.02, 0.02)$	$(-0.02, 0.03)$	$(-0.02, 0.01)$
KOOS	$(0.03, 0.15)$	$(-0.17, -0.06)$	$(-0.12, -0.03)$	$(-0.10, -0.01)$	$(-0.20, -0.11)$	$(-0.08, 0.04)$
MCS	$(-0.01, 0.12)$	$(0.01, 0.09)$	$(-0.01, 0.05)$	$(-0.01, 0.05)$	$(0.00, 0.08)$	$(-0.02, 0.05)$
NRS	$(-0.16, -0.01)$	$(0.07, 0.21)$	$(0.03, 0.14)$	$(0.01, 0.12)$	$(0.13, 0.23)$	$(-0.05, 0.10)$
PASE	$(-0.42, -0.26)$	$(-0.05, 0.12)$	$(-0.01, 0.12)$	$(-0.04, 0.10)$	$(0.06, 0.17)$	$(-0.12, 0.06)$
PCS	$(-0.08, 0.05)$	$(-0.17, -0.05)$	$(-0.10, -0.01)$	$(-0.09, 0.01)$	$(-0.16, -0.07)$	$(-0.09, 0.04)$

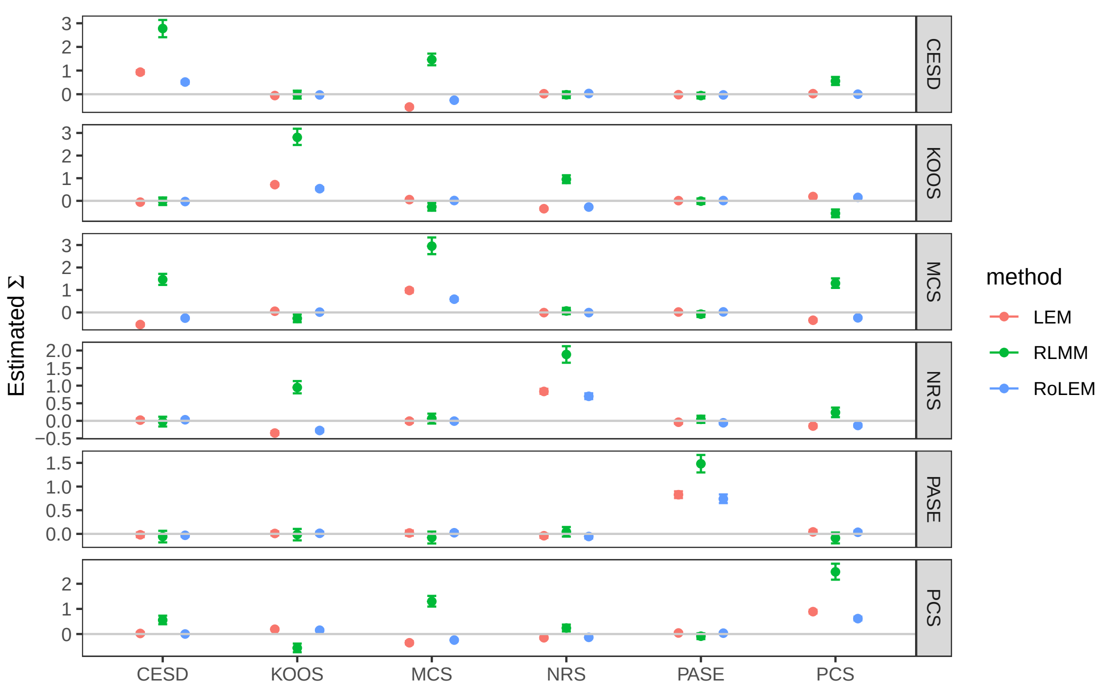

Figure S8: The posterior means of Σ_ε and their associated 95% HPD intervals obtained from LEM, RLMM, and RoLEM.

Table S16: Posterior Means of Σ_ε for LEM, RLMM, and RoLEM

	CESD	KOOS	MCS	NRS	PASE	PCS
			LEM			
CESD	0.93	−0.06	−0.54	0.02	−0.02	0.02
KOOS	−0.06	0.72	0.05	−0.35	0.01	0.19
MCS	−0.54	0.05	0.98	−0.01	0.02	−0.35
NRS	0.02	−0.35	−0.01	0.84	−0.04	−0.15
PASE	−0.02	0.01	0.02	−0.04	0.83	0.04
PCS	0.02	0.19	−0.35	−0.15	0.04	0.89
			RLMM			
CESD	2.78	−0.02	1.46	−0.02	−0.06	0.55
KOOS	−0.02	2.81	−0.26	0.95	−0.01	−0.55
MCS	1.46	−0.26	2.95	0.06	−0.08	1.30
NRS	−0.02	0.95	0.06	1.89	0.04	0.24
PASE	−0.06	−0.01	−0.08	0.04	1.48	−0.09
PCS	0.55	−0.55	1.30	0.24	−0.09	2.48
			RoLEM			
CESD	0.52	−0.03	−0.25	0.03	−0.03	0.00
KOOS	−0.03	0.54	0.01	−0.27	0.01	0.15
MCS	−0.25	0.01	0.59	−0.01	0.02	−0.24
NRS	0.03	−0.27	−0.01	0.70	−0.06	−0.13
PASE	−0.03	0.01	0.02	−0.06	0.74	0.03
PCS	0.00	0.15	−0.24	−0.13	0.03	0.61

Table S17: 95% HPD Intervals of Σ_ε for LEM, RLMM, and RoLEM

	CESD	KOOS	MCS	NRS	PASE	PCS
			LEM			
CESD	$(0.85, 1.01)$	$(-0.10, -0.01)$	$(-0.60, -0.48)$	$(-0.03, 0.07)$	$(-0.07, 0.03)$	$(-0.03, 0.08)$
KOOS	$(-0.10, -0.01)$	$(0.66, 0.77)$	$(0.01, 0.09)$	$(-0.40, -0.30)$	$(-0.03, 0.05)$	$(0.15, 0.23)$
MCS	$(-0.60, -0.48)$	$(0.01, 0.09)$	$(0.89, 1.06)$	$(-0.06, 0.04)$	$(-0.04, 0.07)$	$(-0.41, -0.28)$
NRS	$(-0.03, 0.07)$	$(-0.40, -0.30)$	$(-0.06, 0.04)$	$(0.77, 0.91)$	$(-0.09, 0.01)$	$(-0.19, -0.10)$
PASE	$(-0.07, 0.03)$	$(-0.03, 0.05)$	$(-0.04, 0.07)$	$(-0.09, 0.01)$	$(0.76, 0.90)$	$(0.00, 0.08)$
PCS	$(-0.03, 0.08)$	$(0.15, 0.23)$	$(-0.41, -0.28)$	$(-0.19, -0.10)$	$(0.00, 0.08)$	$(0.82, 0.96)$
			RLMM			
CESD	$(2.41, 3.14)$	$(-0.18, 0.15)$	$(1.22, 1.72)$	$(-0.16, 0.12)$	$(-0.18, 0.06)$	$(0.39, 0.73)$
KOOS	$(-0.18, 0.15)$	$(2.47, 3.19)$	$(-0.44, -0.09)$	$(0.78, 1.13)$	$(-0.14, 0.10)$	$(-0.72, -0.38)$
MCS	$(1.22, 1.72)$	$(-0.44, -0.09)$	$(2.59, 3.34)$	$(-0.08, 0.20)$	$(-0.20, 0.05)$	$(1.09, 1.52)$
NRS	$(-0.16, 0.12)$	$(0.78, 1.13)$	$(-0.08, 0.20)$	$(1.65, 2.12)$	$(-0.06, 0.14)$	$(0.10, 0.38)$
PASE	$(-0.18, 0.06)$	$(-0.14, 0.10)$	$(-0.20, 0.05)$	$(-0.06, 0.14)$	$(1.30, 1.67)$	$(-0.20, 0.03)$
PCS	$(0.39, 0.73)$	$(-0.72, -0.38)$	$(1.09, 1.52)$	$(0.10, 0.38)$	$(-0.20, 0.03)$	$(2.16, 2.80)$
			RoLEM			
CESD	$(0.44, 0.58)$	$(-0.06, 0.00)$	$(-0.30, -0.21)$	$(-0.00, 0.06)$	$(-0.06, -0.00)$	$(-0.03, 0.03)$
KOOS	$(-0.06, 0.00)$	$(0.48, 0.60)$	$(-0.02, 0.05)$	$(-0.32, -0.23)$	$(-0.02, 0.05)$	$(0.12, 0.19)$
MCS	$(-0.30, -0.21)$	$(-0.02, 0.05)$	$(0.52, 0.66)$	$(-0.05, 0.03)$	$(-0.01, 0.06)$	$(-0.28, -0.20)$
NRS	$(-0.00, 0.06)$	$(-0.32, -0.23)$	$(-0.05, 0.03)$	$(0.62, 0.78)$	$(-0.10, -0.02)$	$(-0.17, -0.09)$
PASE	$(-0.06, -0.00)$	$(-0.02, 0.05)$	$(-0.01, 0.06)$	$(-0.10, -0.02)$	$(0.65, 0.83)$	$(-0.00, 0.07)$
PCS	$(-0.03, 0.03)$	$(0.12, 0.19)$	$(-0.28, -0.20)$	$(-0.17, -0.09)$	$(-0.00, 0.07)$	$(0.54, 0.69)$