Dynamic Programming: Theory and Applications

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Outline

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1. Asset Pricing
   - Model
   - Estimation
2. Cake Eating
   - Discrete Choice
   - Continuous Choice

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Model

- Price an asset (tree) with stochastic dividends (fruit)

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- Price an asset (tree) with stochastic dividends (fruit)
- Recursive Asset Pricing:

$$q(d) = d + \beta E_{d'|d} q(d') \qquad \forall d \in D \tag{1}$$

- here:
  - d is the current dividend (state)
  - $q(d)$ is the unknown asset price (value function)
  - $\beta \in [0, 1)$
  - stationarity
  - d follows a first-order Markov process by assumption
  - D is the set of possible realizations of d .. often discrete
- goal is to find $q(d)$ solving (1)

Solving the functional equation

- Assume two states: $d \in (d_L, d_H)$
- transition matrix is Π: π_{ij} is the probability of going from state i to j
- Solution solves two equations:

$$\begin{aligned} q(d_L) &= d_L + \beta[\pi_{LL} q(d_L) + (1 - \pi_{LL}) q(d_H)] \\ q(d_H) &= d_H + \beta[\pi_{HH} q(d_H) + (1 - \pi_{HH}) q(d_L)] \end{aligned} \tag{2}$$

- Easy: solve by substitution

Guess/Verify

- assume $d_{t+1} = \rho d_t + \varepsilon_{t+1}$ so $E_{d_{t+1}|d_t} d_{t+1} = \rho d_t$
- $|\rho| < 1$
- Guess/Verify: $q(d) = \xi d$
  - $$\xi d = d + \beta \xi E_{d'|d} d'$$
    for all d
  - $$\xi d = d + \beta \xi \rho d$$
    for all d
  - $$\xi = \frac{1}{1 - \beta\rho}$$
- fails if $d_{t+1} = \mu_d + \rho d_t + \varepsilon_{t+1}$
- guess/verify only works for special problems

Value Function Iteration

- Overkill for this problem, essential for others
- iteration indexed by i
- initial guess, $q^0(d) \qquad \forall d \in D$
- D is discrete, transition matrix Π
- iterate

$$q^{i+1}(d) = d + \beta E_{d'|d} q^i(d') \qquad \forall d \in D \tag{3}$$

- will converge to solution of the functional equation regardless of $q^0(d)$ if discounting and monotonicity (Blackwell)

Estimating Parameters

- Asset Pricing Model

$$q(d) = u(d) + \beta E_{d'|d} q(d') \qquad \forall d \in D \tag{4}$$

- Data: (q_t, d_t) for $t = 1, 2, 3 \ldots T$
- Want to estimate: $\Theta = (\Pi, \beta, u(\cdot))$
- How can we do that?
- Estimate Π directly from observations on dividends ...
  - count transitions
  - use Tauchen Procedure to create a discrete representation of the AR(1) process
- Estimate $(\beta, u(\cdot))$ using: GMM, SMM, non-linear least squares

SMM example

- estimate β, assume $u(d) = d$ and Π known
- compute a data moment: eg. mean of q, denoted μ^d
- estimation loop
  - solve (4) for a given β to get $q_\Theta(d)$
  - simulate a data set using Π and $q_\Theta(d)$
  - calculate the mean of q from the simulated data, $\mu^s(\beta)$
  - find β to minimize $(\mu^s(\beta) - \mu^d)^2$
- Exercise: Graph simulated and data moments as a function of β

Extensions

- estimate more parameters
- overidentified model:
  - more moments than parameters
  - test fit of theory as well as estimate parameters
- Indirect Inference
  - obtain moments from a regression: $q_t = \alpha_0 + \alpha_1 d_t + \alpha_2 {d_t}^2$
  - match these regression coefficients in the simulated data
- GMM
  - if there was a choice
  - obtain ex post error from: $\varepsilon_{t+1}(\Theta) = q_t - u(d_t) - \beta q_{t+1}$
  - theory implies $E_t \varepsilon_{t+1}(\Theta^*) = 0$
  - use sample analogue to estimate Θ

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Dynamic Discrete Choice

$$V(W,\varepsilon) = \max(V^E(W,\varepsilon), V^N(W,\varepsilon)) \tag{5}$$

$$V^E(W,\varepsilon) = \varepsilon u(W) \tag{6}$$

and

$$V^N(W,\varepsilon) = \beta E_{\varepsilon'|\varepsilon} V(\rho W,\varepsilon'). \tag{7}$$

for all (W,ε)

Policy Function

- $Z(W,\varepsilon) \in \{0,1\}$, probability of eating the cake
- Leading Example
  - Assume ε is iid, $\rho = 1$.
  - $Z(W,\varepsilon) = 1$ iff $\varepsilon \geq \varepsilon^*$ for all W.
  - usual "reservation wage" structure

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Estimation

- Estimate $\Theta \equiv (\beta, \Pi, u(\cdot))$
- $\varepsilon^*(\Theta)$: actions depend on parameters
- What can we identify here? Not Much!

Two State Example

- $\varepsilon \in \{\varepsilon_L, \varepsilon_H\}$
- $Z(W, \varepsilon_H) = 1$ for sure
- $Z(W, \varepsilon_L) =???$
- create program
  - to illustrate VFI approach for this example
  - study policy function and identification

Continuous Choice

$$V(W,\varepsilon) = \max_{W'} \varepsilon u(W - W') + \beta E_{\varepsilon'|\varepsilon} V(W',\varepsilon') \qquad \forall (W,\varepsilon) \quad (8)$$

- ε is a taste shock: exogenous state
- W is the size of the cake: endogenous state
- control is future cake (or could use consumption)
- policy function: $W' = \phi(W,\varepsilon)$
- characterize solution through an Euler equation

Conditions for Optimality

- FOC

$$\varepsilon u'(W - W') = \beta E_{\varepsilon'|\varepsilon} V_1(W', \varepsilon') \tag{9}$$

- Envelope Condition

$$V_1(W, \varepsilon) = \varepsilon u'(W - W') \tag{10}$$

- Euler Equation

$$\varepsilon u'(W - W') = \beta E_{\varepsilon'|\varepsilon} \varepsilon' u'(W' - W") \tag{11}$$

- Condition for ex-ante optimality: no errors anticipated

Estimation Approach

- estimate $\Theta = (\beta, u(\cdot))$
- Ex-post Error in Time Domain:

$$\zeta_{t+1}(\Theta) = \frac{\beta \varepsilon_{t+1} u'(c_{t+1})}{\varepsilon_t u'(c_t)} - 1 \qquad (12)$$

- Orthogonality: $E_t \zeta_{t+1}(\Theta^*) = 0$
- estimate Θ from sample analogue ala Hansen-Singleton
- under conditions for doing this in the next lecture