HOMEWORK_13

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Q7.9

(1)

证明：

只需证明$E(Y(t)|Y(s))=Y(s)$。其中$s\le t$。有

$$\begin{aligned}
E(Y(t)|Y(s)) =& E(\int_0^t h(u)dB(u)|\int_0^s h(u)dB(u)) \quad (1)\\
=& \int_0^s h(u)dB(u)+E(\int_s^t h(u)dB(u)|\int_0^s h(u)dB(u)) \quad (2)\\
=& Y(s)+E(\int_s^t h(u)dB(u))=Y(s) \quad (3)
\end{aligned}$$

所以$\{Y(t):t\ge 0\}$是鞅。

(2)

证明：

取一个分割$0=t_0<t_1<\cdots t_{m_n}=t$，由伊藤积分的定义，有：

$$\begin{aligned}
E[\exp\left(\lambda\int_0^t h(s)dB(s)\right)] =& E[\exp\left(\lambda\lim_{n\to\infty}\sum_{k=0}^{m_n-1}h(t_k)(B(t_{k+1})-B(t_k))\right)]\\
=& E[\lim_{n\to\infty}\prod_{k=0}^{m_n-1}\exp\left(\lambda h(t_k)(B(t_{k+1})-B(t_k))\right)]\\
=& \lim_{n\to\infty}E\prod_{k=0}^{m_n-1}\exp\left(\lambda h(t_k)(B(t_{k+1})-B(t_k))\right)\\
=& \lim_{n\to\infty}\prod_{k=0}^{m_n-1}E\exp\left(\lambda h(t_k)(B(t_{k+1})-B(t_k))\right) \quad (\because \text{每一项都彼此独立})\\
=& \lim_{n\to\infty}\prod_{k=0}^{m_n-1}\exp\left(\frac{\lambda^2}{2}h(t_k)^2(t_{k+1}-t_k)\right) \quad (\because (B(t_{k+1})-B(t_k))\sim\mathcal{N}(0,t_{k+1}-t_k))\\
=& \lim_{n\to\infty}\exp\left(\frac{\lambda^2}{2}\sum_{k=0}^{m_n-1}h(t_k)^2(t_{k+1}-t_k)\right)\\
=& \exp\left(\frac{\lambda^2}{2}\lim_{n\to\infty}\sum_{k=0}^{m_n-1}h(t_k)^2(t_{k+1}-t_k)\right)\\
=& \exp\left(\frac{\lambda^2}{2}\int_0^t h^2(s)ds\right)
\end{aligned}$$

得证。（上式中的$\lim_{n\to\infty}$表示均方意义下）

Q7.11

证明：

取一个分割$0=t_0<t_1<\cdots t_{m_n}=t$，由伊藤积分的定义，有

$$
\begin{aligned}
\int_0^t sdB(s) &= \lim_{n\to\infty}\sum_{k=0}^{m_n-1} t_k(B(t_{k+1})-B(t_k)) && (1\\
&= \lim_{n\to\infty}\sum_{k=0}^{m_n-1} [(t_{k+1}B(t_{k+1})-t_kB(t_k))-B(t_{k+1})(t_{k+1}-t_k)] && (1\\
&= \lim_{n\to\infty}\sum_{k=0}^{m_n-1} (t_{k+1}B(t_{k+1})-t_kB(t_k))-\lim_{n\to\infty}\sum_{k=0}^{m_n-1} B(t_{k+1})(t_{k+1}-t_k) && (1\\
&=tB(t)-\lim_{n\to\infty}\sum_{k=0}^{m_n-1} B(t_{k+1})(t_{k+1}-t_k) && (1\\
&=tB(t)-\int_0^t B(s)ds && (1\\
&(\text{这里}\int_0^t B(s)ds\text{就是普通积分了，分段后取右端点即有}\int_0^t B(s)ds=\lim_{n\to\infty}\sum_k B(t_{k+1})(t_{k+1}-t_k)) && (1
\end{aligned}
$$

得证。（上式中的$\lim_{n\to\infty}$表示均方意义下）

Q7.12（2）

证明：

首先

$$
\begin{aligned}
B^3(t_{k+1})-B^3(t_k) =&B^2(t_k)(B(t_{k+1})-B(t_k))+(B(t_{k+1})+2B(t_k))(B(t_{k+1})-B(t_k))^2 && (18)\\
=&B^2(t_k)(B(t_{k+1})-B(t_k))+3B(t_k)(B(t_{k+1})-B(t_k))^2+(B(t_{k+1})-B(t_k))^3 && (19)
\end{aligned}
$$

取一个分割$0=t_0<t_1<\cdots t_{m_n}=t$，由伊藤积分的定义，有

$$
\int_0^t B^2(s)dB(s) \overset{m.s.}{=} \lim_{n\to\infty}\sum_{k=0}^{m_n-1} B^2(t_k)(B(t_{k+1})-B(t_k)) \tag{20}
$$

令$\lambda\triangleq \max_{0\le k\le m_n-1}(t_{k+1}-t_k)$

$$
\begin{aligned}
&\lim_{n\to\infty} E[\sum_{k=0}^{m_n-1} B(t_k)(B(t_{k+1})-B(t_k))^2-\sum_{k=0}^{m_n-1} B(t_k)(t_{k+1}-t_k)]^2 && (21)\\
=&\lim_{n\to\infty} E[\left(\sum_{k=0}^{m_n-1} B(t_k)(B(t_{k+1})-B(t_k))^2\right)^2+\left(\sum_{k=0}^{m_n-1} B(t_k)(t_{k+1}-t_k)\right)^2 && (22)\\
&-2\sum_{k=0}^{m_n-1} B(t_k)(B(t_{k+1})-B(t_k))^2\sum_{k=0}^{m_n-1} B(t_k)(t_{k+1}-t_k)] && (23)\\
=&\lim_{n\to\infty} E[\sum_{k=0}^{m_n-1} B^2(t_k)(B(t_{k+1})-B(t_k))^4+2\sum_{0\le i<j\le m_n-1} B(t_i)B(t_j)(B(t_{i+1})-B(t_i))^2(B(t_j)-B(t_{j+1}))^2 && (24)\\
&+\sum_{k=0}^{m_n-1} B^2(t_k)(t_{k+1}-t_k)^2+2\sum_{0\le i<j\le m_n-1} B(t_i)B(t_j)(t_{i+1}-t_i)(t_{j+1}-t_j) && (25)\\
&-2\left[\sum_{k=0}^{m_n-1} B(t_k)^2(B(t_{k+1})-B(t_k))^2(t_{k+1}-t_k)+\sum_{0\le i,j\le m_n-1,i\ne j} B(t_i)(B(t_{i+1})-B(t_i))^2B(t_j)(t_{j+1}-t_j)\right]] && (26)\\
=&\lim_{n\to\infty} 2\sum_{k=0}^{m_n-1} t_k(t_{k+1}-t_k)^2 && (27)\\
\le&2\lambda t\lim_{n\to\infty}\sum_{k=0}^{m_n-1}(t_{k+1}-t_k)=2\lambda t^2\to 0\quad(\lambda\to 0) && (28)
\end{aligned}
$$

因此

$$\lim_{n\to\infty}\sum_{k=0}^{m_n-1}B(t_k)(B(t_{k+1})-B(t_k))^2\overset{m.s.}{=}\lim_{n\to\infty}\sum_{k=0}^{m_n-1}B(t_k)(t_{k+1}-t_k)^{=}\int_0^t B(s)ds \tag{29}$$

另一方面，有

$$\lim_{n\to\infty}E\left[\sum_{k=0}^{m_n-1}(B(t_{k+1})-B(t_k))^3\right]^2 \tag{30}$$

$$=\lim_{n\to\infty}E\left[\sum_{k=0}^{m_n-1}(B(t_{k+1})-B(t_k))^6+2\sum_{0\le i<j\le m_n-1}(B(t_{i+1})-B(t_i))^3(B(t_{j+1})-B(t_j))^3\right] \tag{31}$$

$$=\lim_{n\to\infty}\sum_{k=0}^{m_n-1}E(B(t_{k+1})-B(t_k))^6+2\sum_{0\le i<j\le m_n-1}E(B(t_{i+1})-B(t_i))^3(B(t_{j+1})-B(t_j))^3 \tag{32}$$

$$=\lim_{n\to\infty}\sum_{k=0}^{m_n-1}15(t_{k+1}-t_k)^3 \tag{33}$$

$$\le 15\lambda^2\lim_{n\to\infty}\sum_{k=0}^{m_n-1}(t_{k+1}-t_k)=15\lambda^2 t\to 0\quad(\lambda\to 0) \tag{34}$$

因此

$$\lim_{n\to\infty}\sum_{k=0}^{m_n-1}(B(t_{k+1})-B(t_k))^3\overset{m.s.}{=}0 \tag{35}$$

所以

$$B^3(t)=\sum_{k=0}^{m_n-1}(B^3(t_{k+1})-B^3(t_k)) \tag{36}$$

$$=\sum_{k=0}^{m_n-1}B^2(t_k)(B(t_{k+1})-B(t_k))+3B(t_k)(B(t_{k+1})-B(t_k))^2+(B(t_{k+1})-B(t_k))^3 \tag{37}$$

两边同时令$n\to\infty$，考虑均方意义下，有

$$B^3(t)=\int_0^t B^2(s)dB(s)+3\int_0^t B(s)ds \tag{38}$$

也即

$$\int_0^t B^2(s)dB(s)=\frac{1}{3}B^3(t)-\int_0^t B(s)ds \tag{39}$$

得证。