AGGREGATE SHOCKS AND DEBT MANAGEMENT I

Fiscal and Monetary Policy 2024

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PLAN

- How should the government change taxes and debt in response to changes in purchases?
- For example: should it finance war spending in the same way as public education?
- This is a normative question, but to the extent real world governments behave according to prescriptions of models we will see in class, it will also be a positive theory of government debt and taxes.

PLAN

- We will study optimal taxation and debt policy in two environments:
  - incomplete markets: the government can buy or sell only a limited set of securities, often only a single risk-free security.
  - complete markets: the government can buy or sell claims contingent on all possible states of the world.
- First study some reduced form models, then proceed to optimal taxation in a competitive equilibrium (Lucas and Stokey, 1983).

REDUCED FORM MODELS

INCOMPLETE MARKETS

- We study the following problem based on Barro (1979):
  - the government uses taxes τ_t to finance *stochastic* purchases g_t;
  - $D(\tau_t)$ is the deadweight loss of taxes;
  - the government wants to finance g_t in a way that minimizes the expected value of present discounted deadweight loss of taxes:

$$\min_{\{\tau_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\left(\frac{1}{1+r} \right)^t D(\tau_t) \right]$$

    where $1/(1+r)$ is the (gross) rate at which the government discounts the future.

INCOMPLETE MARKETS

- Assume that g_t is governed by an N state Markov chain.
- The state of the world is given by s_t that follows a Markov chain with a transition probability matrix P:

$$P_{ij} = P\left\{s_{t+1} = \bar{s}_j \mid s_t = \bar{s}_i\right\}.$$

- Government spending $\{g_t\}$ obeys

$$g_t = \begin{cases} g_1 & \text{if } s_t = \bar{s}_1 \\ \vdots & \\ g_N & \text{if } s_t = \bar{s}_N. \end{cases}$$

INCOMPLETE MARKETS

- The government can issue only one period, non-state contingent debt (incomplete markets) at the price $1/\left(1+r\right)$, equal to the discount rate.
- Government problem:

$$\min_{\{\tau_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\left(\frac{1}{1+r} \right)^t D\left(\tau_t\right) \right]$$

$$\text{s.t.} \quad g_t + b_t = \frac{1}{1+r} b_{t+1} + \tau_t$$

and, in addition, we assume

$$\mathbb{E}_0 \left[\left(\frac{1}{1+r} \right)^t b_t^2 \right] \leq \infty.$$

DIGRESSION ON BELLMAN EQUATIONS

- To solve the above problem write it in a *recursive* form:

$$
\begin{aligned}
V(b, g) &= \min_{\tau} D(\tau) + \frac{1}{1+r} \mathbb{E}\left[V\left(b', g'\right) \mid g\right] \\
\text{s.t.} \quad g + b &= \frac{1}{1+r} b' + \tau
\end{aligned}
$$

  where x denotes current variables (x_t) and x' denotes variables in the next period (x_{t+1}).
- It looks like a two period problem (instead of infinite horizon) – the “only” issue is that we do not know $V(b, g)$ (and it appears on the right hand side).
- More on that in, for example, my Quantitative Economics course.

DIGRESSION ON BELLMAN EQUATIONS

- Write

$$V\left(b,g\right) = \min_{\tau} D\left(\tau\right) + \frac{1}{1+r}\mathbb{E}\left[V\left(\left(1+r\right)\left(g+b-\tau\right),g'\right) \mid g\right]$$

  and differentiate with respect to τ to get the first order condition:

$$D'\left(\tau\right) = \mathbb{E}\left[\frac{\partial V\left(b',g'\right)}{\partial b'} \,\middle|\, g\right]$$

- How to get $V'\left(b',g'\right) := \partial V\left(b',g'\right)/\partial b'$?

DIGRESSION ON BELLMAN EQUATIONS

- Write

$$V(b,g) = D\left(g + b - \frac{1}{1+r}b'\right) + \frac{1}{1+r}\mathbb{E}\left[V\left(b',g'\right) \mid g\right]$$

  (note: there is no $\min_\tau$) and differentiate with respect to b:

$$V'(b,g) = D'(\tau).$$

  This is the envelope condition and it implies

$$V'\left(b',g'\right) = D'\left(\tau'\right).$$

- Important: do not write $b' = b'(b,g)$ or anything like that, these terms drop out anyway (why?).

DIGRESSION ON BELLMAN EQUATIONS

- The first order condition

$$D'(\tau) = \mathbb{E}\left[\frac{\partial V(b', g')}{\partial b'} \,\middle|\, g\right]$$

together with the envelope condition

$$V'(b', g') = D'(\tau')$$

give us the Euler equation

$$D'(\tau_t) = \mathbb{E}_t\left[D'(\tau_{t+1})\right]$$

INCOMPLETE MARKETS

- Solution to the government's problem is characterized by

$$D'\left(\tau\right) = \mathbb{E}_t\left[D'\left(\tau'\right)\right].$$

- Tax smoothing: spread out distortionary effects of taxes across time.
- Marginal distortion of taxes today should be equal to the expected marginal distortion of taxes tomorrow.

INCOMPLETE MARKETS

- There are two special cases we will consider in more detail:
  - Special case 1 - no uncertainty (Barro, 1979):

$$\tau_t = \tau_{t+1}$$

  - Special case 2 - quadratic deadweight loss ($D\left(\tau_t\right) = a\tau_t^2 + \gamma$) :

$$\tau_t = \mathbb{E}_t\left[\tau_{t+1}\right]$$

SPECIAL CASE 1: PERFECT FORESIGHT

- With no uncertainty about future g_t (perfect foresight), taxes are constant.
- τ (a single value) is chosen to satisfy the budget constraint:

$$\tau = \frac{r}{1+r}\left[\sum_{t=0}^{\infty}(1+r)^t g_t + b_0\right]$$

- Debt absorbs all fluctuations in government spending:

$$b_{t+1} - b_t = r\,(b_t - b_0) + (1+r)\,g_t - r\sum_{s=0}^{\infty}(1+r)^s g_s.$$

SPECIAL CASE 1: PERFECT FORESIGHT

- If $g_t = g$ is constant, then

$$\tau = g + \frac{r}{1+r} b_0, \quad b_t = b_0;$$

  debt remains constant at its initial level.

- If $g_0 = g + \epsilon$ and $g_t = g$ for all $t \geq 1$ then

$$\tau = g + \frac{r}{1+r}\left(\epsilon + b_0\right), \quad b_t = b_0 + \epsilon;$$

  debt increases by ϵ and stays elevated forever.

- The level of debt b_t is irrelevant. What matters for debt is transitory fluctuations in g_t, not the average level of g_t.
- Intuition valid also with trend growth of GDP and government spending (see Barro's paper).

SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- What does the condition

$$\tau_t = \mathbb{E}_t\left[\tau_{t+1}\right]$$

  tell us about the optimal tax policy?

- τ_t is a random walk:

$$\begin{aligned}\tau_{t+1} - \tau_t &= \tau_{t+1} - \mathbb{E}_t\left[\tau_{t+1}\right] \\ &= \tau_{t+1} - \left(\tau_{t+1} - \epsilon_{t+1}\right) \\ &= \epsilon_{t+1}\end{aligned}$$

  where ϵ_{t+1} is white noise.

SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- Taxes tomorrow expected to be the same as today, τ_t moves only because of changes in expected future g_t.
- Whenever new information is revealed, taxes immediately jump to a new level.
- To see this, use the budget constraint together with $\mathbb{E}_t\left[\tau_{t+1}\right] = \tau_t$ to get

$$\tau_t = \frac{r}{1+r} b_t + \frac{r}{1+r} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \mathbb{E}_t\left[g_{t+s}\right].$$

so

$$\tau_t - \mathbb{E}_{t-1}\left[\tau_t\right] = \frac{r}{1+r} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s \left(\mathbb{E}_t\left[g_{t+s}\right] - \mathbb{E}_{t-1}\left[g_{t+s}\right]\right)$$

SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- The size of the jump in τ_t depends on the size of the change in expected future g_t:
  - If there is a permanent shift in g_t, $\mathbb{E}_t\left[g_{t+s}\right] - \mathbb{E}_{t-1}\left[g_{t+s}\right] = \epsilon$ for all s, taxes change by exactly the same amount ϵ.
  - If the change is temporary, $\mathbb{E}_t\left[g_{t+s}\right] - \mathbb{E}_{t-1}\left[g_{t+s}\right] = \epsilon$ only for $s = 0$, taxes change by $\epsilon r / \left(1 + r\right)$.
- Remaining needed resources raised through debt issuance:
  - Debt remains constant for a permanent change in g_t.
  - Debt changes by $\epsilon / \left(1 + r\right)$ for a temporary change in g_t.
- Implication: finance education by taxes, finance war by debt.

SPECIAL CASE 2 - QUADRATIC DEADWEIGHT LOSS

- Debt also inherits the random walk property of taxes:

$$\mathbb{E}_t \left[b_{t+1} \right] = b_t$$

- There is nothing that acts as a force that would push debt to a particular level:
  - No debt target;
  - No "raise taxes if debt is too high" rule;
  - Debt is a random walk, so its variance goes to infinity unless all shocks to g_t are fully permanent.

COMPLETE MARKETS

- This is a simplified version Lucas and Stokey (1983), we will see the full version later.
- In the complete markets case, the government can issue state-contingent securities.
- This means that the government can issue a security that pays 1 unit of consumption in state s in the next period, and it can do it for all possible states.
- We call such a security an Arrow security.
- Let s_t denote the state in period t and $s^t := (s_0, s_1, \cdots, s_t)$ denote the history of states up to period t.
- Let $q\left(s^{t+1} \mid s^t\right)$ denote the period t price of a security that pays 1 unit of goods for a particular history s^{t+1} in period $t+1$ (and zero in other states).

COMPLETE MARKETS

- To facilitate comparison with the incomplete markets case we will assume

$$q\left(s^{t+1} \mid s^t\right) = \frac{1}{1+r}\pi\left(s^{t+1} \mid s^t\right)$$

  where $\pi\left(s^{t+1} \mid s^t\right)$ is the conditional probability of history s^{t+1} given history s^t.

- The above assumption implies:

$$\sum_{s^{t+1} \geq s^t} q\left(s^{t+1} \mid s^t\right) = \frac{1}{1+r},$$

  the price of a portfolio that pays one unit of goods for sure is equal to $(1+r)^{-1}$.

COMPLETE MARKETS

- The problem of the government is now

$$
\begin{aligned}
& \min_{\{\tau\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \pi\left(s^t\right)\left(\frac{1}{1+r}\right)^t D\left(\tau\left(s^t\right)\right) \\
\text{s.t.} \quad \forall s^t \quad & g\left(s^t\right) + b\left(s^t\right) = \tau\left(s^t\right) + \sum_{s^{t+1} \geq s^t} q\left(s^{t+1} \mid s^t\right) b\left(s^{t+1}\right) \\
& b\left(s_0\right) = b_0
\end{aligned}
$$

where $\pi\left(s^t\right)$ is the probability of history s^t, and $b\left(s^t\right)$ is the quantitity of (negative) Arrow securities that pay in history s^t.

- We can write a Lagrangian:

$$\sum_{t=0}^{\infty}\sum_{s^t}\pi\left(s^t\right)\left(\frac{1}{1+r}\right)^t D\left(\tau\left(s^t\right)\right)$$
$$+\sum_{t=0}^{\infty}\sum_{s^t}\lambda\left(s^t\right)\left[g\left(s^t\right)+b\left(s^t\right)-\tau\left(s^t\right)-\sum_{s^{t+1}\geq s^t}q\left(s^{t+1}\mid s^t\right)b\left(s^{t+1}\right)\right]$$

COMPLETE MARKETS

- First order condition with respect to $\tau\left(s^t\right)$:

$$\pi\left(s^t\right)\left(\frac{1}{1+r}\right)^t D'\left(\tau\left(s^t\right)\right) = \lambda\left(s^t\right)$$

- First order condition with respect to $b\left(s^{t+1}\right)$:

$$q\left(s^{t+1} \mid s^t\right)\lambda\left(s^t\right) = \lambda\left(s^{t+1}\right)$$

COMPLETE MARKETS

- Given the assumption

$$q\left(s^{t+1} \mid s^t\right) = \frac{1}{1+r}\pi\left(s^{t+1} \mid s^t\right)$$

we have

$$\frac{1}{1+r}\pi\left(s^{t+1} \mid s^t\right)\lambda\left(s^t\right) = \lambda\left(s^{t+1}\right)$$

so

$$\begin{aligned}\frac{1}{1+r}\pi\left(s^{t+1} \mid s^t\right)\pi\left(s^t\right)\left(\frac{1}{1+r}\right)^t D'\left(\tau\left(s^t\right)\right) \\ = \pi\left(s^{t+1}\right)\left(\frac{1}{1+r}\right)^{t+1} D'\left(\tau\left(s^{t+1}\right)\right)\end{aligned}$$

COMPLETE MARKETS

- It simplifies to

$$D'\left(\tau\left(s^{t}\right)\right) = D'\left(\tau\left(s^{t+1}\right)\right)$$

  i.e.

$$\tau\left(s^{t}\right) = \tau\left(s^{t+1}\right).$$

- This is the key result: perfect smoothing of taxes across states.
- The result holds regardless of the form of $D\left(\tau\right)$ or the nature of risk.
- Compare it with incomplete markets case: it was true only with perfect foresight.

COMPLETE MARKETS

- What about debt?
- Guess that $b\left(s^{t+1}\right)$ depends only on the future state s_{t+1}, and not on the history s^t.
- Intuitively: the government cares only about the future state, not about the history (why?).
- Use this guess together with $\tau\left(s^t\right) = \tau$ and the initial condition $b\left(s_0\right) = b_0$ to solve for τ and optimal debt policy.
- This also verifies the guess.

COMPLETE MARKETS

- Since $b\left(s^{t+1}\right) = b\left(s_{t+1}\right)$, debt neither accumulates, nor decumulates, nor drifts – it switches between different levels.

COMPARISON

- Incomplete: taxes drift over time as a random walk; the level of taxes at time t depends on the level of debt that the government brings into the period as well as the expected discounted present value of government purchases.
- Complete: taxes are constant, regardless of the state of the world; the level of taxes at time t is independent of the level of debt that the government brings into the period.

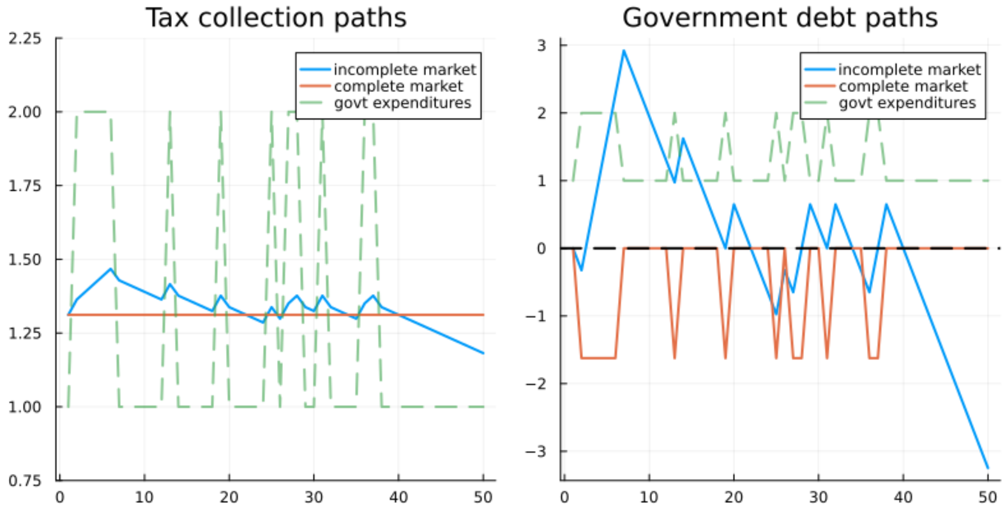
COMPARISON

- Incomplete: debt drifts upward over time in response to large purchases and drifts downward over time in response to low purchases.
- Complete: debt osciallates between several levelsl; this is akin to the government purchasing insurance that protects against the need to raise taxes too high or issue too much debt in the high government expenditure event.

NUMERICAL EXAMPLE

- Let $(1+r)^{-1} = 0.96$ and there are two levels of government purchases: $g_1 = 1$ and $g_2 = 2$ with with transition probabilities $P_{12} = 0.2$ and $P_{21} = 0.4$.
- The initial condition is $b_0 = 0$, and we start with g_1.

NUMERICAL EXAMPLE



Tax and debt policy in complete and incomplete markets. Computed using QuantEcon Julia package.

LUCAS AND STOKEY (1983)

SETUP

- We study the following problem based on Lucas and Stokey (1983):
  - the representative household has preferences over consumption and labor (leisure);
  - production technology is linear in labor, there is no capital;
  - the government uses distortionary tax rate τ_t on labor income to finance *stochastic* purchases g_t (we assume g_t is a Markov process);
  - the households and the government can trade a full set of Arrow securities (state-contingent claims).
  - the government wants to finance g_t in a way that maximizes the welfare of the representative household.
- General equilibrium.

SETUP

- Price of securities no longer exogenous.
- The government's cares about welfare of the representative household.
- Think of the problem in two steps:
  1. for given government policies there exists some competitive equilibrium;
  2. the government picks policies that result in the "best" equilibrium.
- Key difference: policies not only have to satisfy the government's budget constraint, but also household's optimality conditions.
- Primal approach: we will look directly for allocations that maximize welfare and then think of policies that implement that.

Competitive equilibrium

A competitive equilibrium given government policies $\tau\left(s^t\right)$ and $g\left(s^t\right)$ is a set of allocations $\left(c\left(s^t\right), \ell\left(s^t\right), a\left(s^{t+1}\right)\right)$ and prices $\left(q\left(s^{t+1} \mid s^t\right), w\left(s^t\right)\right)$ such that:

1. Given prices, allocations solve the household problem;
2. Given prices, allocations solve the firm problem;
3. The government budget constraint is satisfied is each state;
4. All markets clear in each state.

- Note: this is not a proper definition of a competitive equilibrium, but it will do for our purposes.

HOUSEHOLD PROBLEM

- Household chooses consumption, labor supply and portfolio of Arrow securities to maximize expected utility:
- Household problem:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c\left(s^t\right), \ell\left(s^t\right)\right)$$

$$\begin{aligned}
\text{s.t.} \quad \forall s^t \quad & c\left(s^t\right) + \sum_{s^{t+1} \geq s^t} q\left(s^{t+1} \mid s^t\right) a\left(s^{t+1}\right) \\
& = \left(1 - \tau\left(s^t\right)\right) w\left(s^t\right) \ell\left(s^t\right) + a\left(s^t\right) \\
& a\left(s^0\right) = a_0
\end{aligned}$$

where $u(\cdot)$ is the utility function, $c(\cdot)$ consumption and $\ell(\cdot)$ labor.

HOUSEHOLD PROBLEM

- The sequence of household budget constraints can be written as a single constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} q\left(s^t\right) c\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q\left(s^t\right)\left(1-\tau\left(s^t\right)\right) w\left(s^t\right) \ell\left(s^t\right) + q\left(s^0\right) a_0$$

  where $q\left(s^t\right)$ is now the time-0 price of one unit of goods in state s^t.
- Why? Complete markets are equivalent with Arrow securities and time-0 trading.

HOUSEHOLD PROBLEM

- Household problem:

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c\left(s^t\right), \ell\left(s^t\right)\right)$$

$$\text{s.t.} \sum_{t=0}^{\infty} \sum_{s^t} q\left(s^t\right) c\left(s^t\right) = \sum_{t=0}^{\infty} \sum_{s^t} q\left(s^t\right)\left(1-\tau\left(s^t\right)\right) w\left(s^t\right) \ell\left(s^t\right) + q\left(s^0\right) a_0$$

- This is nice because there is only one constraint.

FIRST ORDER CONDITIONS

- Write down the Lagrangian and differentiate with respect to $c\left(s^t\right)$ and $\ell\left(s^t\right)$ to get the first order conditions:

$$
\begin{aligned}
\left[c\left(s^t\right):\right] \quad & \beta^t \pi\left(s^t\right) u_c\left(c\left(s^t\right), \ell\left(s^t\right)\right) = \lambda q\left(s^t\right) \\
\left[\ell\left(s^t\right):\right] \quad & \beta^t \pi\left(s^t\right) u_\ell\left(c\left(s^t\right), \ell\left(s^t\right)\right) = -\lambda q\left(s^t\right)\left(1-\tau\left(s^t\right)\right) w\left(s^t\right)
\end{aligned}
$$

- Note: the Lagrange multiplier λ is the same for all states s^t.
- To simplify notation

$$
\begin{aligned}
\left[c\left(s^t\right):\right] \quad & \beta^t \pi\left(s^t\right) u_c\left(s^t\right) = \lambda q\left(s^t\right) \\
\left[\ell\left(s^t\right):\right] \quad & \beta^t \pi\left(s^t\right) u_\ell\left(s^t\right) = -\lambda q\left(s^t\right)\left(1-\tau\left(s^t\right)\right) w\left(s^t\right)
\end{aligned}
$$

IMPLEMENTABILITY CONSTRAINT

- The first order conditions:

$$\left[c\left(s^t\right):\right]\ \beta^t\pi\left(s^t\right)u_c\left(s^t\right) = \lambda q\left(s^t\right)$$
$$\left[\ell\left(s^t\right):\right]\ \beta^t\pi\left(s^t\right)u_\ell\left(s^t\right) = -\lambda q\left(s^t\right)\left(1-\tau\left(s^t\right)\right)w\left(s^t\right)$$

together with the budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} q\left(s^t\right)c\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q\left(s^t\right)\left(1-\tau\left(s^t\right)\right)w\left(s^t\right)\ell\left(s^t\right) + q\left(s^0\right)a_0$$

allow us to write the implementability constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)\left[u_c\left(s^t\right)c\left(s^t\right) + u_\ell\left(s^t\right)\ell\left(s^t\right)\right] = u_c\left(s^0\right)a_0$$

IMPLEMENTABILITY CONSTRAINT

- The implementability constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) \left[u_c\left(s^t\right) c\left(s^t\right) + u_\ell\left(s^t\right) \ell\left(s^t\right)\right] = u_c\left(s^0\right) a_0$$

  summarizes household optimality conditions and the budget constraint.
- The government can only choose allocations that satisfy this constraint.
- Captures the notion that the government’s choices result in some optimal (given these choices) household behavior.

FIRM PROBLEM

- Output in this economy is produced by a representative price-taking firm.
- Linear production function that uses labor as an input.
- Firm problem is almost trivial – it chooses labor to maximize profits:

$$\max_{\ell(s^t)} A\ell\left(s^t\right) - w\left(s^t\right)\ell\left(s^t\right).$$

- In a competitive equilibrium we must have

$$w\left(s^t\right) = A.$$

GOVERNMENT BUDGET CONSTRAINT

- The implementability constraint is the household budget constraint (+ optimality conditions)
- The government should also be constrained by its budget constraint.
- We can ignore it and focus directly on the resource constraint of the economy:

$$A\ell\left(s^t\right) = g\left(s^t\right) + c\left(s^t\right)$$

  where $A\ell\left(s^t\right)$ is the linear production technology.
- Why? Walras' law – the government budget constraint is redundant.

GOVERNMENT PROBLEM

- The government chooses allocations $c\left(s^t\right), \ell\left(s^t\right)$ to maximize the household's utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(c\left(s^t\right), \ell\left(s^t\right)\right)$$

subject to the implementability constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) \left[u_c\left(s^t\right) c\left(s^t\right) + u_\ell\left(s^t\right) \ell\left(s^t\right)\right] = u_c\left(s^0\right) a_0$$

and resource constraints:

$$\forall s^t \quad A\ell\left(s^t\right) = g\left(s^t\right) + c\left(s^t\right)$$

GOVERNMENT PROBLEM

- Full commitment: the government announces its state-contingent plans at time 0 and cannot change them later.
- Everyone else trusts the government and knows it will stick to its plans.

OPTIMALITY CONDITIONS

- The first order conditions for $s^t > s^0$ (we will return to s^0 later):

$$
\begin{aligned}
u_c\left(s^t\right) + \eta\left[u_{cc}\left(s^t\right)c\left(s^t\right) + u_c\left(s^t\right) + u_{c\ell}\left(s^t\right)\ell\left(s^t\right)\right] &= \mu\left(s^t\right) \\
u_\ell\left(s^t\right) + \eta\left[u_{c\ell}\left(s^t\right)c\left(s^t\right) + u_\ell\left(s^t\right) + u_{\ell\ell}\left(s^t\right)\ell\left(s^t\right)\right] &= -\mu\left(s^t\right)A
\end{aligned}
$$

  where η is the Lagrange multiplier on the implementability constraint and $\mu\left(s^t\right)$ is the Lagrange multiplier on the resource constraint.

- There is also a resource constraint for each state:

$$
A\ell\left(s^t\right) = g\left(s^t\right) + c\left(s^t\right)
$$

- This gives us, for each state, three equations in three unknowns: $c\left(s^t\right), \ell\left(s^t\right), \mu\left(s^t\right)$.

OPTIMALITY CONDITIONS

- We also have a fourth unknown: η and one more equation: the implementability constraint.
- Key: η is the only link between different states.
- Once we know it, $c(s^t), \ell(s^t), \mu(s^t)$ are determined for each s^t from 2 first order conditions and the resource constraint.
- For two different states with the same $g(s^t)$, the solution must be the same!

IMPLICATIONS

- What are the implications for taxes?
- From household first order condition we have

$$\left(1-\tau\left(s^t\right)\right) w\left(s^t\right) = -\frac{u_\ell\left(s^t\right)}{u_c\left(s^t\right)}$$

and linear production technology implies $w\left(s^t\right) = A$ so

$$1-\tau\left(s^t\right) = -\frac{u_\ell\left(c\left(s^t\right), \ell\left(s^t\right)\right)}{u_c\left(c\left(s^t\right), \ell\left(s^t\right)\right)}\frac{1}{A}.$$

- We have solved for $c\left(s^t\right), \ell\left(s^t\right)$, we can get $\tau\left(s^t\right)$!

IMPLICATIONS

- Since for two different states with the same $g\left(s^t\right)$ consumption and labor are the same – taxes must be the same!
- Conclusion: very different from Barro (1979) – no history dependence.
- If government purchases are i.i.d., taxes are i.i.d. as well.
- Contrast with the incomplete markets case: taxes were a random walk.
- Contrast with the reduced form complete markets case: taxes were constant.

IMPLICATIONS

- Why is the result different from the reduced form complete markets case?
- No full tax smoothing (unless with particular household preferences), because of a different objective.
- When g_t is high, the government could want to increase labor supply to finance it without a large drop in consumption.

EXAMPLE

- Let $u(\cdot) = \frac{c^{1-\gamma}}{1-\gamma} - \psi \ell$.
- Then $u_c = c^{-\gamma}$, $u_{cc} = -\gamma c^{-\gamma-1}$, $u_\ell = -\psi$, $u_{\ell\ell} = 0$, $u_{c\ell} = 0$.
- The first order conditions

$$
\begin{aligned}
u_c\left(s^t\right) + \eta\left[u_{cc}\left(s^t\right)c\left(s^t\right) + u_c\left(s^t\right) + u_{c\ell}\left(s^t\right)\ell\left(s^t\right)\right] &= \mu\left(s^t\right) \\
u_\ell\left(s^t\right) + \eta\left[u_{c\ell}\left(s^t\right)c\left(s^t\right) + u_\ell\left(s^t\right) + u_{\ell\ell}\left(s^t\right)\ell\left(s^t\right)\right] &= -\mu\left(s^t\right)A
\end{aligned}
$$

are

$$
\begin{aligned}
\left[1 + \eta\left(1-\gamma\right)\right]c\left(s^t\right)^{-\gamma} &= \mu\left(s^t\right) \\
\left[1 + \eta\right]\psi &= \mu\left(s^t\right)A
\end{aligned}
$$

EXAMPLE

- For each state $s^t > s^0$ we have:

$$\frac{1+\eta}{\left[1+\eta\left(1-\gamma\right)\right]A}\psi = c\left(s^t\right)^{-\gamma}$$

  so consumption is constant.

- Taxes satisfy

$$1-\tau\left(s^t\right) = \frac{\psi}{c\left(s^t\right)^{-\gamma}}\frac{1}{A}:$$

  since $c\left(s^t\right)$ is constant, taxes are constant as well.

- With these preferences households supply labor fully elastically, fluctuations in g_t are absorbed by changes in production.

LESSONS

- Rembember: the above result is not general.
- What is general is that here allocations and taxes are history independent – they are fully determined by the current state.
- What is the optimal path of debt? Similar logic as in the reduced form case: state-contingent debt that depends only on the next period state.

TIME-0

- We ignored one important issue: first order conditions are different for period 0.
- This is because of the $u_c\left(s^0\right)a_0$ term in the implementability constraint.
- First order conditions for s^0 are:

$$u_c\left(s^0\right)+\eta\left[u_{cc}\left(s^0\right)c\left(s^0\right)+u_c\left(s^0\right)+u_{c\ell}\left(s^0\right)\ell\left(s^0\right)-u_{cc}\left(s^0\right)a_0\right]=\mu\left(s^0\right)$$
$$u_\ell\left(s^0\right)+\eta\left[u_{c\ell}\left(s^0\right)c\left(s^0\right)+u_\ell\left(s^0\right)+u_{\ell\ell}\left(s^0\right)\ell\left(s^0\right)-u_{c\ell}\left(s^0\right)a_0\right]=-\mu\left(s^0\right)A$$

- Time-0 allocations will be different from other periods, even with the same level of g_t.

TIME-0

- The government has incentives to play with taxes in period 0 to change $q\left(s^0\right)$ and thus affect the value of debt/assets.
  - For example: it could try to make debt due at time 0 worthless.
  - This is good, because it allows the government to create lower tax distortions in the future.
- No similar effect in other periods, forward looking agents take it into account.

TIME INCONSISTENCY

- Time consistency problem: if the government could re-optimize at future dates, period 1 would be like period 0.
- Same logic applies to all other periods.
- But forward-looking agents would know the government is tempted to do it every period!
- This is actually the essence of Lucas and Stokey (1983).
  - Is there a way to design debt portfolio (maturities etc.) to ensure the government will not be tempted to re-optimize?