# Notes on Firm Investment

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January 24, 2024

# Outline

1. Firm Dynamics

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2. Overview of Investment
   - Basic Optimization Model

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3. Models and Estimation
   - Profit Function
   - No Adjustment Costs
   - Q theory
   - Machine Replacement
   - Continuous Choice

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# Sources of Firm Dynamics

- dynamic factor (capital and labor) demand
- inventories
- prices
- training
- R & D
- advertising
- strategic interactions

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## Overview of Investment

- example of dynamic factor demand
- costs of adjustment imply dynamics
- lumpy adjustment of capital and labor at the plant level
- aggregate is smoother
- what are the adjustment costs?
    - what models do we consider?
    - how do we estimate the parameters?
    - what do we find?

$$V(A,K) = max_{K'}\ \pi(A,K) - p(K' - (1-\delta)K) - C(A,K',K) + \beta E_{A'|A} V(A',K') \quad (1)$$

for all $(A, K)$.

Key Elements

- discount at $\beta(\cdot)$: can allow state dependent SDF
- $\pi(A, K)$ is a reduced form profit function from optimization over flexible factors. This function is strictly increasing and concave. $A = (a, \varepsilon)$.
- $C(\cdot)$ is a general cost of adjustment function
- $A$ plays two roles: influences profits and provides information
- assume problem is bounded and so there exists a solution to the functional equation.

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## Specifications

- No Adjustment Costs
- Quadratic Adjustment Costs
  - Q - Theory
  - Euler Equation Estimation
  - Cooper-Ejarque on SMM
- Nonconvex Adjustment Costs
  - Dynamic Discrete Choice Problem
  - No Q, No Euler
  - Cooper-Haltiwanger on SMM

# Profit Function

$\Pi(A, K) = AK^{\alpha}$

$$\Pi(A, K) = max_x R(\tilde{A}, K, x) - \omega x \qquad (2)$$

where $x$ are flexible factors at a cost of $\omega$ and $R()$ is the revenue function

- with CRS production and constant elasticity demand: $\Pi(A, K) = AK^{\alpha}$
- $\alpha$ reflects factor shares and demand elasticity (CH, RES)
- $A$ combines TFP and Demand Shocks and factor prices

# Profit Function

How to estimate $\Pi(A, K) = AK^{\alpha}$?

- OLS
  - omitted variable bias
  - could use as a moment in SMM
- Quasi-first differences and GMM (Cameron and Trevedi)
- Proxy for Current Productivity

# Quasi-first differences and GMM

- Profits in logs

$$\pi_{it} = a_t + \varepsilon_{it} + \alpha k_{it}$$

- $\varepsilon_{it}$ has serial correlation $\rho$
- take difference using $\rho * \pi_{t-1}$

$$\pi_{it} = \rho * \pi_{it-1} + a_t - \rho * a_{t-1} + \varepsilon_{it} - \rho * \varepsilon_{it-1} + \alpha(k_{it} - \rho * k_{it-1})$$

$$= \rho * \pi_{it-1} + a_t - \rho * a_{t-1} + \alpha(k_{it} - \rho * k_{it-1}) + \zeta_{it}$$

- key is that $\zeta_{it} = \varepsilon_{it} - \rho * \varepsilon_{it-1}$ is a period $t$ innovation uncorrelated with anything prior to period $t$: $E_{t-1}\zeta_{it} = 0$.

# GMM Production Function: No Aggregate Shock

- Orthogonality Condition: firm $i$, instrument $j$, time $t$

$$Ez_{jit-1} \times \overbrace{[\pi_{it} - (\rho * \pi_{it-1} + \alpha(k_{it} - \rho k_{it-1}))]}^{error_{it}} = 0$$

- This holds at truth: $\Theta^* = (\alpha^* \rho^*)$
- moment

$$m^{ji}(\Theta) = \frac{\sum_t (z_{jit-1} error_{it})}{T-1}$$

- $$m^j(\Theta) = \frac{\sum_i m^{ji}(\Theta)}{I}$$

- estimator

$$\hat{\Theta} = \text{argmin}_{\Theta} mWm'$$

- where
  - $$m(\Theta) = (m^1(\Theta), m^2(\Theta) \ldots m^J(\Theta))$$

    is $1xJ$
  - $W$ is a $JxJ$ weighting matrix

# Proxy Method

- Proxy for Current Productivity:
  - $x_{it} = \phi(\varepsilon_{it})$ from optimal choice
  - invert to obtain $\varepsilon_{it} = \phi^{-1}(x_{it})$
  - approximate $\phi^{-1}(x_{it})$
- Condition on Investment Choices (Olley-Pakes)
- Condition on material demand (Levinson-Petrin)
- Parameterize conditional expectation of productivity using lagged inputs (Wooldridge)
- Estimate parameters with GMM

## No Adjustment Costs

- FOC
$$\beta E_{A',p'|A,p} V_k(A',K',p') = p \tag{3}$$
- Euler
$$\beta E_{(A',p'|A,p)}[\Pi_k(A',K') + (1-\delta)p'] = p. \tag{4}$$
- Specify function $\Pi(A,K) = AK^{\alpha}$
- All data or $\Theta$
- Estimate Parameters via GMM
- key is $\Pi_k(A',K') = \alpha \frac{\Pi(A',K')}{K'}$ is inferred given $\alpha$ without observing $A'$

## GMM: No Adjustment Costs

- Euler

$$\beta E_{(A',p'|A,p)}[\Pi_k(A',K') + (1-\delta)p'] = p. \tag{5}$$

- *Ex post*

$$\varepsilon(\Theta) = \beta[\Pi_k(A',K') + (1-\delta)p'] - p. \tag{6}$$

- using time and firm subscripts

$$\varepsilon_{it+1}(\Theta) = \beta[\Pi_k(A_{it+1},K_{it+1}) + (1-\delta)p_{t+1}] - p_t \tag{7}$$

- Theory Implies $E_t[\varepsilon_{it+1}(\Theta)] = 0$ at $\Theta = \Theta^*$
- assume firms are homogenous: i.e. same $\Theta^*$. else estimate by sector

## Methodology: (Overview Only)

- impose sample analogue of $E_t[\varepsilon_{it+1}(\Theta^*)] = 0$
- compute *ex post* errors from data given $\Theta$
- $m_T^{ij}(\Theta) \equiv \frac{1}{T}\sum(Z_t^j \times \varepsilon_{it+1}(\Theta))$ for instrument $j = 1, 2, 3, \ldots J$
- solve $J \equiv min_\Theta m_T(\Theta)' W m_T(\Theta)$ where
  - $m_T$ is vector ($Jx1$) of moments, sample length of $T$
  - $W$ is a weighting matrix; usually the inverse of the VarCov
  - this minimization yields consistent estimates of $\Theta^*$
  - implement iteratively: $W = I$, estimate parameters, calculate VarCov
- See Hamilton GMM Chpt. or Hansen notes for proofs of these properties

## Quadratic Adjustment Costs

- FOC

$$C_{K'}(K',K)+p=\beta E_{(A',p'|A,p)}V_{K'}(A',K',p'). \tag{8}$$

- Euler

$$\begin{aligned}C_{K'}(K',K)+p=\beta E_{(A',p'|A,p)}\{\Pi_K(K',A')+\\ p'(1-\delta)-C_{K'}(K'',K')\}.\end{aligned} \tag{9}$$

- Quadratic Adjustment Costs

$$C(\cdot)=\frac{\gamma}{2}\left(\frac{K'-(1-\delta)K}{K}\right)^2 K \tag{10}$$

## Q Theory

Assume

- $\Pi(A, K) = AK$ perfect competition plus CRS
- quadratic adjustment costs
- $V(A, K) = \phi(A)K$ (Guess and Verify from FOC)
  - $V_K(A, K) = \frac{V(A,K)}{K} = \phi(A)$
  - $$i = \frac{I}{K} = \frac{1}{\gamma}(\beta E_{A'|A} V_K(A', K') - p) = \frac{1}{\gamma}(\beta E_{A'|A}\phi(A') - p) \tag{11}$$
  - from this $V(A, K)$ in (1) is proportional to $K$.

## Empirical Implementation Findings

- Many panel studies
- study:

$$i_{it} = \eta_0 + \eta_1 q_{it} + \eta_2 Z_{it} + error_{it}$$

- $q_{it}$ value of firm divided by capital (average Q), $Z_{it}$ are other variables (eg cash flow)
- theory predicts: $\eta_1 = \frac{1}{\gamma}$ and $\eta_2 = 0$ for any instrument
- Find huge adjustment costs: $\gamma$ big
- Financial Frictions: cash flow is significant in explaining investment, contrary to theory

## Cooper-Ejarque (RED)

- $\Pi(A, K) = AK^{\alpha}$
- SMM matching Q regressions, average Q, etc.
- Find:
  - $\alpha < 1$: market power
  - match Q regression coefficients well
  - no evidence of financial frictions
- earlier findings of financial frictions explained by market power: Average Q $\neq$ Marginal Q

## GMM Approach

- start with Euler

$$C_{K'}(K', K) + p = \beta E_{(A',p'|A,p)}\{\Pi_K(K', A') + p'(1-\delta) - C_{K'}(K'', K')\}. \tag{12}$$

- calculate *ex post* error using form of adjustment cost
- impose orthodogonality
- follow the recipe

# Simple Discrete Choice: Machine Replacement Problem

$$V_\Theta(A, K) = \max\{V^i_\Theta(A, K), V^a_\Theta(A, K)\}$$

inactive

$$V^i_\Theta(A, K) = \Pi(A, K) + \beta E_{A'|A} V(A', K(1-\delta))$$

and

active

$$V^a_\Theta(A, K) = \Pi(A, 1)\lambda - p + \beta E_{A'|A} V(A', (1-\delta)).$$

for all $(A, K)$

- $\lambda$ is an opportunity cost of lumpy investment
- $p$ is the cost of new equipment (fixed cost)
- no time to build
- $A = (a, \varepsilon)$
- value function iteration solution
- parameter vector is $\Theta$

## Special Case

- $A$ is not stochastic
- Optimal Solution is $K^*_\Theta$ such that
  - inactive (no replacement) for $K > K^*_\Theta$
  - replace if $K \leq K^*_\Theta$
  - interesting case has $1 > K^*_\Theta > 0$
- deterministic replacement cycle
- $\Theta$ determines $K^*$: Identification Problem

## Aggregate Implications: No Shocks

- investment rate determined by initial cross sectional distribution by age
- state space is a ladder
- cycles at one extreme, constant investment rate at the other

## More general Problem: Solution and Identification

- $A$ is iid for this part
- Optimal Stopping problem
- $\Delta(A, K) \equiv V^a(A, K) - V^i(A, K)$ is increasing in $K$ given $A$
- Optimal Solution is $K^*_\Theta(A)$ such that replace iff $K \leq K^*_\Theta(A)$ in state $(A, K)$
- identify $\Theta$ through dependence of $K^*(A)$ on $A$.
- $z(a, \varepsilon, K) = 1$ denotes replacement

## Estimation

- no continuous choice so no GMM approach
- identify $\Theta$ through dependence of $K^*_\Theta(A)$ on $A$.
- Moments for SMM
  - average time to replacement
  - logit: probability of replacement as a function of $(A, K)$
  - hazard: probability of replacement as a function of capital vintage
  - etc.

## Aggregate Implications: Shocks

- $I_t$ is aggregate investment rate in period $t$
- depends on probability of adjustment and distribution of $K$.

$$I_t = \sum_K H(A_t, K)\Gamma_t(K). \tag{13}$$

where

$$H(A, K) = \int_\varepsilon z(A, \varepsilon, K)dG(\varepsilon)$$

- The evolution of the cross sectional distribution of capital is given by:

$$\Gamma_{t+1}((1-\delta)K) = (1 - H(A_t, K))\Gamma_t(K) \tag{14}$$

## Continuous Choice

- Cooper-Haltiwanger estimate model with lumpy and continuous choice.
- Choice is:

$$V(A,K) = max_{K'}\ \pi(A,K) - p(K' - (1-\delta)K) - C(A,K',K) + \beta E_{A'|A} V(A',K') \tag{15}$$

  for all $(A,K)$.
- Adjustment costs
  - quadratic
  - fixed cost
  - opportunity cost
  - irreversibility $p^s < p^b = 1$

## Find

- opportunity cost model fits moments better
- irreversibility and small quadratic adjustment costs
- $\alpha \approx 0.60$
- tough to match intermediate capital expenditures

## General Equilibrium

- What happens to the convexities when there is aggregation across plants?
- Cooper-Haltiwanger:
  - fixed $\beta$
  - consider representation firm with quadratic adjustment costs
  - find parameter to match aggregate time series of model with non-convexity at the plant level
  - $R^2$ about 0.85
- Thomas (JPE)
  - RBC with micro non-convexities
  - indistinguishable from standard RBC
  - Key is countercyclical stochastic discount factor