# AGGREGATE SHOCKS, TAXES, AND DEBT MANAGEMENT I

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Piotr Żoch

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# PLAN

- How should the government change taxes and debt in response to changes in purchases?
- For example: should it finance war spending in the same way as public education?
- This is a normative question, but to the extent real world governments behave according to prescriptions of models we will see in class, it will also be a positive theory of government debt and taxes.

# PLAN

- We will study optimal taxation and debt policy in two environments:
  - incomplete markets: the government can buy or sell only a limited set of securities, often only a single risk-free security.
  - complete markets: the government can buy or sell claims contingent on all possible states of the world.
- We will first study some reduced form models and then proceed to optimal taxation in a competitive equilibrium (Lucas and Stokey, 1983).

# REDUCED FORM MODELS

# INCOMPLETE MARKETS

- We study the following problem based on Barro (1979):
  - the government uses taxes $\tau_t$ to finance *stochastic* purchases $g_t$;
  - $D(\tau_t)$ is the deadweight loss of taxes;
  - the government wants to finance $g_t$ in a way that minimizes the expected value of present discounted deadweight loss of taxes:

$$\min_{\{\tau_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left[ \left( \frac{1}{1+r} \right)^t D(\tau_t) \right]$$

where $1/(1+r)$ is the (gross) rate at which the government discounts the future.

# INCOMPLETE MARKETS

- Assume that $g_t$ is governed by an $N$ state Markov chain.
- The state of the world is given by $s_t$ that follows a Markov chain with a transition probability matrix $P$:

$$P_{ij} = P\left\{s_{t+1} = \bar{s}_j \mid s_t = \bar{s}_i\right\}.$$

- Government spending $\{g_t\}$ obeys

$$g_t = \begin{cases} g_1 & \text{if } s_t = \bar{s}_1 \\ \vdots & \\ g_N & \text{if } s_t = \bar{s}_N. \end{cases}$$

## INCOMPLETE MARKETS

- The government can issue only one period, non-state contingent debt (incomplete markets) at price $1/(1+r)$, equal to the discount rate.
- Government problem:

$$\min_{\{\tau_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left[ \left( \frac{1}{1+r} \right)^t D(\tau_t) \right]$$

$$\text{s.t.} \quad g_t + b_t = \frac{1}{1+r} b_{t+1} + \tau_t$$

and, in addition, we assume

$$\mathbb{E}_0 \left[ \left( \frac{1}{1+r} \right)^t b_t^2 \right] \leq \infty$$

## DIGRESSION ON BELLMAN EQUATIONS

- How to solve the above problem?
- My recommendation is to write it in a *recursive* form:

$$
\begin{aligned}
V(b, g) &= \min_{\tau} D(\tau) + \frac{1}{1+r} \mathbb{E}\left[V\left(b', g'\right) \mid g\right] \\
\text{s.t.} \quad g + b &= \frac{1}{1+r} b' + \tau
\end{aligned}
$$

  where $x$ denotes current variables ($x_t$) and $x'$ denotes variables in the next period ($x_{t+1}$).
- It looks like a two period problem (instead of infinite horizon) – the "only" issue is that we do not know $V(b, g)$ (and it appears on the right hand side).
- More on that in, for example, my Quantitative Economics course.

# DIGRESSION ON BELLMAN EQUATIONS

- Write

$$V(b, g) = \min_{\tau} D(\tau) + \frac{1}{1+r} \mathbb{E}\left[V\left((1+r)(g+b-\tau), g'\right) \mid g\right]$$

and differentiate with respect to $\tau$ to get the first order condition:

$$D'(\tau) = \mathbb{E}\left[\frac{\partial V\left(b', g'\right)}{\partial \tau} \,\middle|\, g\right]$$

- How to get $V'\left(b', g'\right) := \partial V\left(b', g'\right) / \partial \tau$?

# DIGRESSION ON BELLMAN EQUATIONS

- Write

$$V(b,g) = D\left(g + b - \frac{1}{1+r}b'\right) + \frac{1}{1+r}\mathbb{E}\left[V\left(b',g'\right) \mid g\right]$$

(note: there is no $\min_\tau$) and differentiate with respect to $b$:

$$V'(b,g) = D'(\tau)\,.$$

This is the envelope condition and it implies

$$V'\left(b',g'\right) = D'\left(\tau'\right)\,.$$

- Important: do not write $b' = b'(b,g)$ or anything like that, these terms drop out anyway (why?).

# DIGRESSION ON BELLMAN EQUATIONS

- The first order condition

$$D'\left(\tau\right) = \mathbb{E}\left[\frac{\partial V\left(b', g'\right)}{\partial \tau} \;\middle|\; g\right]$$

together with the envelope condition

$$V'\left(b', g'\right) = D'\left(\tau'\right)$$

give us the Euler equation

$$D'\left(\tau_t\right) = \mathbb{E}_t\left[D'\left(\tau_{t+1}\right)\right]$$

# INCOMPLETE MARKETS

- Solution to the government's problem is characterized by the first order condition

$$D'(\tau) = \mathbb{E}_t\left[D'\left(\tau'\right)\right].$$

  - Tax smoothing: spread out distortionary effects of taxes across time.

- There are two important special cases:
  - Special case 1 - no uncertainty (Barro, 1979):

$$\tau_t = \tau_{t+1}$$

  - Special case 2 - quadratic deadweight loss $\left(D\left(\tau_t\right) = a\tau_t^2 + \gamma\right)$ :

$$\tau_t = \mathbb{E}_t\left(\tau_{t+1}\right)$$

# SPECIAL CASE 1: PERFECT FORESIGHT

- With no uncertainty about future $g_t$ (perfect foresight), taxes are constant.
- $\tau$ is chosen to satisfy the budget constraint:

$$\tau = \frac{r}{1+r}\left[\sum_{t=0}^{\infty}(1+r)^t g_t + b_0\right]$$

- Debt absorbs all fluctuations in government spending:

$$b_{t+1} - b_t = r\left(b_t - b_0\right) + (1+r)\, g_t - r\sum_{s=0}^{\infty}(1+r)^s g_s.$$

## SPECIAL CASE 1: PERFECT FORESIGHT

- If $g_t = g$ is constant, then

$$\tau = g + \frac{r}{1+r} b_0, \quad b_t = b_0;$$

  debt remains constant at its initial level.

- If $g_0 = g + \epsilon$ and $g_t = g$ for all $t \geq 1$ then

$$\tau = g + \frac{r}{1+r} (\epsilon + b_0), \quad b_t = b_0 + \epsilon;$$

  debt increases by $\epsilon$ and stays elevated forever.

- The level of debt $b_t$ is irrelevant. What matters for debt is transitory fluctuations in $g_t$, not the average level of $g_t$.
- Intuition valid also with trend growth of GDP and government spending (see Barro's paper).

## SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- What does the condition

$$\tau_t = \mathbb{E}_t\left[\tau_{t+1}\right]$$

  tell us about the optimal tax policy?

- $\tau_t$ is a random walk:

$$\begin{aligned}\tau_{t+1} - \tau_t &= \tau_{t+1} - \mathbb{E}_t\left[\tau_{t+1}\right] \\ &= \tau_{t+1} - (\tau_{t+1} - \epsilon_{t+1}) \\ &= \epsilon_{t+1}\end{aligned}$$

  where $\epsilon_{t+1}$ is white noise.

# SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- Taxes tomorrow expected to be the same as today, $\tau_t$ moves only because of changes in expected future $g_t$.
- Whenever new information is revealed, taxes immediately jump to a new level.
- To see this, use the budget constraint together with $\mathbb{E}_t[\tau_{t+1}] = \tau_t$ to get

$$\tau_t = \frac{r}{1+r} b_t + \frac{r}{1+r} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \mathbb{E}_t[g_{t+s}].$$

so

$$\tau_t - \mathbb{E}_{t-1}[\tau_t] = \frac{r}{1+r} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s (\mathbb{E}_t[g_{t+s}] - \mathbb{E}_{t-1}[g_{t+s}])$$

## SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- The size of the jump in $\tau_t$ depends on the size of the change in expected future $g_t$:
  - If there is a permanent shift in $g_t$, $\mathbb{E}_t [g_{t+s}] - \mathbb{E}_{t-1} [g_{t+s}] = \epsilon$ for all $s$, taxes change by exactly the same amount $\epsilon$.
  - If the change is temporary, $\mathbb{E}_t [g_{t+s}] - \mathbb{E}_{t-1} [g_{t+s}] = \epsilon$ only for $s = 0$, taxes change by $\epsilon r/ (1 + r)$.
- Remaining needed resources raised through debt issuance:
  - Debt remains constant for a permanent change in $g_t$.
  - Debt changes by $\epsilon/ (1 + r)$ for a temporary change in $g_t$.
- Implication: finance education by taxes, finance war by debt.

## SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- Debt also inherits the random walk property of taxes:

$$\mathbb{E}_t \left[ b_{t+1} \right] = b_t$$

- There is nothing that acts as a force that would push debt to a particular level:
  - No debt target;
  - No "raise taxes if debt is too high" rule;
  - Debt is a random walk, so its variance goes to infinity unless all shocks to $g_t$ are fully permanent.

## COMPLETE MARKETS

- In the complete markets case, the government can issue state-contingent securities.
- This means that the government can issue a security that pays 1 unit of consumption in state $s$ in the next period, and it can do it for all possible states.
- We call such a security an Arrow security.
- Let $s_t$ denote the state in period $t$ and $s^t := (s_0, s_1, \cdots, s_t)$ denote the history of states up to period $t$.
- Let $q\left(s^{t+1} \mid s^t\right)$ denote the period $t$ price of a security that pays 1 unit of goods for a particular history $s^{t+1}$ in period $t+1$ (and zero in other states).

## COMPLETE MARKETS

- To facilitate comparison with the incomplete markets case we will assume

$$q\left(s^{t+1} \mid s^t\right) = \frac{1}{1+r}\pi\left(s^{t+1} \mid s^t\right)$$

  where $\pi\left(s^{t+1} \mid s^t\right)$ is the conditional probability of history $s^{t+1}$ given history $s^t$.

- The above assumption implies:

$$\sum_{s^{t+1} \geq s^t} q\left(s^{t+1} \mid s^t\right) = \frac{1}{1+r},$$

  the price of a portfolio that pays one unit of goods for sure is equal to $(1+r)^{-1}$.

## COMPLETE MARKETS

- The problem of the government is now

$$\min_{\{\tau\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \pi\left(s^t\right) \left(\frac{1}{1+r}\right)^t D\left(\tau\left(s^t\right)\right)$$

$$\text{s.t.} \quad \forall s^t \quad g\left(s^t\right) + b\left(s^t\right) = \tau\left(s^t\right) + \sum_{s^{t+1} \geq s^t} q\left(s^{t+1} \mid s^t\right) b\left(s^{t+1}\right)$$

$$b\left(s_0\right) = b_0$$

where $\pi\left(s^t\right)$ is the probability of history $s^t$, and $b\left(s^t\right)$ is the quantitity of (negative) Arrow securities that pay in history $s^t$.

- We can write a Lagrangian:

$$\sum_{t=0}^{\infty}\sum_{s^t}\pi\left(s^t\right)\left(\frac{1}{1+r}\right)^t D\left(\tau\left(s^t\right)\right)$$

$$+\sum_{t=0}^{\infty}\sum_{s^t}\lambda\left(s^t\right)\left[g\left(s^t\right)+b\left(s^t\right)-\tau\left(s^t\right)-\sum_{s^{t+1}\geq s^t}q\left(s^{t+1}\mid s^t\right)b\left(s^{t+1}\right)\right]$$

# COMPLETE MARKETS

- First order condition with respect to $\tau\left(s^t\right)$:

$$\pi\left(s^t\right)\left(\frac{1}{1+r}\right)^t D'\left(\tau\left(s^t\right)\right) = \lambda\left(s^t\right)$$

- First order condition with respect to $b\left(s^{t+1}\right)$:

$$q\left(s^{t+1} \mid s^t\right)\lambda\left(s^t\right) = \lambda\left(s^{t+1}\right)$$

## COMPLETE MARKETS

- Given the assumption

$$q\left(s^{t+1} \mid s^{t}\right) = \frac{1}{1+r}\pi\left(s^{t+1} \mid s^{t}\right)$$

we have

$$\frac{1}{1+r}\pi\left(s^{t+1} \mid s^{t}\right)\lambda\left(s^{t}\right) = \lambda\left(s^{t+1}\right)$$

so

$$\begin{aligned}\frac{1}{1+r}\pi\left(s^{t+1} \mid s^{t}\right)\pi\left(s^{t}\right)\left(\frac{1}{1+r}\right)^{t} D'\left(\tau\left(s^{t}\right)\right) \\ = \pi\left(s^{t+1}\right)\left(\frac{1}{1+r}\right)^{t+1} D'\left(\tau\left(s^{t+1}\right)\right)\end{aligned}$$

## COMPLETE MARKETS

- It simplifies to

$$D'\left(\tau\left(s^{t}\right)\right)=D'\left(\tau\left(s^{t+1}\right)\right)$$

  i.e.

$$\tau\left(s^{t}\right)=\tau\left(s^{t+1}\right).$$

- This is the key result: perfect smoothing of taxes across states.
- The result holds regardless of the form of $D(\tau)$ or the nature of risk.
- Compare it with incomplete markets case: it was true only with perfect foresight.

# COMPLETE MARKETS

- What about debt?
- Guess that $b\left(s^{t+1}\right)$ depends only on the future state $s_{t+1}$, and not on the history $s^t$.
- Intuitively: the government cares only about the future state, not about the history (why?).
- Use this guess together with $\tau\left(s^t\right) = \tau$ and the initial condition $b\left(s_0\right) = b_0$ to solve for $\tau$ and optimal debt policy.
- This also verifies the guess.

## COMPLETE MARKETS

- Since $b\left(s^{t+1}\right) = b\left(s_{t+1}\right)$, debt neither accumulates, nor decumulates, nor drifts – it switches between different levels.

# COMPARISON

- Incomplete: taxes drift over time as a random walk; the level of taxes at time $t$ depends on the level of debt that the government brings into the period as well as the expected discounted present value of government purchases.
- Complete: taxes are constant, regardless of the state of the world; the level of taxes at time $t$ is independent of the level of debt that the government brings into the period.

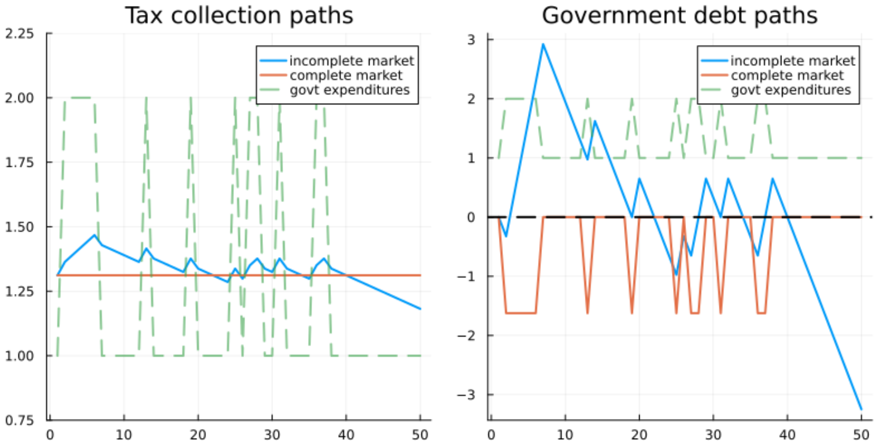
# COMPARISON

- Incomplete: debt drifts upward over time in response to large purchases and drifts downward over time in response to low purchases.
- Complete: debt osciallates between several levelsl; this is akin to the government purchasing insurance that protects against the need to raise taxes too high or issue too much debt in the high government expenditure event.

## NUMERICAL EXAMPLE

- Let $(1+r)^{-1} = 0.96$ and there are two levels of government purchases: $g_1 = 1$ and $g_2 = 2$ with with transition probabilities $P_{12} = 0.2$ and $P_{21} = 0.4$.
- The initial condition is $b_0 = 0$, and we start with $g_1$.

# NUMERICAL EXAMPLE




Tax and debt policy in complete and incomplete markets. Computed using QuantEcon Julia package.