Variational Image Triangulation for Poly-Art Design: Supplemental Materials

A Gradient Derivations

A.0.1 Constant Case

When g_T is constant, its optimal value is such that

$$\begin{aligned}
\frac{\partial}{\partial g_T}\int_T (f-g_T)^2\sigma dA &= 0\\
\int_T 2(f-g_T)\sigma dA &= 0\\
g_T\int_T \sigma dA &= \int_T f\sigma dA\\
g_T &= \frac{\int_T f\sigma dA}{\int_T \sigma dA}
\end{aligned}$$

Let $W_T = \displaystyle\int_T \sigma dA$ denote the saliency-weighted area of triangle T so that

$$g_T = \frac{1}{W_T}\int_T f\sigma dA.$$

We can rewrite the energy function as

$$\begin{aligned}
E(\mathcal{T}) &= \sum_{T\in\mathcal{T}}\int_T (f-g_T)^2\sigma dA\\
&= \sum_{T\in\mathcal{T}}\left(\int_T f^2\sigma dA - 2g_T\int_T f\sigma dA + g_T^2 W_T\right)\\
&= \sum_{T\in\mathcal{T}}\int_T f^2\sigma dA - \sum_{T\in\mathcal{T}}\frac{1}{W_T}\left(\int_T f\sigma dA\right)^2
\end{aligned}$$

where the first sum is a constant value. Thus we only need to take the gradient of the second term.

Applying Reynold's transport Theorem, the gradient of $\displaystyle\int_T f\sigma dA$ is

$$\frac{d}{dt}\int_{T(t)} f\sigma dA = \int_{\partial T(t)} (\mathbf{v}\cdot\mathbf{n}) f\sigma ds$$

because $f\sigma$ do not vary with time. Similarly, the derivative of W_T, is

$$\frac{d}{dt}\int_{T(t)}\sigma dA = \int_{\partial T(t)}(\mathbf{v}\cdot\mathbf{n})\sigma ds$$

because σ is not time varying. With these two pieces, we can compute the gradient

$$\begin{aligned}
\frac{d}{dt}E(\mathcal{T}) &= -\sum_{T\in\mathcal{T}}\frac{d}{dt}\left[\frac{\left(\int_{T(t)} f\sigma dA\right)^2}{W_{T(t)}}\right] \\
&= -\sum_{T\in\mathcal{T}}\frac{1}{W_{T(t)}^2}\left[W_{T(t)}\frac{d}{dt}\left(\int_{T(t)} f\sigma dA\right)^2\right. \\
&\qquad\left. - \left(\int_{T(t)} f\sigma dA\right)^2\frac{d}{dt}W_{T(t)}\right] \\
&= -\sum_{T\in\mathcal{T}}\frac{1}{W_{T(t)}^2}\left[2W_{T(t)}\left(\int_{T(t)} f\sigma dA\right)\left(\int_{\partial T(t)}(\mathbf{v}\cdot\mathbf{n})f\sigma ds\right)\right. \\
&\qquad\left. - \left(\int_{T(t)} f\sigma dA\right)^2\left(\int_{\partial T(t)}(\mathbf{v}\cdot\mathbf{n})\sigma ds\right)\right]
\end{aligned}$$

This produces a gradient flow that gives higher priority to the salient features of the image.

A.0.2 Linear Case

In the piecewise linear case, again we first derive the optimal colors for a fixed triangulation. We have $g_T = c_0\phi_0 + c_1\phi_1 + c_2\phi_2$ and

$$\begin{aligned}
\frac{\partial}{\partial c_j}\int_T (f-g_T)^2\sigma dA &= 0 \\
\int_T 2(f-g_T)\phi_j\sigma dA &= 0 \\
\int_T g_T\phi_j\sigma dA &= \int_T f\phi_j\sigma dA
\end{aligned}$$

Let $S_j = \int_T f\phi_j\sigma dA$, and $S_{ij} = \int_T \phi_i\phi_j\sigma dA$. Expanding $g_T = \sum c_i\phi_i$ we get the system

$$\sum_i c_i S_{ij} = S_j \tag{1}$$

This gives us the optimal linear color scheme for a fixed triangulation as $\{c_i\} = \{S_j\}\{S_{ij}\}^{-1}$.

By color optimality we have

$$\int_T 2(f-g_T)\frac{\partial g_T}{\partial c_j}dA = 0$$
$$\int_T g_T\phi_j dA = \int_T f\phi_j dA$$
$$\sum_j c_j \int_T g_T\phi_j\sigma dA = \sum_j c_j \int_T f\phi_j\sigma dA$$
$$\int_T g_T \sum_j c_j\phi_j\sigma dA = \int_T f \sum_j c_j\phi_j\sigma dA$$
$$\int_T g_T^2\sigma dA = \int_T f g_T\sigma dA$$

Therefore we may rewrite the energy as

$$\begin{aligned} E(\mathcal{T}) &= \sum_{T\in\mathcal{T}}\int_T (f-g_T)^2\sigma dA \\ &= \sum_{T\in\mathcal{T}}\int_T f^2\sigma dA - \sum_{T\in\mathcal{T}}\int_T f g_T\sigma dA \end{aligned}$$

The first sum is constant, so

$$\begin{aligned} \frac{d}{dt}E(\mathcal{T}) &= -\frac{d}{dt}\sum_{T\in\mathcal{T}}\int_T f g_T\sigma dA \\ &= -\sum_{T\in\mathcal{T}}\sum_j \frac{d}{dt}\left(c_j\int_T f\phi_j\sigma dA\right) \\ &= -\sum_{T\in\mathcal{T}}\sum_j \left[\left(\frac{d}{dt}c_j\right)S_j + c_j\left(\frac{d}{dt}S_j\right)\right] \qquad (2) \end{aligned}$$

In order to evaluate this gradient, we need $c_j, S_j, \frac{d}{dt}c_j$, and $\frac{d}{dt}S_j$. We already showed how to compute S_j and c_j, so all we need are the derivatives. The computation for $\frac{d}{dt}S_j$ is exactly the same as in (8) with all integrals weighted by σdA instead of just dA. This comes from straightforward application of Reynold's Transport Theorem . However, the computation for $\frac{d}{dt}c_j$ is now significantly messier.

We can differentiate the system (1) to get a new system

$$\sum_i c_i\left(\frac{d}{dt}S_{ij}\right) + \left(\frac{d}{dt}c_i\right)S_{ij} = \frac{d}{dt}S_j \;\forall j \qquad (3)$$

We have c_i, S_{ij}, and $\frac{d}{dt}S_j$ from previous computations. If we can evaluate $\frac{d}{dt}S_{ij}$, then we can treat (3) as a linear system in $\frac{d}{dt}c_i$ and extract these derivatives by solving the system.

To compute $\frac{d}{dt}S_{ij}$, we need to apply the Reynolds Transport Theorem yet again.

$$\frac{d}{dt}\int_{T(t)}\phi_i\phi_j\sigma dA = \int_{T(t)}\frac{d}{dt}(\phi_i\phi_j)\sigma dA + \int_{\partial T(t)}(\mathbf{v}\cdot\mathbf{n})\phi_i\phi_j\sigma dA$$

The second integral can be evaluated numerically. The first integral is evaluated using

$$\frac{d}{dt}(\phi_i \phi_j) = \phi_i \frac{d\phi_j}{dt} + \phi_j \frac{d\phi_i}{dt}$$

where

$$\frac{d\phi_i(x,y)}{dt} = \frac{\partial \psi_i(u,v)}{\partial u}\frac{du}{dt} + \frac{\partial \psi_i(u,v)}{\partial v}\frac{dv}{dt}.$$

From here we can compute the value of $\frac{d}{dt}(\phi_i \phi_j)$ at each point in triangle T, and thus we can numerically integrate $\int_{T(t)} \frac{d}{dt}(\phi_i \phi_j)\sigma dA$. Then we can obtain $\frac{d}{dt}\int_{T(t)} \phi_i \phi_j \sigma dA$ and use these values to solve system (3) for $\frac{d}{dt}c_i$. Finally, plugging this back into (2) gives the gradient of the saliency-weighted linear approximation.