HOMEWORK_7

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Q6.1

$\forall 0 \le t_0 < t_1 < \cdots < t_{n-1} < t_n < t_{n+1}$以及$\forall i_0, i_1, \cdots, i_n, i_{n+1} \in S$，设$T(i,n) = \begin{cases} 2n, & 2 \mid i \\ 2n+1, & 2 \nmid i \end{cases}$，有

$$\begin{aligned}
&P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n, X_{t_{n-1}} = i_{n-1}, \cdots, X_{t_0} = i_0) && (1)\\
=&P(\cup_{k=0}^{\infty}(N(t_{n+1}) - N(t_n) = T(i_{n+1} - i_n, k)) | \cup_{k=0}^{\infty} (N(t_n) - N(t_{n-1}) = T(i_n - i_{n-1}, k))) && (2)\\
&\cdots P(\cup_{k=0}^{\infty}(N(t_0) = T(i_0 - 1, k)) && (3)\\
=&P(\cup_{k=0}^{\infty}(N(t_{n+1}) - N(t_n) = T(i_{n+1} - i_n, k))) && (4)
\end{aligned}$$

另一方面，

$$\begin{aligned}
&P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n) && (5)\\
=&P(\cup_{k=0}^{\infty}(N(t_{n+1}) - N(t_n) = T(i_{n+1} - i_n, k)) | \cup_{k=0}^{\infty} (N(t_n) = T(i_n, k)) && (6)\\
=&P(\cup_{k=0}^{\infty}(N(t_{n+1}) - N(t_n) = T(i_{n+1} - i_n, k))) && (7)
\end{aligned}$$

所以

$$P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n, X_{t_{n-1}} = i_{n-1}, \cdots, X_{t_0} = i_0) = P(X_{t_{n+1}} = i_{n+1} | X_{t_n} = i_n) \tag{8}$$

也即$\{X(t) : t \ge 0\}$是马尔可夫链。

由上可得，转移概率

$$\begin{aligned}
p_{ij}(t) =&P(X_{t+s} = j | X_s = i) = P(\cup_{k=0}^{\infty}(N(t+s) - N(s) = T(j - i, k))) && (9)\\
=&\sum_{k=0}^{\infty} \frac{(\lambda t)^{T(j-i,k)}}{T(j-i,k)!} e^{-\lambda t} && (10)
\end{aligned}$$

当$j = 0, i = 0$或$j = 1, i = 1$时，$j - i = 0$，$T(j - i, k) = 2k$，从而

$$p_{ij}(t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k}}{(2k)!} e^{-\lambda t} = A(t) \tag{11}$$

当$j = 1, i = 0$或$j = 0, i = 1$时，$2 \nmid j - i$，$T(j - i) = 2k + 1$，

$$p_{ij}(t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(2k+1)!} e^{-\lambda t} = B(t) \tag{12}$$

所以

$$\mathbf{P}(t) = \begin{pmatrix} A(t) & B(t) \\ B(t) & A(t) \end{pmatrix} \tag{13}$$

有$\lim_{t \to 0} A(t) = 1$，$\lim_{t \to 0} B(t) = 0$，因此$\lim_{t \to 0} \mathbf{P}(t) = \mathbf{I}$，因此$\mathbf{P}$是标准的。

考虑$\mathbf{Q}=(q_{ij})$。

当$i=j$时，$i=0,1$，此时

$$q_{ii}=p_{ii}^{'}(0_+)=\left\{-\lambda\sum_{k=0}^{\infty}\frac{(\lambda t)^{(2k)}}{(2k)!}e^{-\lambda t}+\sum_{k=1}^{\infty}\frac{\lambda\cdot(\lambda t)^{2k-1}}{(2k-1)!}e^{-\lambda t}\right\}\Bigg|_{t\to 0_+}=-\lambda \tag{14}$$

当$i\neq j$时，$i=0,j=1$或$i=1,j=0$，此时

$$q_{ij}=p_{ij}^{'}(0_+)=-\lambda\sum_{k=0}^{\infty}\frac{(\lambda t)^{2k+1}}{(2k+1)!}e^{-\lambda t}+\sum_{k=0}^{\infty}\frac{\lambda\cdot(\lambda t)^{2k}}{(2k)!}e^{-\lambda t}\Bigg|_{t\to 0_+}=\lambda \tag{15}$$

从而

$$\mathbf{Q}=\begin{pmatrix}-\lambda & \lambda\\ \lambda & -\lambda\end{pmatrix} \tag{16}$$

Q6.26

证明：首先证明$\forall i\in S$，若$\int_0^\infty P_{ii}(t)dt=\infty$，则$\forall h>0$，均有$\sum_{n=1}^\infty P_{ii}(nh)=\infty$成立。

先证明至少存在一个$h>0$，使得$\sum_{n=1}^\infty P_{ii}(nh)=\infty$成立。反证法。假设对于所有$h>0$，均有$\sum_{n=1}^\infty P_{ii}(nh)<\infty$。令$f(h)=\sum_{n=1}^\infty P_{ii}(nh)$，由于$P_{ii}(nh)$连续，则$f(h)$在$h\in(0,1]$上连续，从而由积分中值定理，有

$$\exists h^{'}\in(0,1],\quad \int_0^1 f(h)dh=f(h^{'})<+\infty \tag{17}$$

另一方面，

$$\begin{aligned}
\int_0^1 f(h)dh &= \int_0^1\sum_{n=1}^\infty P_{ii}(nh)dh && (18)\\
&=\sum_{n=1}^\infty\int_0^1 P_{ii}(nh)dh && (19)\\
&=\sum_{n=1}^\infty\int_0^n\frac{P_{ii}(h)}{n}dh && (20)\\
&=\sum_{n=1}^\infty\frac{1}{n}\sum_{k=1}^n\int_{k-1}^k P_{ii}(h)dh && (21)\\
&=\sum_{n=1}^\infty\frac{1}{n}\sum_{k=1}^n T_k \qquad \text{令}T_k=\int_{k-1}^k P_{ii}(h)dh,k\in\mathbb{N}^+ && (22)\\
&=\sum_{k=1}^\infty(\sum_{n=k}^\infty\frac{1}{n})T_k && (23)
\end{aligned}$$

而考虑到$\sum_{n=k}^\infty\frac{1}{n}$为调和级数的一部分，则$\forall k$，一定存在$m>k$，$\sum_{n=k}^\infty\frac{1}{n}>\sum_{n=k}^m\frac{1}{n}>1$。从而

$$\int_0^1 f(h)dh=\sum_{k=1}^{\infty}(\sum_{n=k}^{\infty}\frac{1}{n})T_k>\sum_{k=1}^{\infty}T_k=\int_0^{\infty}P_{ii}(t)dt=\infty \tag{24}$$

矛盾。从而一定存在$h_0>0$，使得$\sum_{n=1}^{\infty}P_{ii}(nh_0)$成立。考虑对任意的$h>0$，不妨设$h>h_0$，否则取$bh>h_0, b\in\mathbb{N}^+$即可。设$h=kh_0+\epsilon_0$，$0\le\epsilon<h_0$，$k\in\mathbb{N}^+$。

令$h_0^{'}=(k+1)h_0$。则$\forall n\in\mathbb{N}^+$，一定存在某个$t_n\in\mathbb{N}^+$，使得$nh_0^{'}\le t_nh<(n+1)h_0^{'}$。否则有$h>(n+1)h_0^{'}-nh_0^{'}=h_0^{'}=(k+1)h_0>h$，矛盾。考虑数列$\{t_nh\}_{n\ge1}$，设$t_nh=nh_0^{'}+\epsilon_n$，其中$\epsilon_n\in[0,h_0^{'})$。由于$P_{ii}(t)$在$t\in[0,h_0^{'})$上连续，且在$t=0$处右连续，$P_{ii}(0)=1$，$P_{ii}(h_0^{'})>0$，因此$P_{ii}(t)$在区间$[0,h_0^{'})$上存在最小值，设为$M>0$，即$\forall n\in\mathbb{N}^+$，$P_{ii}(\epsilon_n)\ge M$。从而$\forall n\in\mathbb{N}^+$，有

$$P_{ii}(t_nh)=P_{ii}(nh_0^{'}+\epsilon_n)\ge P_{ii}(nh_0^{'})P(\epsilon_n)\ge MP_{ii}(nh_0^{'})=MP_{ii}(n(k+1)h_0) \tag{25}$$

因此

$$\sum_{n=1}^{\infty}P_{ii}(nh)\ge\sum_{n=1}^{\infty}P_{ii}(nt_nh)\ge M\sum_{n=1}^{\infty}P_{ii}(n(k+1)h_0) \tag{26}$$

而考虑一系列概率，

$$\begin{aligned}&\{P_{ii}((n-1)(k+1)h_0+h_0),P_{ii}((n-1)(k+1)h_0+2h_0),\cdots,P_{ii}(n(k+1)h_0)\}\\&=\{P_{ii}((n-1)(k+1)h_0+sh_0)\}_{s\in\{1,2,\cdots,k+1\}},\quad\forall n\ge1\end{aligned} \tag{27}$$

有$\forall s\in\{1,2,\cdots,k\}$

$$\begin{aligned}P_{ii}(n(k+1)h_0)&=P_{ii}((n-1)(k+1)h_0+sh_0+(k+1-s)h_0) &(28)\\&\ge P_{ii}((n-1)(k+1)h_0+sh_0)P_{ii}((k+1-s)h_0) &(29)\\&\ge P_{ii}((n-1)(k+1)h_0+sh_0)\inf\{P_{ii}(th_0)\}_{t\in\{1,2,\cdots,k\}} &(30)\\&=M_kP_{ii}((n-1)(k+1)h_0+sh_0),\quad 令M_k=\inf\{P_{ii}(th_0)\}_{t\in\{1,2,\cdots,k\}}>0 &(31)\end{aligned}$$

从而

$$P_{ii}((n-1)(k+1)h_0+sh_0)\le\frac{1}{M_k}P_{ii}(n(k+1)h_0) \tag{32}$$

进一步地，有

$$\sum_{n=1}^{\infty}P_{ii}((n-1)(k+1)h_0+sh_0)\le\frac{1}{M_k}\sum_{n=1}^{\infty}P_{ii}(n(k+1)h_0),\quad\forall s\in\{1,2\cdots,k\} \tag{33}$$

所以

$$\infty = \sum_{n=1}^{\infty} P_{ii}(nh_0) \tag{34}$$

$$= \sum_{n=1}^{\infty} \sum_{s=1}^{k+1} P_{ii}((n-1)(k+1)h_0 + sh_0) \tag{35}$$

$$= \sum_{s=1}^{k+1} \sum_{n=1}^{\infty} P_{ii}((n-1)(k+1)h_0 + sh_0) \tag{36}$$

$$\leq (\frac{k}{M_k} + 1) \sum_{n=1}^{\infty} P_{ii}(n(k+1)h_0) \tag{37}$$

因此

$$\sum_{n=1}^{\infty} P_{ii}(n(k+1)h_0) = \infty \Longrightarrow \sum_{n=1}^{\infty} P_{ii}(nh) = \infty \tag{38}$$

得证。

反过来，若对某一个$h > 0$，有$\sum_{n=1}^{\infty} P_{ii}(nh) = \infty$，下证明$\int_0^{\infty} P_{ii}(t)dt = \infty$。

设$M_h = \inf \{P_{ii}(t)\}_{t\in[0,h)} > 0$。当$t \in [(n-1)h, nh)$，$n \in \mathbb{N}^+$时，设$t = (n-1)h + \epsilon$，其中$\epsilon \in [0, h)$，此时有

$$P_{ii}(t) = P_{ii}((n-1)h + \epsilon) \geq P_{ii}((n-1)h)P_{ii}(\epsilon) \geq M_h P_{ii}((n-1)h) \tag{39}$$

因此有

$$\int_0^{\infty} P_{ii}(t)dt = \sum_{n=1}^{\infty} \int_{(n-1)h}^{nh} P_{ii}(t)dt \tag{40}$$

$$\geq \sum_{n=1}^{\infty} \int_{(n-1)h}^{nh} M_h P_{ii}((n-1)h)dt \tag{41}$$

$$= hM_h \sum_{n=1}^{\infty} P_{ii}((n-1)h) \tag{42}$$

$$\geq hM_h \sum_{n=1}^{\infty} P_{ii}(nh) = \infty \tag{43}$$

得证。

Q6.2

1. EX_t

   有

$$EX_t = P(X_t = 1) = P_{01}(t) \tag{44}$$

   将Q进行特征值分解，得到

$$Q = P \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{pmatrix} P^{-1} = P\Sigma P^{-1} \tag{45}$$

其中

$$P=\begin{pmatrix}1 & \lambda \\ 1 & -\mu\end{pmatrix},\quad P^{-1}=\frac{1}{\lambda+\mu}\begin{pmatrix}\mu & \lambda \\ 1 & -1\end{pmatrix} \tag{46}$$

所以

$$\begin{aligned}
\mathbb{P}(t) =& e^{tQ}=\sum_{n=0}^{\infty}\frac{(tQ)^n}{n!} &(47)\\
=& \sum_{n=0}^{\infty}\frac{t^n P\Sigma^n P^{-1}}{n!} &(48)\\
=& P\sum_{n=0}^{\infty}\frac{(t\Sigma)^n}{n!}P^{-1} &(49)\\
=& Pe^{t\Sigma}P^{-1} &(50)\\
=& \begin{pmatrix}1 & \lambda \\ 1 & -\mu\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & e^{-(\lambda+\mu)t}\end{pmatrix}\frac{1}{\lambda+\mu}\begin{pmatrix}\mu & \lambda \\ 1 & -1\end{pmatrix} &(51)\\
=& \frac{1}{\lambda+\mu}\begin{pmatrix}\mu+\lambda e^{-(\lambda+\mu)t} & \lambda(1-e^{-(\lambda+\mu)t}) \\ \mu(1-e^{-(\lambda+\mu)t}) & \lambda+\mu e^{-(\lambda+\mu)t}\end{pmatrix} &(52)
\end{aligned}$$

所以

$$EX_t=P_{01}(t)=\frac{\lambda}{\lambda+\mu}-\frac{\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu} \tag{53}$$

2. $E(\tau_1|X_0=0)$。

   因为

$$P(\tau_1<t|X_0=0)=e^{-q_0t} \tag{54}$$

   所以

$$E(\tau_1|X_0=0)=\frac{1}{q_0}=\frac{1}{\lambda} \tag{55}$$

Q6.15

(1)

根据条件概率定义，以及$X=\{X(t):t\geq 0\}$处于稳态，有

$$P\left\{X(t)=i|X(t)\in B\right\}=\frac{P_i}{\sum_{j\in B}P_j} \tag{56}$$

(3)

证明：

有

$$\tilde{F}_i(s) = E\left\{e^{-sT_i}|X(0)=i\right\} \tag{57}$$

$$= \int_0^\infty e^{-st} P(T_i = t|X(0)=i)dt \tag{58}$$

及$P_{ij} = \frac{q_{ij}}{\sum_{j\neq i} q_{ij}} = \frac{q_{ij}}{q_i}$而

$$P(T_i = t|X(0)=i) = \sum_{j\in B}\int_0^t P(\tau_1 = u, X_u = j|X(0)=i)P(T_j = t-u|X(0)=j)du \tag{59}$$

$$+\sum_{j\in G} P(\tau_1 = t, X_t = j|X(0)=i) \tag{60}$$

$$= \sum_{j\in B}\int_0^t q_i e^{-q_i u} P_{ij} P(T_j = t-u|X(0)=j)du + \sum_{j\in G} q_i e^{-q_i t} P_{ij} \tag{61}$$

故

$$\begin{aligned}\tilde{F}_i(s) = &\int_0^\infty e^{-st}\sum_{j\in B}\int_0^t q_i e^{-q_i u} P_{ij} P(T_j = t-u|X(0)=j)dudt \\ &+ \int_0^\infty e^{-st}\sum_{j\in G} q_i e^{-q_i t} P_{ij} dt\end{aligned} \tag{62}$$

而

$$\int_0^\infty e^{-st}\sum_{j\in B}\int_0^t q_i e^{-q_i u} P_{ij} P(T_j = t-u|X(0)=j)dudt \tag{63}$$

$$= q_i \int_0^\infty e^{-(s+q_i)u}\sum_{j\in B} P_{ij}\int_u^\infty e^{-s(t-u)} P(T_j = t-i|X(0)=j)dtdu \tag{64}$$

$$= q_i \int_0^\infty e^{-(s+q_i)u}\sum_{j\in B} P_{ij}\int_0^\infty e^{-st} P(T_j = t|X(0)=j)dtdu \tag{65}$$

$$= q_i \int_0^\infty e^{-(s+q_i)u}\sum_{j\in B} P_{ij}\tilde{F}_j(s)du \tag{66}$$

$$= q_i \sum_{j\in B} P_{ij}\tilde{F}_j(s)(\int_0^\infty e^{-(s+q_i)u}du) = \frac{q_i}{q_i+s}\sum_{j\in B} P_{ij}\tilde{F}_j(s) \tag{67}$$

另一方面,

$$\int_0^\infty e^{-st}\sum_{j\in G} q_i e^{-q_i t} P_{ij} dt \tag{68}$$

$$= q_i \sum_{j\in G} P_{ij}\int_0^\infty e^{-(s+q_i)t}dt \tag{69}$$

$$= \frac{q_i}{q_i+s}\sum_{j\in G} P_{ij} \tag{70}$$

将二者相加,得到

$$\tilde{F}(s) = q_i(q_i+s)^{-1}[\sum_{j\in B} P_{ij}\tilde{F}_j(s) + \sum_{j\in G} P_{ij}] \tag{71}$$

得证。

Q6.17

(1)

考虑生灭过程，即有马尔可夫链$X=\{X(t):t\geq 0\}$，$S=\{0,1,2\cdots,\}$，若$P(t)=(P_{ij}(t))$满足当h充分小时，

$$\begin{cases} P_{i,i+1}(h)=\lambda_i h+o(h), & \lambda_i\geq 0, i\geq 0 \\ P_{i,i-1}(h)=\mu_u h+o(h), & \mu_i\geq 0, i\geq 1 \\ P_{ii}(h)=1-(\lambda_i+\mu_i)h+o(h), & \mu_0=0, i\geq 0 \\ \sum_{|j-i|\geq 2} P_{ij}(h)=o(h), & i\geq 0 \end{cases} \tag{72}$$

有X是常返链⇔平稳分布唯一存在。根据课本定理6.4.2，知生灭过程X存在唯一平稳分布⇔ $\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}<\infty$。因此$X$是非常返链⇔ $\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}=\infty$。

根据题意，有$\lambda_n=n\lambda+\delta$，$\delta>0$，$\mu_n=n\mu$，$\mu_0=0$。

1. $\lambda>\mu$。此时

$$\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}=\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{\lambda_i}{\mu_{i+1}} \tag{73}$$

$$=\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{i\lambda+\delta}{(i+1)\mu} \tag{74}$$

$$>\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{i\lambda}{(i+1)\mu} \tag{75}$$

取充分大的N，使得$1>\frac{N}{N+1}>\frac{\mu}{\lambda}$，记$\prod_{i=0}^{N-1}\frac{i\lambda}{(i+1)\mu}$，则

$$\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}>\sum_{k=N+1}^{\infty}S\prod_{i=N}^{k-1}\frac{i}{i+1}\cdot\frac{\lambda}{\mu}>\sum_{k=N+1}^{\infty}S=\infty \tag{76}$$

2. $\lambda=\mu<\delta$。此时

$$\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}=\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{i\lambda+\delta}{(i+1)\lambda} \tag{77}$$

$$>\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\frac{i\lambda+\lambda}{(i+1)\lambda} \tag{78}$$

$$=\sum_{k=1}^{\infty}1=\infty \tag{79}$$

综上，当$\lambda>\mu$或$\lambda=\mu<\delta$时，$\sum_{k=1}^{\infty}\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}=\infty$，则链为非常返的，得证。