Problem Set 1

Fiscal and Monetary Policy, Fall 2023

November 29, 2023

This problem set consists of two problems. Please submit it before 17.12.2023 23:59:59.

Problem 1: Lucas and Stokey

Consider an economy with a representative agent who maximizes the expected discounted sum of future utility. The agent's problem is

$$
\begin{aligned}
&\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c\left(s^t\right), \ell\left(s^t\right)\right) \\
\text{s.t.} \quad \forall s^t \quad & c\left(s^t\right) + \sum_{s^{t+1} \geq s^t} q\left(s^{t+1} \mid s^t\right) a\left(s^{t+1}\right) \\
&= \left(1 - \tau\left(s^t\right)\right) w\left(s^t\right) \ell\left(s^t\right) + a\left(s^t\right) \\
& a\left(s^0\right) = a_0
\end{aligned}
$$

We specialize the period utility function to be of the form[1]

$$u(c, \ell) = c - \frac{1}{2}\ell^2.$$

[1]

Note that marginal utility of consumption is constant and equal to one. There will be no incentive to smooth consumption, but there will be incentive to smooth labor supply. This period utility function will make it easier to solve the problem.

The representative firm chooses labor to maximize profits

$$\max_{\ell(s^t)} A\ell\left(s^t\right) - w\left(s^t\right)\ell\left(s^t\right),$$

so in a competitive equilibrium the wage rate is equal to the marginal product of labor:

$$w\left(s^t\right) = A.$$

In each period there are two possible states of the world. They differ in the level of government purchases $g\left(s^t\right)$. Both states of the world are equally likely every period. In the first state of the world, government purchases are equal to 0. In the second state of the world, government purchases are equal to g_H. We will call the first state of the world "peace" and the second state of the world "war".

The initial stock of government debt is equal to $b_0 = 0$. This means that $a_0 = 0$. The initial state of the world is "peace". This means that $g\left(s^0\right) = 0$.

Your task is to find the Ramsey equilibrium.
Preliminaries.

1. Derive the implementability constraint. This requires you to find first order conditions of the agent's problem and the firm's problem and use them in the budget constraint of the household. Use the fact there is no initial government debt.[2]

2. Write down the problem of the Ramsey planner.[3]

3. Find the first order conditions of the Ramsey planner's problem.[4]

Finding the Ramsey allocation.

4. Now you should be able to see that for each state of the world there are three equations (first order condition for consumption, first order condition for labor and the feasibility constraint) in three unknowns (consumption, labor and the multiplier on the feasibility constraint). The problem is that there is a multiplier on the implementability constraint that we do not know. You have to use these three equations to express $c\left(s^t\right)$ and $\ell\left(s^t\right)$, and $\mu\left(s^t\right)$ as functions of the multiplier on the implementability constraint η and government purchases $g\left(s^t\right)$.

5. How does labor and consumption move across states of the world? Why? Are they constant? Do they depend on the level of government purchases?

6. Use the first order conditions of the representative agent's problem to find the tax rate $\tau\left(s^t\right)$ in "war" and "peace".

7. Now the crucial step. We want to recover $q\left(s^{t+1} \mid s^t\right)$. These are the prices of one-period Arrow securities. In each state of the world the representative agent and the government trade two Arrow securities: one that pays one unit of goods if the next period state is "peace" and one that pays one unit of goods if the next period state is "war". From the first order conditions of the representative agent we know that

$$q\left(s^{t+1} \mid s^t\right) = \beta\pi\left(s^{t+1} \mid s_t\right)\frac{u_c\left(c\left(s^{t+1}\right)\right)}{u_c\left(c\left(s^t\right)\right)},$$

where $\pi\left(s^{t+1} \mid s_t\right)$ is the conditional probability of state tomorrow beign s^{t+1} given state today being s^t. Provide expressions for $q\,(\text{war}|\text{peace})$, $q\,(\text{peace}|\text{peace})$. $q\,(\text{peace}|\text{war})$ and $q\,(\text{war}|\text{war})$.

8. Show that the prices of Arrow securities do not depend on the current state of the world. Show that they do not depend on the future state of the world either. We can express them as q.

[2] Just use the formula from the slides. Calculate the derivative of the period utility function with respect to consumption and labor. Plug in. At this stage it is useful to keep the notation pretty general – do not use the fact that there are only two possible states.

[3] Again, just use the formula from the slides, but with the correct utility function.

[4] Take the derivatives of the Lagrangian of the planner's problem with respect to $c\left(s^t\right)$, $\ell\left(s^t\right)$. For state of the world you will have two first order conditions and one feasibility constraint with its own multiplier $\mu\left(s^t\right)$. In addition, there will be the implementability constraint with a single multiplier η.

9. We are ready to figure out portfolios held in this economy. In each state the budget constraint of the representative agent is

$$c\left(s^t\right)+\sum_{s^{t+1}\geq s^t} q\left(s^{t+1}\mid s^t\right) a\left(s^{t+1}\right)=\left(1-\tau\left(s^t\right)\right) w\left(s^t\right) \ell\left(s^t\right)+a\left(s^t\right)$$

which in our example is

$$\begin{aligned}c^{\text{war}}+qa^{\text{peace}}+qa^{\text{war}}&=\left(1-\tau^{\text{war}}\right) A\ell^{\text{war}}+a^{\text{war}}\\ c^{\text{peace}}+qa^{\text{peace}}+qa^{\text{war}}&=\left(1-\tau^{\text{peace}}\right) A\ell^{\text{peace}}+a^{\text{peace}}\end{aligned}$$

Above we used the fact that the amount of two types of assets bought in the "war" state will be the same as in the "peace" state. This follows from what you had to show in the previous part. The above system of equations has two unknowns: a^{peace} and a^{war}. Solve for them.

10. We see that the agent buys the same portolio of assets in every state of the world

$$q\left(a^{\text{peace}}+a^{\text{war}}\right)$$

The return on this portfolio depends on the realization of the state tomorrow. If there is a "war" tomorrow, the return is

$$\frac{a^{\text{war}}}{q\left(a^{\text{peace}}+a^{\text{war}}\right)}.$$

Similarly, if there is a "peace" tomorrow, the return is

$$\frac{a^{\text{peace}}}{q\left(a^{\text{peace}}+a^{\text{war}}\right)}.$$

11. Is the return on the portfolio the same in all states of the world? Why? If it is not, is it larger in war or in peace? What is the expected return on the portfolio?

Optional: finding η

12. Our answers depended on η. Use the formulas you derived in previous steps in the implementability constraint. This will give you an equation that relates η and government purchases in all states of the world at all times. Use the assumption about the process for government purchases to express η as a function of g_H and $\pi\left(s^t\right)$. Remember that the initial state of the world is "peace".[5]

13. We can now go back to the formulas for consumption and labor and express them as functions of g_H and $\pi\left(s^t\right)$. After this step we know the exact values of consumption and labor in all states of the world at all times. Denote consumption in the state with high government purchases as c_{war} and consumption in the state with

[5] You will have to solve a quadratic equation. Remember that η is a multiplier on a constraint. It puts some restrictions on the sign. Which root should you pick? It might be useful consider the case with government purchases equal to 0 in all states of the world first. Consumption should be equal to $A\ell$ in this case. What would that imply for η?

low government purchases as c_{peace}. Denote labor in the state with high government purchases as ℓ_{war} and labor in the state with low government purchases as ℓ_{peace}.

14. Assume now that $\beta = 0.95, g_H = 0.1, A = 1$. Calculate consumption, labor, tax rates, securitiy prices and returns on the portfolio in each state of the world.

Problem 2: IGBC and Bohn (1998)

There is budget constraint of the government:

$$\frac{B_t}{Y_t} = \frac{G_t - T_t}{Y_t} + (1+r)\frac{B_{t-1}}{Y_{t-1}}\frac{Y_{t-1}}{Y_t}.$$

Let $X_{t-1,t} := \frac{Y_t}{Y_{t-1}}$. Assume that $X_{t-1,t}$ is always 1. Assume that $r > 0$. For simplicity, define $b_t := \frac{B_t}{Y_t}$, $g_t := \frac{G_t}{Y_t}$, $\tau_t := \frac{T_t}{Y_t}$. These are ratios of various variables to GDP. We can write (given the assumption that $X_{t-1,t} = 1$):

$$b_t = g_t - \tau_t + (1+r)\, b_{t-1}.$$

Suppose the government follows a policy rule of the form

$$\tau_t - g_t = \rho b_{t-1} + \bar{s}.$$

The parameter ρ is the strength of the response. Assume that the stochastic discount factor is equal to $\frac{1}{1+r}$. We say that the IGBC holds if the following condition is satisfied:

$$b_t = \sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^j \mathbb{E}_t \left(g_{t+j} - \tau_{t+j}\right).$$

This is derived from solving the budget constraint forward and using $\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T \mathbb{E}_t\, b_T = 0$.

Let $b_0 > 0$ be the initial debt to GDP. Assume perfect foresight (so that you can drop $\mathbb{E}_t$).

1. Use the policy rule to express b_t as a function of b_{t-1} and constants.
2. Use the policy rule to express b_t as a function of initial debt b_0 and constants.
3. Express primary surplus $\tau_t - g_t$ as a function of b_0 and constants.
4. What is $\lim_{t\to\infty} b_t$ if $\rho > r$? What if $\rho < r$?
5. What is $\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t b_t$ if $\rho > r$? What if $\rho < r$? Does the answer depend on b_0?
6. Use your answers to the previous question to show conditions under which the IGBC holds.