INTRO TO DYNAMIC PROGRAMMING

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A TYPICAL PROBLEM

Many problems seen in economics have a common structure:

- We observe the current state X_t at time t.
- We choose an action A_t at time t.
- We get a reward R_t at time t.
- The state progresses to X_{t+1} at time $t + 1$.

If the largest possible t is $T < \infty$, then we have a finite horizon problem.

Otherwise we have an infinite horizon problem.

EXAMPLE

Consider a problem of a firm that produces a good. The firm wants to maximize the expected present discounted value of profits:

$$\mathbb{E} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right) \pi_t$$

- X_t is the current state of the firm. It can be the current level of capital, the current level of inventory, prices set by competitors...
- A_t is the action taken by the firm. It can be the level of production, the level of future inventory, the price of the good...
- R_t is the reward. Here it is the profit of the firm, π_t.
- X_{t+1} is the state of the firm in the next period. It can depend on the current state X_t and the action A_t taken.

EXAMPLE

- This is potentially an extremely complicated problem.
- For example: the state can include the demand for the good – and it could be random.
- Find actions for all possible future states...
- We will learn tools that can help us solve such problems.

PLAN

- Today we will study an example: McCall's job search model (1970).
- Exposition based on Stachurski and Sargent (2023).
- Next time: more general theory of dynamic programming.

MCCALL'S JOB SEARCH MODEL

TWO-PERIOD PROBLEM

- An unemployed agent receives a job offer at wage W_t.
- She can either accept the offer or reject it.
- If she accepts, she gets this wage permenantly.
- If she rejects, she gets unemployment benefit c.
- Wage offers are independent and identically distributed (i.i.d.) and nonnegative, with distribution ϕ:
  - $\mathbb{W} \subset \mathbb{R}_+$ is a finite set of possible wages.
  - $\phi : \mathbb{W} \to [0, 1]$ is a probability mass function, $\phi(w)$ is the probability of getting a wage w.
- The agent is risk-neutral and impatient. The utlity of getting y tomorrow is βy, with $\beta \in (0, 1)$.

TWO-PERIOD PROBLEM

- The agent lives for two periods and starts unemployed
- The question is: is it better to accept or wait for a better offer?
- What is the lowest wage that the agent should accept?
- We will start analyzing the problem by looking at the second period, $t = 2$: backward induction.

PERIOD T=1

- Suppose the agent is unemployed at $t = 2$.
- She gets a wage offer W_2.
- She can either accept or reject the offer.
- Since this is the last period of her life, she will accept if and only if

$$W_2 \geq c.$$

PERIOD T=2

- The agent gets a wage offer W_1.
- She can either (a) accept and get W_1 forever, or (b) reject and get c in period $t = 1$ and then get the maximum of W_2 and c in period $t = 2$.
- The utility of (a) is $W_1 + \beta W_1$. We call it the stopping value.
- The utility of (b) is $h_1 := c + \beta \, \mathbb{E} \max \{W_2, c\}$. We call it the continuation value.

$$h_1 = c + \beta \sum_{w' \in \mathrm{W}} \max v_2(w') \phi(w'), \quad v_2(w') := \max \{w', c\}$$

- The agent will accept if and only if the stopping value is greater than the continuation value:

$$W_1 + \beta W_1 \geq h_1.$$

VALUE FUNCTION

- The key object in dynamic programming is the value function.
- It is a function that maps the state to the maximum expected present discounted value of future rewards.
- In our example, the state is the time t and the wage offer w.
- $v_2(w)$ is the value function at time $t = 2$ and wage w: the largest possible reward that the agent can get if she starts unemployed at $t = 2$ and gets a wage offer w.
- The time 1 value function is

$$v_1(w) := \max \left\{ w + \beta w, c + \beta \sum_{w' \in \mathbb{W}} v_2\left(w'\right) \phi\left(w'\right) \right\}.$$

TWO-PERIOD EXAMPLE

- This particular problem is easy to solve.
- Accept if

$$w \geq \frac{h_1}{1 + \beta}$$

so the value function is

$$v_1(w) = \begin{cases} (1 + \beta)\, w & \text{if } w \geq \frac{h_1}{1+\beta} \\ h_1 & \text{otherwise.} \end{cases}$$

- In context of this example, we call $w^* \coloneqq \frac{h_1}{1+\beta}$ the reservation wage.
- We see that since h_1 is increasing in c, the reservation wage is higher when the unemployment benefit is higher.

THREE-PERIOD EXAMPLE

- Extend the model by one period, $t = 0$.
- The value function at $t = 0$ is

$$v_0(w) := \max \left\{ w + \beta w + \beta^2 w, c + \beta \sum_{w' \in \mathsf{W}} v_1(w') \phi(w') \right\}.$$

where the formula for v_1 is from the previous slide.

- Key insight: at $t = 0$ it is like a two-period problem.
- All information about the future is summarized in the value function at $t = 1$.
- This is the standard approach: convert a complicated dynamic optimization problem into a sequence of two-period problems.

BELLMAN EQUATION

- Recall we had

$$
\begin{aligned}
v_2(w) &= \max\{w, c\} \\
v_1(w) &= \max\left\{w + \beta w, c + \beta \sum_{w' \in W} v_2\left(w'\right)\phi\left(w'\right)\right\} \\
v_0(w) &= \max\left\{w + \beta w + \beta^2 w, c + \beta \sum_{w' \in W} v_1\left(w'\right)\phi\left(w'\right)\right\}.
\end{aligned}
$$

- The recursive relationships between the value functions are called the Bellman equations.
- Warning: these equations are (in general) functional equations. We need to find functions, not numbers. Here it is easy: we have a finite set of possible wages – treat functions as vectors.

INFINITE HORIZON

- The three-period problem is also easy to solve.
- In fact, we can use the same approach (backward induction) as before for any finite horizon problem.
- What if the horizon is infinite? We no longer have the terminal period.
- Dynamic programming makes this problem tractable.

INFINITE HORIZON

- The objective function is

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t R_t,$$

  where $R_t \in \{c, W_t\}$.
- Let $\beta \in (0, 1)$ be the discount factor. $c > 0$.
- The wage process satisfies $(W_t)_{t\geq 0} \overset{iid}{\sim} \phi$ where $\phi \in \mathcal{D}(\mathrm{W})$ and $\mathrm{W} \subset \mathbb{R}_+$ with $|\mathrm{W}| < \infty$.
- For any finite or countable set F, $\mathcal{D}(\mathrm{F})$ is the set of distributions on F.

INFINITE HORIZON

- What is stopping value?
- If the worker accepts wage w she gets

$$w + \beta w + \beta^2 w + \beta^3 w + \cdots = \frac{w}{1-\beta}.$$

- What is the continuation value?
- If the worker rejects wage w she gets

$$c + \beta \sum_{w' \in \mathrm{W}} v(w') \phi(w').$$

note that the value function is the same in all periods – there is always infinite remaining future.

INFINITE HORIZON

- Bellman equation is:

$$v(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \sum_{w' \in W} v\left(w'\right) \phi\left(w'\right) \right\}.$$

- Principle of optimality, Bellman (1960): *An optimal policy has the property that whatever the initial state and the initial decisions it must constitute an optimal policy with regards to the state resulting from the first decision.*
- This is not that trivial, we will return to it (and prove it!) later.
- Intepretation: value function satisfies the Bellman equation.

CHALLENGE

- Bellman equation is:

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in W} v\left(w'\right)\phi\left(w'\right)\right\}.$$

- Once we have $v(w)$ we can characterize the optimal choice of the agent.
- Q: how to find $v(w)$? It is a function!
- Q: is there a solution?
- Q: is the solution unique?
- Q: what are the properties of the solution?
- A: we will learn how to answer these questions.

APPROACH I

- This particular problem is relatively easy.
- Recall how we defined the continuation value:

$$h^* := c + \beta \sum_{w' \in W} \max \left\{ \frac{w'}{1-\beta}, h^* \right\} \phi(w')$$

- We can write the value function as

$$v(w) = \max \left\{ \frac{w}{1-\beta}, h^* \right\}.$$

- Key: h^* is a scalar, not a function. When would is break down?
- Find h^* directly, by solving the equation numerically.
- Then use h^* to get the value function.

APPROACH I

Algorithm Solving for v directly

procedure MCCALL

  $k \leftarrow 1, \epsilon \leftarrow \tau + 1, h_k \leftarrow c$

  while $\epsilon > \tau$ **do**

    $h_{k+1} \leftarrow c + \beta \sum_{w' \in \mathrm{W}} \max \left\{ \frac{w'}{1-\beta}, h_k \right\} \phi\left(w'\right)$

    $\epsilon \leftarrow \left|h_{k+1} - h_k\right|, k \leftarrow k + 1$

  end while

  for $w \in \mathrm{W}$ **do**

    $v(w) \leftarrow \max \left\{ \frac{w}{1-\beta}, h_k \right\}$

  end for

end procedure

APPROACH II

- We know how to solve a finite horizon problem – backward induction.
- Maybe we can get an approximate solution to the infinite horizon problem by considering a finite horizon problem with a very large number of periods?
- We will prove that it actually works.

MATHEMATICAL DETOUR: BANACH'S CONTRACTION MAPPING THEOREM

EXISTENCE AND UNIQUENESS

- Before solving any problem it is useful to know if there is a solution at all.
- We will use a powerful theorem that will help us answer this question.
- First we introduce a concept of a fixed point.
- Let U be any nonempty set. We call T a self-map on U if $T : U \to U$.
- For a self-map T on U, we say that a point $u^* \in U$ is a fixed point of T if $Tu^* = u^*$.

FIXED POINT

- Some examples of fixed points:
  - $U = \mathbb{R}$, $T(u) = 2u + 3$. Then $u^* = -3$ is a fixed point of T.
  - $U = [0, 1]$, $T(u) = u$. Then every $u \in U$ is a fixed point of T.
  - $U = \mathbb{R}$, $T(u) = u + 1$. There is no fixed point of T.
- Global stability: a self-map T on U is globally stable on U if T has a unique fixed point u^* in U and $T^k u \to u^*$ for all $u \in U$.

METRIC SPACE

- A metric space is a set U, together with a metric (distance function) ρ, $\rho : U \times U \to \mathbb{R}$, such that for all $u, v, w \in U$:
  - $\rho(u, v) \geq 0$ (nonnegativity), with equality if and only if $u = v$.
  - $\rho(u, v) = \rho(v, u)$ (symmetry).
  - $\rho(u, v) \leq \rho(u, w) + \rho(w, v)$ (triangle inequality).

METRIC SPACE

- Convergence: a sequence $\{x_n\}_{n=0}^{\infty}$ in U converges to $x \in U$, if for each $\epsilon > 0$, there exists N_ϵ such that $\rho(x_n, n) < \epsilon, \forall n \geq N_\epsilon$.
- Cauchy sequence: A sequence $\{x_n\}_{n=0}^{\infty}$ in U is Cauchy if for each $\epsilon > 0$, there exists N_ϵ such that $\rho(x_n, x_m) < \epsilon, \forall n, m \geq N_\epsilon$.
- Complete metric space: A metric space (U, ρ) is complete if every Cauchy sequence in U converges to an element in U.
- Example: $\mathbb{R}$ with $\rho(u, v) = |u - v|$ is a complete metric space.

NORMED VECTOR SPACE

- A normed vector space is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ together with a norm $\|\cdot\|$, $\|\cdot\| : V \to \mathbb{R}$, such that for all $u, v \in V$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$:
  - $\|u\| \geq 0$ (nonnegativity), with equality if and only if $u = 0$.
  - $\|\alpha u\| = |\alpha| \, \|u\|$ (absolute homogeneity).
  - $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality).
- Some norms: $\|u\|_1 = \sum_{i=1}^n |u_i|$ (Manhattan), $\|u\|_2 = \left(\sum_{i=1}^n u_i^2\right)^{1/2}$ (Euclidean), $\|u\|_\infty = \max_{i=1,\ldots,n} |u_i|$ (supremum).
- We will only focus on real vector spaces.
- A normed vector space is a metric space with $\rho(u, v) = \|u - v\|$.
- A complete normed vector space is called a Banach space.

NORMED VECTOR SPACE

- Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $C(X)$ be the set of bounded continuous functions on X with the supremum norm, $\| f \|_\infty = \sup_{x \in X} | f(x)|$. Then C is a Banach space (complete normed vector space).

CONTRACTION

- Let (U, ρ) be a metric space and T a self-map on U. T is a contraction mapping (with modulus λ) if for some $\lambda \in (0, 1)$

$$\rho(Tu, Tv) \leq \lambda\rho(u, v), \qquad \text{for all } u, v \in U.$$

- If T is a contraction on U, then T is uniformly continuous on U.

BANACH'S CONTRACTION MAPPING THEOREM

Theorem (Banach's contraction mapping theorem)

If (U, ρ) is a complete metric space and a self-map T is a contraction mapping with modulus λ, then:

- *T has a unique fixed point u^* in U, and*
- *for any $u_0 \in U$, $\rho\left(T^k u_0, u^*\right) \leq \lambda^k \rho(u_0, u^*)$ for all $k \in \mathbb{N}$.*

BANACH'S CONTRACTION MAPPING THEOREM

- By the contraction property of T we have:

$$\rho(u_2, u_1) = \rho(Tu_1, Tu_0) \leq \lambda\rho(u_1, u_0).$$

By induction:

$$\rho(u_{k+1}, u_k) \leq \lambda^k \rho(u_1, u_0), n = 1, 2, \ldots.$$

Using it and the triangle inequality, for $m > n$

$$\begin{aligned}
\rho(u_m, u_n) &\leq \rho(u_m, u_{m-1}) + \rho(u_{m-1}, u_{m-2}) + \cdots + \rho(u_{n+1}, u_n) \\
&\leq \left[\lambda^{m-1} + \lambda^{m-2} + \ldots + \lambda^n\right] \rho(u_1, u_0) \\
&\leq \frac{\lambda^n}{1 - \lambda} \rho(u_1, u_0).
\end{aligned}$$

PROOF

- From

$$\rho(u_{k+1}, u_k) \leq \frac{\lambda^n}{1-\lambda}\rho(u_1, u_0)$$

  we see that $\{u_k\}_{k=0}^{\infty}$ is a Cauchy sequence.
- Since U is complete, $\{u_k\}_{k=0}^{\infty}$ converges to some $u^* \in U$.
- We now show that u^* is a fixed point of T. For all n and all $u_0 \in U$,

$$\begin{aligned}\rho\left(Tu^*, u^*\right) &\leq \rho\left(Tu^*, T^n u_0\right) + \rho\left(T^n u_0, u^*\right)\\ &\leq \lambda\rho\left(u^*, T^{n-1}u_0\right) + \rho\left(T^n u_0, u^*\right).\end{aligned}$$

- By the previous result both terms on the right hand side go to 0 as $n \to \infty$.

PROOF

- We need to show there is no other $\hat{u}$ such that $T\hat{u} = \hat{u}$.
- Suppose there is such $\hat{u} \neq u^*$. Take $\rho(\hat{u}, u^*) = \delta > 0$ Then

$$\delta = \rho\left(u^*, \hat{u}\right) = \rho\left(Tu^*, T\hat{u}\right) \leq \lambda\rho\left(u^*, \hat{u}\right) = \lambda\delta.$$

but $\delta \leq \delta\lambda$ cannot hold because $\lambda < 1$!

PROOF

- To prove "for any $u_0 \in U, \rho\left(T^k u_0, u^*\right) \leq \lambda^k \rho(u_0, u^*)$ for all $k \in \mathbb{N}$" notice that for any $n \geq 1$:

$$\rho\left(T^k u_0, u^*\right) = \rho\left(T\left(T^{k-1} u_0\right), Tu^*\right) \leq \lambda \rho\left(T^{k-1} u_0, u^*\right).$$

- We can also show that

$$\rho\left(T^k u_0, u^*\right) \leq \frac{1}{1-\lambda} \rho\left(T^k u_0, T^{k-1} u_0\right).$$

BANACH'S CONTRACTION MAPPING THEOREM

- Banach's contraction mapping theorem is a very powerful result.
- First, we can use it to show that a unique solution exists (fixed point).
- Second, it gives us a way to find the fixed point – we can iterate the contraction mapping (successive approximation / fixed point iteration)
- It proves that the fixed point iteration converges. It also gives us a bound on the rate of convergence (λ, the modulus of contraction).
- We often use it to prove the existence of a solution to a functional equation or to find a stationary distribution.

AN EXAMPLE

- Let $U = \mathbb{R}$ and $\|\cdot\| = |\cdot|$, and T be a self-map on U defined by $T(u) = 0.5u + 3$.
- U with $\rho(u, v) = |u - v|$ is a Banach space.
- We have

$$\begin{aligned} \rho(T(u), T(v)) &= \left|T(u) - T(v)\right| \\ &= \left|0.5u + 3 - 0.5v - 3\right| \\ &= 0.5 \cdot \left|u - v\right| \\ &= 0.5 \cdot \rho(u, v). \end{aligned}$$

  so T is a contraction with modulus $\lambda = 0.5$.
- By the CMT, there exists a unique fixed point of T on U: $u^* = 6$.

AN EXAMPLE

- Let $U = \mathbb{R}$ and $\|\cdot\| = |\cdot|$, and T be a self-map on U defined by $T(u) = 0.5u + 3$.
- U with $\rho(u, v) = |u - v|$ is a Banach space.
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  so T is a contraction with modulus $\lambda = 0.5$.
- By the CMT, there exists a unique fixed point of T on U: $u^* = 6$.

AN EXAMPLE

- Another application: Picard-Lindelöf theorem.
- Suppose we have an initival value problem:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0.$$

- If $f(t, \cdot)$ is continuous and bounded and $f(t, \cdot)$ is Lipschitz continuous in y with Lipschitz constant L for every $t \in [t_0 - \alpha, t_0 + \alpha]$, then there exists a unique solution to the problem in the neighborhood of t_0.

BACK TO JOB SEARCH MODEL

BELLMAN EQUATION

- We want to solve the Bellman equation:

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathrm{W}} v\left(w'\right)\phi\left(w'\right)\right\}.$$

- Introduce a Bellman operator defined at $v \in \mathbb{R}^W$ as

$$(Tv)(w) := \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w' \in \mathrm{W}} v\left(w'\right)\phi\left(w'\right)\right\}.$$

- Note: here we treat v as a vector, not a function. We can do it, because W is finite.
- Let $V := \mathbb{R}_+^W$ and let $\|\cdot\|_\infty$ be the supremum norm. Note that V with this norm is a Banach space.

BELLMAN EQUATION

- Notice that

$$\left|\max\{a, x\} - \max\{a, y\}\right| \leq |x - y| \quad \text{for all } a, x, y \in \mathbb{R}.$$

- Take any $f, g \in V$ and fix any $w \in \mathrm{W}$. Use the above to get

$$\begin{aligned}\left|(Tf)(w) - (Tg)(w)\right| v(w) &\leq \beta \left| \sum_{w' \in \mathrm{W}} \left[f\left(w'\right) - g\left(w'\right) \right] \right| \\ &\leq \beta \, \| f - g \|_\infty\end{aligned}$$

- Take the supremum over w to get

$$\| Tf - Tg \|_\infty \leq \beta \, \| f - g \|_\infty .$$

- This proves T is a contraction with modulus β.

BELLMAN EQUATION

- There exists a unique fixed point v^* of T in V.
- The fixed point of the Bellman operator solves the Bellman equation.
- The solution to the Bellman equation is a fixed point of the Bellman operator.
- We can obtain v^* by iterating the Bellman operator:

$$v_{k+1} = Tv_k, \quad k = 0, 1, \ldots .$$

- We can start with any $v_0 \in V$.

APPROACH II

Algorithm Value function iteration

procedure VFI
    $k \leftarrow 1, \epsilon \leftarrow \tau + 1, v_k \leftarrow v_{init}$
    while $\epsilon > \tau$ **do**
        for $w \in W$ **do**
            $v_{k+1}(w) \leftarrow (Tv_k)(w)$
        end for
        $\epsilon \leftarrow \|v_{k+1} - v_k\|_\infty, k \leftarrow k + 1$
    end while
end procedure

OPTIMAL CHOICES

- Once we have v^* we can characterize the optimal choice of the agent.
- We can calculate the continuation value h^*:

$$h^* := c + \beta \sum_{w' \in \mathbb{W}} v^*(w') \phi(w').$$

- Reject the offer if $w/(1-\beta) < h^*$, accept otherwise.
- Denote rejection given wage W_t as $A_t = 0$ and acceptance as $A_t = 1$.

OPTIMAL CHOICES

- Let $A_t = \sigma_t(W_t)$ be the optimal choice of the agent at time t given wage offer W_t.
- We call σ_t the (time t) policy function.
- In this particular case the policy function is

$$\sigma_t(w) = \begin{cases} 1 & \text{if } \frac{w}{1-\beta} \geq h^* \\ 0 & \text{otherwise.} \end{cases}$$

- The policy function here depends only on the current state (wage).
- We call such a policy function (depending on the current state only) Markov policy.

OPTIMAL CHOICES

- A policy is an "instruction manual" for the agent: what to do in each state.
- For an agent following $\sigma \in \Sigma$, if the current wage offer is w, the agent will responds with $\sigma(w) \in \{0, 1\}$.
- For each $v \in V$, a v – greedy policy is a $\sigma \in \Sigma$ satisfying

$$\sigma(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geq c + \beta \sum_{w' \in \mathrm{W}} v\left(w'\right)\phi\left(w'\right)\right\} \quad \text{for all } w \in \mathrm{W}.$$

- The recommendation is: adopt a v^* – greedy policy (notice the superscript!).
- This is a restatement of Bellman's principle of optimality.

LOOKING FORWARD

- Here we have a finite state space – we can treat v as a vector.
- We only used a fraction of the power of dynamic programming...
- What if we move away from finite state and action spaces?
- What are the conditions under which the Bellman operator is a contraction? Is there an easy way to check it?
- Does the principle of optimality hold in general?