OPTIMAL MATURITY STRUCTURE

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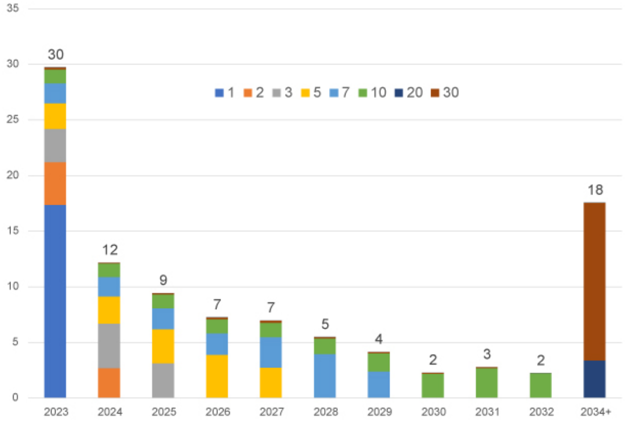
PLAN

- Our theories dealt with one period debt only, yet we see that countries issues debt with various maturities.
- Today we derive some implications for the optimal maturity structure.
- Key idea (Angeletos (2002), Buera and Nicolini (2004)): with many different maturities of risk-free debt it is possible to replicate the optimal complete markets allocation.
- How to do it – this has implications for the optimal maturity structure.

MATURITY STRUCTURE IN THE US

Figure 1



NOTE: We plot the fixed-rate marketable US debt (FRMUSD) maturing in each year. The colors represent the debt's maturity in years. E.g., The 1-year debt in blue contains all notes and bills with maturity of 1 year or less.

SOURCE: US Treasury quantities and rates in December 2022 are from the Monthly Statement of Public Debt (MSPD) published by the US Treasury Department (available on fiscaldata.treasury.gov/).

Source: here

MATURITY DETERMINATION

NOTATION

- Let there be $J > 1$ different maturities of risk-free zero coupon real government debt.
- $b^j\left(s^t\right)$ is bond issued in state s^t that promises to pay one unit of goods in period $t+j$ (j periods from now).
- The household budget constraint is now

$$c\left(s^t\right)+\sum_{j=1}^{J} p^j\left(s^t\right) b^j\left(s^t\right)=\left(1-\tau\left(s^t\right)\right) w\left(s^t\right) \ell\left(s^t\right) + \sum_{j=0}^{J} p^j\left(s^t\right) b^j\left(s^{t-1}\right).$$

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NOTATION

- We could also write the budget constraint as

$$c_t + \sum_{j=1}^{J} p_t^j b_t^j = (1 - \tau_t)\, w_t \ell_t + \sum_{j=0}^{J} p_t^j b_{t-1}^j$$

  because nothing is state-contingent.
- We do not do it because want to be able to compare the result with the complete markets case.
- Recall: our goal is to show that with a rich enough maturity structure we implement the complete markets optimal alocation.

ENVIRONMENT

- The rest of the economy is as in Lecture 2 (Lucas and Stokey (1983)):
  - The representative household maximizes expected utility;
  - Linear production function that uses labor only;
  - Government purchases are Markov.
- The only other difference is the government budget constraint:

$$
\begin{aligned}
g\left(s^t\right) + \sum_{j=0}^{J} p^j\left(s^t\right) b^j\left(s^{t-1}\right) = \tau\left(s^t\right) w\left(s^t\right) \ell\left(s^t\right) \\
+ \sum_{j=1}^{J} p^j\left(s^t\right) b^j\left(s^t\right).
\end{aligned}
$$

FIRST ORDER CONDITIONS

- We have $p^0\left(s^t\right) = 1$.
- The other prices are determined by first order conditions of the household.

$$p^j\left(s^t\right) = \beta^j \frac{\mathbb{E}\left[u_c\left(s^{t+j}\right) \mid s^t\right]}{u_c\left(s^t\right)}.$$

- Use these in the budget constraint.

IMPLEMENTABILITY CONSTRAINT

- The implementability constraint is now

$$\overbrace{\sum_{i\geq 0, s^{t+i}\geq s^t} \beta^i \pi\left(s^{t+i} \mid s^t\right) \left[\frac{u_c\left(s^{t+i}\right)}{u_c\left(s^t\right)} c\left(s^{t+i}\right) + \frac{u_\ell\left(s^{t+i}\right)}{u_c\left(s^t\right)} \ell\left(s^{t+i}\right)\right]}^{z\left(s^t\right)}$$

$$= \underbrace{\sum_{j=0}^{J-1} \frac{\mathbb{E}\left[u_c\left(s^{t+j}\right) \mid s^t\right]}{u_c\left(s^t\right)} b^j\left(s^{t-1}\right)}_{a(s^t)\cdot b(s^{t-1})} \qquad \text{for all } s^t.$$

- We obtained it by substituting the prices into the budget constraint and summing forward.

IMPLEMENTABILITY CONSTRAINT

- Complete markets:

$$\sum_{i,s^i} \beta^i \pi\left(s^i\right) \left[\frac{u_c\left(s^i\right)}{u_c\left(s^0\right)} c\left(s^i\right) + \frac{u_\ell\left(s^i\right)}{u_c\left(s^0\right)} \ell\left(s^i\right)\right] = b_0.$$

- Incomplete markets:

$$\sum_{i\geq 0, s^{t+i}\geq s^t} \beta^i \pi\left(s^{t+i} \mid s^t\right) \left[\frac{u_c\left(s^{t+i}\right)}{u_c\left(s^t\right)} c\left(s^{t+i}\right) + \frac{u_\ell\left(s^{t+i}\right)}{u_c\left(s^t\right)} \ell\left(s^{t+i}\right)\right]$$
$$= \sum_{j=0}^{J-1} \frac{\mathbb{E}\left[u_c\left(s^{t+j}\right) \mid s^t\right]}{u_c\left(s^t\right)} b^j\left(s^{t-1}\right) \quad \text{for all } s^t.$$

- Idea: to find b that satisfies the incomplete markets IC with c, ℓ from the complete markets allocation.

IMPLEMENTABILITY CONSTRAINT

- We have S distinct states and J distinct maturities.
- Let Z be a $S \times 1$ vector of $z\left(s^t\right)$, A be a $S \times J$ matrix of $a\left(s^t\right)$ and B be a $J \times 1$ vector $b\left(s^{t-1}\right)$.
- Notice that A is a matrix of prices of bonds: J types of bonds in S number of states.
- The implementability constraint is

$$Z = A \cdot B.$$

IMPLEMENTING COMPLETE MARKETS ALLOCATION

- Pick Z and A from the complete markets allocation.
- We need to find B such that $Z = A \cdot B$. It has to be true for all s^t and $t \geq 0$.
- For example: it requires that we can invert A – this suggest that, among other things, $J = M$ (otherwise it is not square).
- With $J > M$ more than one solution.
- Angeletos (2002): if the number of maturities is at least as large as the number of states, then we can always get arbitrarily close to complete market allocation.
- Formally: see Angeletos (2002) for details (and a rigorous proof)

PREDICTIONS FOR THE MATURITY STRUCTURE

EXAMPLE

- Suppose there are two i.i.d. states (L and H) with $g_H > g_L = 0$ and two maturities.
- Let $b_0 = 0$.
- From Lucas and Stokey (1983) we know that with complete markets there is $z_t(g_H) = z_H, z_t(g_L) = z_L$ with $z_L > z_H = 0$.

EXAMPLE

- We have to solve

$$\begin{bmatrix} 0 \\ z_L \end{bmatrix} = \begin{bmatrix} 1 & \beta \frac{\mathbb{E}\, u_c}{u_c(g_H)} \\ 1 & \beta \frac{\mathbb{E}\, u_c}{u_c(g_L)} \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$

- This can be inverted as long as $u_c(g_H) \neq u_c(g_L)$ (otherwise the matrix is singular).
- Even if $u_c(g_H) = u_c(g_L)$, we can still get arbitrarily close to the complete markets allocation – just set taxes to be slightly different than with complete markets.

EXAMPLE

- Usually we have

$$u_c(g_L) < u_c(g_H)$$

  either due to difference in consumption or labor supply.
- Therefore

$$\det A = \beta \frac{\mathbb{E}\, u_c}{u_c(g_H) u_c(g_L)} [u_c(g_H) - u_c(g_L)] > 0.$$

  and the optimal debt porfolio is

$$\begin{bmatrix} b^1 \\ b^2 \end{bmatrix} = \frac{z_L}{\det A} \begin{bmatrix} -u_c(g_L)\, \mathbb{E}\, u_c \\ 1 \end{bmatrix}$$

EXAMPLE

- In this example $b^1 < 0$ and $b_2 > 0$.
- The government buys the short maturity debt and sells the long maturity debt.
- Intuition: in states with high government purchases the value of portfolio increases (short term debt becomes cheaper relative to long term debt).
- What determines the optimal size of portfolio?

OPTIMAL PORTFOLIO SIZE

- Notice that if $\det A$ is small, then the optimal portfolio size is large.
- $\det A$ is small if $u_c(g_H) - u_c(g_L)$ is small – if marginal utility of consumption is similar in both states.
- Variance of consumption in the data is small.
- This approach suggests very large portfolio sizes!

BUERA AND NICOLINI (2004)

Table 2
Debt positions and interest rates for the four states examples

Risk aversion	Debt positions $\left(\frac{p^j(g_{\text{low}}, s_{\text{high}})b^j}{y_{g_{\text{low}}, s_{\text{high}}}}\right)$				Short term interest rates (%)			
					$g_{\text{low}}, s_{\text{high}}$	$g_{\text{low}}, s_{\text{low}}$	$g_{\text{high}}, s_{\text{high}}$	$g_{\text{high}}, s_{\text{low}}$
Business cycle examples								
Our calibration	$j = 0.25$	$j = 1$	$j = 2$	$j = 30$				
0.50	−21.83	146.08	322.70	198.46	1.03	3.93	1.15	4.06
2.00	−10.10	65.61	−140.41	84.92	−1.15	7.68	−0.66	8.21
10.00	−9.13	60.22	−129.22	78.13	−0.44	6.16	0.07	6.73
Calibration of Chari et al. (1995)	$j = 1$	$j = 4$	$j = 13$	$j = 30$				
1.00	−24.62	71.96	−171.89	125.12	1.81	2.04	1.99	2.23
2.00	−13.58	45.01	−125.93	95.07	0.93	2.08	1.90	3.12
9.00	−7.49	20.52	−43.39	30.00	0.41	2.58	1.25	3.54

Note: In columns 2–5, we report the maturities implementing the Ramsey allocation that minimized the average value of the absolute value of the positions. The business cycle examples were calibrated to match the US postwar experience. Our calibration was done taking a quarter as the time period, while Chari et al. (1995) use the year. In our (Chari et al. (1995)) calibration the short-term interest rate corresponds to a 3-month (1-year) bond. In both cases, interest rates correspond to yearly returns.

WHAT IS DRIVING THIS RESULT?

- We can rewrite the system of equations as

$$\begin{bmatrix} 0 \\ z_L \end{bmatrix} = \begin{bmatrix} 1 & R_L^{-1} \\ 1 & R_H^{-1} \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$

  where we simply used the fact that short-term interest rates satisfy $1 = \beta R_X \frac{\mathbb{E}\, u_c}{u_c(g_X)}$.
- These returns are perfectly correlated with shocks (high when g is high) and their volatility is the same as the volatility of u_c.
- Since the volatility of u_c is small, the volatility of short-term returns is small.
- Buying short-term debt is a good hedge against shocks, the risk is very small. Issue long-term debt to finance it!

PROBLEMS...

- Is it true that returns on short-term debt are perfectly correlated with shocks in the data?
- Is it true that their volatility is the same as the volatility of u_c?
- No.

PROBLEMS...

- In the data returns are very weakly correlated with consumption growth.
- Suppose we estimate

$$\ln c_{t+1} - \ln c_t = \alpha_0 + \alpha_1 R_t + e_t$$

- In our model $\alpha_1 \approx 1$.
- In the data $\alpha_1 \approx 0$.

PROBLEMS...

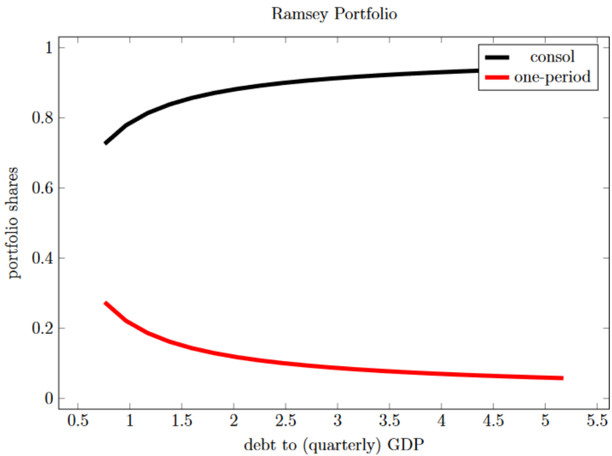
Maturity	**Returns**	
	mean (%)	**std. dev (%)**
less than 6 months	1.31	2.82
6-12 months	1.59	3.88
12-18 months	1.85	5.13
18-24 months	1.96	6.20
24-30 months	2.18	7.48
30-36 months	2.32	8.36
36-42 months	2.41	9.14
42-48 months	2.49	9.99
48-54 months	2.65	10.57
54-60 months	2.11	12.19
60-120 months	2.77	13.42
greater than 120 months	3.89	21.23

- Volatility of returns is high.
- Standard deviation of $\ln c_t$ is around 1% – much lower than the volatility of returns...

SUMMARY

- Portolio suggested by Angeleton (2002) and Buera and Nicolini (2004) is bad given the data.
- It would be very risky for the government.
- We do not see anything resembling this porfolio in the data!
- The problem was that this simple model is terrible at explaining asset prices.
- Bhandari et al. in several papers show that with preferences matching asset prices much better the optimal maturity structure is much more reasonable.

BEGS (2017)



Source: Bhandari et al. (2017)

BEGS (2017)

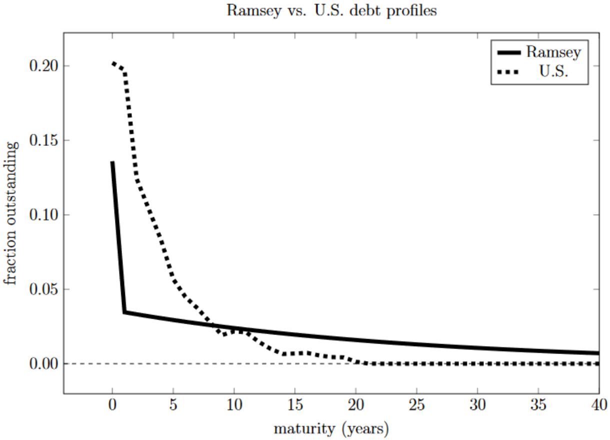


Figure III: Fraction of debt outstanding by maturity (years). For the U.S., we compute outstanding debt by bins of (Macaulay) duration and average for the sample period 1960-2015. The Ramsey debt profile is constructed using a replicating portfolio of zero-coupun bonds of all maturities.

Source: Bhandari et al. (2017)

BEGS (2017)

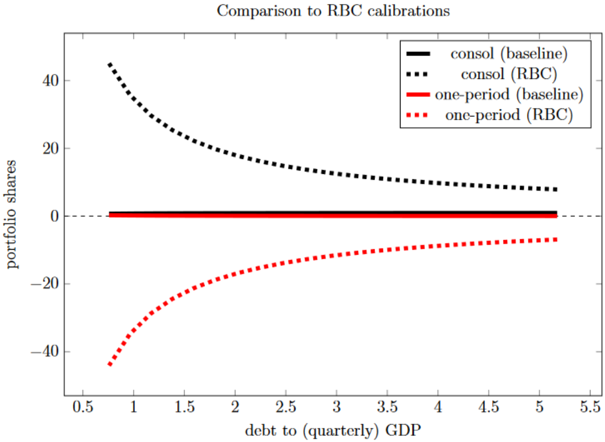


Figure VI: Portfolio shares for the baseline calibration (solid lines) and the RBC calibration (dashed lines)

Source: Bhandari et al. (2017)