HOMEWORK_11

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Q5.4

有$B(t) \sim \mathcal{N}(0,t)$。所以

$$P(|B(t)| \leq s) = P(-s \leq B(t) \leq s) = \int_{-s}^{s} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx, \quad \forall s \geq 0 \tag{1}$$

所以

$$f_{|B(t)|}(x) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathit{1}_{\{x \geq 0\}} \tag{2}$$

另外

$$\begin{aligned} P(M(t) \leq s) =& 1 - P(M(t) > s) = 1 - P(T_s < t) = 1 - 2(1 - \Phi(\frac{s}{\sqrt{t}})) = 2\Phi(\frac{s}{\sqrt{t}}) - 1 & (3) \\ =& \int_{-s}^{s} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = P(|B(t)| \leq s) & (4) \end{aligned}$$

且根据对称性

$$P(|\min_{0 \leq s \leq t} B(s)| \leq a) = P(|M(t)| \leq a) = P(M(t) \leq a) \quad (\because M(t) \geq B_0 = 0) \tag{5}$$

所以$|\min_{0 \leq s \leq t} B(s)|$和$M(t)$都与$|B(t)|$同分布，二者的分布函数与$|B(t)|$的相同。

最后

$$\begin{aligned} \delta(t) =& M(t) - B(t) = \max_{0 \leq s \leq t} B(s) - B(t) & (6) \\ =& \max_{0 \leq s \leq t} (B(s) - B(t)) & (7) \end{aligned}$$

所以

$$\begin{aligned} P(\delta(t) \leq a) &= P(\max_{0 \leq s \leq t} (B(s) - B(t)) \leq a) & (8) \\ =& P(\max_{0 \leq s \leq t} B(s-t) \leq a) & (9) \\ =& P(\max_{0 \leq s \leq t} B(-s) \leq a) = P(\max_{0 \leq s \leq t} B(s) \leq a) = P(M(t) \leq a) & (10) \end{aligned}$$

因此$\delta(t)$与$M(t)$是同分布，它的分布函数也为

$$f_{\delta(t)}(x) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathit{1}_{\{x \geq 0\}} \tag{11}$$

Q5.21

证明定理5.2.4a

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} B(t_{k-1} + \theta(t_k - t_{k-1}))(B(t_k) - B(t_{k-1})) \overset{m.s}{=} \frac{1}{2} B^2(t) + \frac{1}{2}(2\theta - 1)t \tag{12}$$

其中$0 \leq \theta \leq 1$。

证明:

令$C_k = B(t_{k-1} + \theta(t_k - t_{k-1})) - B(t_{k-1})$，$D_k = B(t_k) - B(t_{k-1} + \theta(t_k - t_{k-1}))$，$1 \leq k \leq n$，有

$$\begin{aligned}
&\sum_{k=1}^{n} B(t_{k-1} + \theta(t_k - t_{k-1}))(B(t_k) - B(t_{k-1})) \qquad (13)\\
=&\frac{1}{2}\sum_{k=1}^{n}[B^2(t_k) - B^2(t_{k-1}) + (B(t_{k-1} + \theta(t_k - t_{k-1})) - B(t_{k-1}))^2 \qquad (14)\\
&-(B(t_k) - B(t_{k-1} + \theta(t_k - t_{k-1})))^2] \qquad (15)\\
=&\frac{1}{2}\left[B^2(t) + \sum_{k=1}^{n}(C_k^2 - D_k^2)\right] \qquad (16)
\end{aligned}$$

而

$$\begin{aligned}
&\sum_{k=1}^{n}(C_k^2 + D_k^2) \qquad (17)\\
=&\sum_{k=1}^{n}(B(t_{k-1} + \theta(t_k - t_{k-1})) - B(t_{k-1}))^2 + (B(t_k) - B(t_{k-1} + \theta(t_k - t_{k-1})))^2 \qquad (18)
\end{aligned}$$

其中$\forall 1 \leq k \leq n$,

$$\begin{aligned}
(t_{k-1} + \theta(t_k - t_{k-1})) - t_{k-1} = \theta(t_k - t_{k-1}) < t_k - t_{k-1} < \lambda\\
t_k - (t_{k-1} + \theta(t_k - t_{k-1})) = (1-\theta)(t_k - t_{k-1}) < t_k - t_{k-1} < \lambda
\end{aligned} \qquad (19)$$

因此根据定理5.2.3，有

$$\lim_{\lambda \to 0}\sum_{k=1}^{n}(C_k^2 + D_k^2) \overset{m.s}{=} t \Leftrightarrow \lim_{\lambda \to 0} E\left[\sum_{k=1}^{n}(C_k^2 + D_k^2) - t\right]^2 = 0 \qquad (20)$$

另一方面，$\forall 1 \leq k \leq n$，$C_k \sim B(\theta(t_k - t_{k-1}))$，$D_k \sim B((1-\theta)(t_k - t_{k-1}))$。因此

$$VarD_k = \frac{1-\theta}{\theta} VarC_k \qquad (21)$$

而$ED_k = EC_k = 0$，因此

$$ED_k^2 = VarD_k = \frac{1-\theta}{\theta} VarC_k = \frac{1-\theta}{\theta} EC_k^2 \qquad (22)$$

所以

$$E\sum_{k=1}^{n} D_k^2 = \frac{1-\theta}{\theta} E\sum_{k=1}^{n} C_k^2 \qquad (23)$$

及

$$E(\sum_{k=1}^{n} D_k^2)^2 = E(\sum_{k=1}^{n} D_k^4 + 2\sum_{i<j} D_i^2 D_j^2) \tag{24}$$

$$= 3(VarD_k)^2 + 2\sum_{i<j} ED_i^2 ED_j^2 \tag{25}$$

$$= (\frac{1-\theta}{\theta})^2 \cdot [3(VarC_k)^2 + 2\sum_{i<j} EC_i^2 EC_j^2] \tag{26}$$

$$= (\frac{1-\theta}{\theta})^2 E(\sum_{k=1}^{n} C_k^2)^2 \tag{27}$$

所以

$$0 = \lim_{\lambda\to 0} E\left[\sum_{k=1}^{n}(C_k^2 + D_k^2) - t\right]^2 \tag{28}$$

$$= \lim_{\lambda\to 0} E\left[(\sum_{k=1}^{n} C_k^2)^2 + (\sum_{k=1}^{n} D_k^2)^2 + t^2 + 2(\sum_{k=1}^{n} C_k^2)(\sum_{k=1}^{n} D_k^2) - 2t(\sum_{k=1}^{n} C_k^2) - 2t(\sum_{k=1}^{n} D_k^2)\right] \tag{29}$$

$$= \lim_{\lambda\to 0} E\left[[1 + (\frac{1-\theta}{\theta})^2](\sum_{k=1}^{n} C_k^2)^2 + t^2 + 2\frac{1-\theta}{\theta}(\sum_{k=1}^{n} C_k^2)^2 - 2t\frac{1}{\theta}(\sum_{k=1}^{n} C_k^2)\right] \tag{30}$$

$$= \lim_{\lambda\to 0} E\left[\frac{1}{\theta^2}(\sum_{k=1}^{n} C_k^2)^2 - 2\frac{t}{\theta}(\sum_{k=1}^{n} C_k^2) + t^2\right] \tag{31}$$

$$= \frac{1}{\theta^2}\lim_{\lambda\to 0} E\left[\sum_{k=1}^{n} C_k^2 - \theta t\right]^2 \tag{32}$$

所以

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} C_k^2 \overset{m.s}{=} \theta t \tag{33}$$

同理可知

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} D_k^2 \overset{m.s}{=} (1-\theta)t \tag{34}$$

而$E\sum_{k=1}^{n}(C_k^2 - D_k^2) = \frac{2\theta-1}{\theta} E\sum_{k=1}^{n} C_k^2$，进一步地，我们有

$$\lim_{\lambda\to 0}\sum_{k=1}^{n}(C_k^2 - D_k^2) \overset{m.s.}{=} (2\theta-1)t \tag{35}$$

因此

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} B(t_{k-1} + \theta(t_k - t_{k-1}))(B(t_k) - B(t_{k-1})) \tag{36}$$

$$= \lim_{\lambda\to 0}\frac{1}{2}\left[B^2(t) + \sum_{k=1}^{n}(C_k^2 - D_k^2)\right] \tag{37}$$

$$\overset{m.s.}{=} \frac{1}{2}B^2(t) + \frac{1}{2}(2\theta-1)t \tag{38}$$

得证。

Q5.22

(2)

记

$$X=\{X(t)=e^{\lambda B(t)-\frac{1}{2}\lambda^2 t},t\geq 0\} \tag{39}$$

首先，$|EX(t)|\leq e^{\lambda|B(t)|}e^{-\frac{1}{2}\lambda^2 t}<+\infty$。其次，$\forall 0\leq t_0<t_1<\cdots<t_n<t_{n+1}$，有

$$\begin{aligned}
&E(X(t_{n+1})|X(t_n),\cdots,X(t_0)) &(40)\\
=&E(X(t_{n+1})|X(t_n)) &(41)\\
=&E(e^{\lambda B(t_{n+1})-\frac{1}{2}\lambda^2 t_{n+1}}|X(t_n)) &(42)\\
=&E(X(t_n)e^{\lambda(B(t_{n+1})-B(t_n))-\frac{1}{2}\lambda^2(t_{n+1}-t_n)}|X(t_n)) &(43)\\
=&X(t_n)e^{-\frac{1}{2}\lambda^2(t_{n+1}-t_n)}Ee^{\lambda(B(t_{n+1})-B(t_n))} &(44)
\end{aligned}$$

而

$$\begin{aligned}
&Ee^{\lambda(B(t_{n+1})-B(t_n))} &(45)\\
=&\int_{-\infty}^{\infty}e^{\lambda s}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}e^{-\frac{s^2}{2(t_{n+1}-t_n)}}ds &(46)\\
=&\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}e^{-\frac{(s-\lambda(t_{n+1}-t_n))^2}{2(t_{n+1}-t_n)}+\frac{\lambda^2(t_{n+1}-t_n)}{2}}ds &(47)\\
=&e^{\frac{\lambda^2(t_{n+1}-t_n)}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}e^{-\frac{(s-\lambda(t_{n+1}-t_n))^2}{2(t_{n+1}-t_n)}}ds &(48)\\
=&e^{\frac{1}{2}\lambda^2(t_{n+1}-t_n)} &(49)
\end{aligned}$$

所以

$$E(X(t_{n+1})|X(t_n),\cdots,X(t_0))=X(t_n) \tag{50}$$

因此X是鞅。

(3)

记

$$Y=\{Y(t)=B^2(t)-t,t\geq 0\} \tag{51}$$

首先，$|EY(t)|\leq|B(t)|^2+|t|<+\infty$。其次，$\forall 0\leq t_0<t_1<\cdots<t_n<t_{n+1}$，有

$$\begin{aligned}
&E(Y(t_{n+1})|Y(t_n),Y(t_{n-1}),\cdots,Y(t_0)) &(52)\\
=&E(Y(t_{n+1})|Y(t_n)) &(53)\\
=&E(B^2(t_{n+1})-t_{n+1}|Y(t_n)) &(54)\\
=&E(B^2(t_n)+2B(t_n)(B(t_{n+1})-B(t_n))+(B(t_{n+1})-B(t_n))^2-t_{n+1}|Y(t_n)) &(55)\\
=&Y(t_n)+2B(t_n)E(B(t_{n+1})-B(t_n))+E(B(t_{n+1})-B(t_n))^2+t_n-t_{n+1} &(56)\\
=&Y(t_n)+0+(t_{n+1}-t_n)+t_n-t_{n+1} &(57)\\
=&Y(t_n) &(58)
\end{aligned}$$

所以Y也是鞅。

Q5.5

求出(S_{t_1}, S_{t_2})的联合概率密度函数。设$t_1 < t_2$,

首先，$\{S_t : t \geq 0\}$是一个正态过程，有

$$\begin{aligned} ES_t = E\int_0^t B_u du = \int_0^t EB_u du = 0 \\ Cov(S_{t_1}, S_{t_2}) = \frac{t_1^2}{2}(t_2 - \frac{t_1}{3}) \\ ES_t^2 = E\int_0^t \int_0^t B_u B_v dudv = Cov(S_t, S_t) = \frac{t^3}{3} \end{aligned} \tag{59}$$

因此(S_{t_1}, S_{t_2})的联合概率密度函数就是二维高斯分布的联合概率密度函数，其中$\sigma_1 = \sqrt{\frac{t_1^3}{3}} = t_1\sqrt{\frac{t_1}{3}}$, $\sigma_2 = \sqrt{\frac{t_2^3}{3}} = t_2\sqrt{\frac{t_2}{3}}$，因此相关系数$\rho = \frac{Cov(S_{t_1}, S_{t_2})}{\sigma_1, \sigma_2} = \frac{t_1(3t_2 - t_1)}{2t_2\sqrt{t_1 t_2}}$，则

$$f_{S_{t_1}, S_{t_2}}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma^2}\right]} \tag{60}$$

Q5.9

有$\eta(t) = e^{B(t)}$,

有

$$\begin{aligned} & E[\eta(t+h) - \eta(t) | \eta(t) = x] && (61) \\ =& E[e^{B(t+h)} - e^{B(t)} | e^{B(t)} = x] && (62) \\ =& xE[e^{B(t+h)-B(t)} - 1 | e^{B(t)} = x] && (63) \\ =& x(E[e^{B(t+h)-B(t)}] - 1) && (64) \\ =& x(e^{\frac{h}{2}} - 1) && (65) \end{aligned}$$

所以

$$\begin{aligned} \alpha(x) &= \lim_{h\downarrow 0} \frac{E[\eta(t+h) - \eta(t) | \eta(t) = x]}{h} && (66) \\ &= \lim_{h\downarrow 0} \frac{x(e^{\frac{h}{2}} - 1)}{h} && (67) \\ &= x \lim_{h\downarrow 0} \frac{e^{\frac{h}{2}} - 1}{h} && (68) \\ &= x \lim_{h\downarrow 0} \frac{e^{\frac{h}{2}} \cdot \frac{1}{2}}{1} = \frac{x}{2} && (69) \end{aligned}$$

另一方面，

$$\begin{aligned} & E[(\eta(t+h) - \eta(t))^2 | \eta(t) = x] && (70) \\ =& E[(e^{B(t+h)} - e^{B(t)})^2 | e^{B(t)} = x] && (71) \\ =& E[e^{2B(t+h)} - 2e^{B(t+h)+B(t)} + e^{2B(t)} | e^{B(t)} = x] && (72) \\ =& E[(e^{B(t)} e^{C(h)})^2 - 2(e^{B(t)} e^{C(h)}) e^{B(t)} + e^{2B(t)} | e^{B(t)} = x], \quad \text{其中} C(h) = B(t+h) - B(t) && (73) \\ =& x^2 E(e^{C(h)})^2 - 2x^2 E e^{C(h)} + x^2 && (74) \end{aligned}$$

而$C(h) = B(t+h) - B(t) \overset{d}{=} B(h)$，因此

$$\begin{aligned}Ee^{C(h)} &= e^{\frac{h}{2}} \\ E(e^{C(h)})^2 &= e^{2h}\end{aligned} \tag{75}$$

所以

$$E[(\eta(t+h)-\eta(t))^2|\eta(t)=x] \tag{76}$$
$$=x^2e^{2h}-2x^2e^{\frac{h}{2}}+x^2 \tag{77}$$
$$=x^2(e^{2h}-2e^{\frac{h}{2}}+1) \tag{78}$$

因此

$$\beta(x)=\lim_{h\downarrow 0}\frac{E[(\eta(t+h)-\eta(t))^2|\eta(t)=x]}{h} \tag{79}$$
$$=x^2\lim_{h\downarrow 0}\frac{e^{2h}-2e^{\frac{h}{2}}+1}{h} \tag{80}$$
$$=x^2\lim_{h\downarrow 0}\frac{e^{2h}\cdot 2-2e^{\frac{h}{2}}\cdot\frac{1}{2}}{1}=x^2 \tag{81}$$

Q5.16

$$T_x=\inf\{t\geq 0:B_t=x\} \tag{82}$$

则

$$P(T_1<T_{-1}<T_2)=P(T_1<T_{-1},T_{-1}<T_2) \tag{83}$$
$$=P(T_{-1}<T_2|T_1<T_{-1})P(T_1<T_{-1}) \tag{84}$$

而根据布朗运动的对称性，$P(T_1<T_{-1})=P(T_{-1}<T_1)=\frac{1}{2}$。

令$C=\{B_{t+T_1}:t\geq 0\}$，$U_x=\inf\{t\geq 0:C_t=x\}$。则$C_0=B_{T_1}=1$，

$$P(T_{-1}<T_2|T_1<T_{-1})=P(U_{-1}<U_2|U_1<U_{-1}) \tag{85}$$
$$=P(U_{-1}<U_2) \tag{86}$$
$$=P(U_{-1}<U_2|U_0<U_2)\cdot P(U_0<U_2)+P(U_{-1}<U_2|U_0\geq U_2)P(U_0\geq U_2) \tag{87}$$
$$=P(U_{-1}<U_2|U_0<U_2)\cdot\frac{1}{2}+0 \tag{88}$$
$$=\frac{1}{2}P(U_{-1}<U_2|U_0<U_2) \tag{89}$$

令$D=\{C_{t+U_0}:t\geq 0\}$，$V_x=\inf\{t\geq 0:D_t=x\}$，则$D_0=C_{U_0}=0$。

则

$$P(U_{-1}<U_2|U_0<U_2)=P(V_{-1}<V_2|V_0<V_2) \tag{90}$$
$$=P(V_{-1}<V_2) \tag{91}$$
$$=P(V_1<V_{-2}) \tag{92}$$
$$=1-P(V_{-2}<V_1) \tag{93}$$
$$=1-P(V_{-1}<V_2|V_1<V_{-1}) \tag{94}$$
$$=1-P(T_{-1}<T_2|T_1<T_{-1}) \tag{95}$$

所以

$$P(T_{-1}<T_2|T_1<T_{-1})=\frac{1}{2}(1-P(T_{-1}<T_2|T_1<T_{-1}))\Longrightarrow P(T_{-1}<T_2|T_1<T_{-1})=\frac{1}{3} \tag{96}$$

所以

$$P(T_1 < T_{-1} < T_2) = P(T_{-1} < T_2 | T_1 < T_{-1}) P(T_1 < T_{-1}) = \frac{1}{6} \tag{97}$$