HOMEWORK_12

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Q5.12

证明：令$X(t)=B(t)-\alpha t$为一个漂移参数为$-\alpha<0$的布朗运动。根据定理5.6.2的推论，有

$$
\begin{aligned}
P(B(t)\le \alpha t+\beta,\forall t\ge 0|B(0)=x)=&P(\max_{t\ge 0}(B(t)-\alpha t)\le +\beta|B(0)=x) && (1)\\
=&P(\max_{t\ge 0}X(t)\le \beta|B(0)=x) && (2)\\
=&P(\max_{t\ge 0}X(t)\le \beta-x|B(0)=0) && (3)\\
=&1-P(\max_{t\ge 0}X(t)\ge \beta-x|B(0)=0)) && (4)\\
=&1-\exp\{2\mu(\beta-x)\} && (5)\\
=&1-\exp\{-2\alpha(\beta-x)\} && (6)
\end{aligned}
$$

得证。

Q5.15

证明：有$X(s)=B(s)+\mu s$，令$T_x=\inf\{t\ge 0,X(t)=x\}$。则

$$
\begin{aligned}
&P\left\{\max_{0\le s\le h}|X(s)|>x|B(0)=0\right\} && (7)\\
=&P\{T_x\wedge T_{-x}\le h|B(0)=0\} && (8)\\
\le&P(T_x\le h|B(0)=0)+P(T_{-x}\le h|B(0)=0) && (9)
\end{aligned}
$$

取h，使得$0<h<\frac{x}{|\mu|}$，则

$$
\begin{aligned}
P(T_x\le h|B(0)=0)=&P(\max_{0\le s\le h}X(s)\ge x|B(0)=0) && (10)\\
=&P(\max_{0\le s\le h}(B(s)+\mu s)\ge x|B(0)=0) && (11)\\
\le&P(\max_{0\le s\le h}B(s)\ge (x-|\mu|h)|B(0)=0) && (12)\\
=&2P(B(h)\ge (x-|\mu|h)) && (13)\\
=&2\frac{1}{\sqrt{2\pi h}}\int_{x-|\mu|h}^{\infty}e^{-\frac{y^2}{2h}}dy && (14)\\
\le&\frac{2}{\sqrt{2\pi h}}\int_{x-|\mu|h}^{\infty}\frac{y^4}{(x-|\mu|h)^4}e^{-\frac{y^2}{2h}}dy && (15)\\
\le&\frac{2}{\sqrt{2\pi h}}\int_{-\infty}^{\infty}\frac{y^4}{(x-|\mu|h)^4}e^{-\frac{y^2}{2h}}dy && (16)\\
=&\frac{2}{(x-|\mu|h)^4}\int_{-\infty}^{\infty}y^4\frac{1}{\sqrt{2\pi h}}e^{-\frac{y^2}{2h}}dy && (17)\\
=&\frac{6h^2}{(x-|\mu|h)^4}=o(h) && (18)
\end{aligned}
$$

另一方面，令$Y(s)=-X(s)=-B(s)-\mu s$，$T_x^{(y)}=\inf\{s\ge 0,Y(s)=x\}$则

$$
P(T_{-x}\le h|B(0)=0)=P(T_x^{(y)}\le h|B(0)=0)=o(h)\quad\text{（根据上面得到的结论）}\qquad (19)
$$

所以

$$
P\left\{\max_{0\le s\le h}|X(s)|>x|B(0)=0\right\}\le P(T_x\le h|B(0)=0)+P(T_{-x}\le h|B(0)=0)=o(h)\qquad (20)
$$

得证。

Q7.1

(1)

首先，有$X_n(\omega)=\sqrt{n}1_{\{0\le\omega\le\frac{1}{n}\}}=\begin{cases}\sqrt{n}, & 0\le\omega\le\frac{1}{n}\\ 0, & other\end{cases}$。

因此当$n<\epsilon^2$时，$\forall\omega\in\Omega$，$X_n(\omega)\le\sqrt{n}<\epsilon$，$P(X_n\ge\epsilon)=0$。

当$n\ge\epsilon^2$时，$X_n(\omega)\ge\epsilon\Leftrightarrow 0\le\omega\le\frac{1}{n}$，故$P(X_n\ge\epsilon)=P(\omega\in[0,\frac{1}{n}])=\frac{1}{n}$。

所以$\forall\epsilon>0$,

$$P(X_n\ge\epsilon)=\begin{cases}0, & n<\epsilon^2\\ \frac{1}{n}, & n\ge\epsilon^2\end{cases}\tag{21}$$

令$\epsilon\to 0$，此时$\forall n\ge 1$，$P(X_n>0)=\frac{1}{n}\Leftrightarrow P(X_n\le 0)=1-\frac{1}{n}$，则

$$\lim_{n\to\infty}P(X_n\le 0)=1\tag{22}$$

而$X_n\ge 0$，因此

$$\lim_{n\to\infty}P(X_n=0)=1\Leftrightarrow\lim_{n\to\infty}X_n\overset{P}{=}0\tag{23}$$

(2)

假设$\lim_{n\to\infty}X_n$在均方收敛的意义下存在，设$\lim_{n\to\infty}X_n\overset{m.s.}{=}k$，则

$$\begin{aligned}0&=\lim_{n\to\infty}(E(X_n-k)^2)^{\frac{1}{2}} &(24)\\ &=\lim_{n\to\infty}\left((\sqrt{n}-k)^2\cdot P(\omega\in[0,\frac{1}{n}])+0\cdot P(\omega\notin[0,\frac{1}{n}])\right)^{\frac{1}{2}} &(25)\\ &=\lim_{n\to\infty}\frac{|\sqrt{n}-k|}{\sqrt{n}} &(26)\\ &=\lim_{n\to\infty}|1-\frac{k}{\sqrt{n}}|=1 &(27)\end{aligned}$$

矛盾。因此$\lim_{n\to\infty}X_n$在均方收敛的意义下不存在。

Q7.3

(1)

有$\rho=\frac{cov(X,Y)}{\sigma_1\sigma_2}=\frac{EXY}{\sigma_1\sigma_2}$,

因此

$$EX(t)=E(X+tY)=EX+tEY=0\tag{28}$$

$$\begin{aligned}cov(X(s),X(t))=&EX(s)X(t)-EX(s)EX(t) &(29)\\ =&EX(s)X(t) &(30)\\ =&E(X+sY)(X+tY) &(31)\\ =&E(X^2+(t+s)XY+stY^2) &(32)\\ =&EX^2+(t+s)EXY+stEY^2 &(33)\\ =&\sigma_1^2+(s+t)\rho\sigma_1\sigma_2+st\sigma_2^2 &(34)\end{aligned}$$

(3)

我们先证明第三小题。

证明:

首先，

$$EX^2(t)=E(X+tY)^2=EX^2+2tEXY+t^2EY^2=\sigma_1^2+2t\rho\sigma_1\sigma_2+t^2\sigma_2^2 \tag{35}$$

先证$X(t)$在$t>0$上均方连续。有$\forall t>0$，$h>0$，

$$\begin{aligned}E(X(t+h)-X(t))^2 =&EX^2(t+h)+EX^2(t)-2EX(t+h)X(t) &(36)\\ =&\sigma_1^2+2(t+h)\rho\sigma_1\sigma_2+(t+h)^2\sigma_2^2 &(37)\\ &+\sigma_1^2+2t\rho\sigma_1\sigma_2+t^2\sigma_2^2 &(38)\\ &-2(\sigma_1^2+(2t+h)\rho\sigma_1\sigma_2+t(t+h)\sigma_2^2) &(39)\\ =&h^2\sigma_2^2 &(40)\end{aligned}$$

所以

$$\lim_{h\to 0}E(X(t+h)-X(t))^2=\lim_{h\to 0}h^2\sigma_2^2=0 \tag{41}$$

所以$X(t)$在$t>0$上均方连续。

再证明$X(t)$在$t>0$上均方可导。$\forall t>0$，有

$$\begin{aligned}&\lim_{h,l\to 0}\frac{R(t_0+h,t_0+l)-R(t_0,t_0+l)-R(t_0+h,t_0)+R(t_0,t_0)}{hl} &(42)\\ =&\lim_{h,l\to 0}\frac{cov(X(t_0+h),X(t_0+l))-cov(X(t_0),X(t_0+l))-cov(X(t_0+h),X(t_0))+cov(X(t_0),X(t_0))}{hl} &(43)\\ =&\lim_{h,l\to 0}\frac{hl\sigma_2^2}{hl}=\sigma_2^2 &(44)\end{aligned}$$

存在。因此$X(t)$在$t>0$上均方可导。

(2)

因为$X(t)$在$t>0$上均方连续，所以

$$EY(t)=E\int_0^t X(u)du=\int_0^t EX(u)du=0 \tag{45}$$

$$\begin{aligned}EZ(t)=&E\int_0^t X^2(u)du=\int_0^t EX^2(u)du &(46)\\ =&\int_0^t \sigma_1^2+2u\rho\sigma_1\sigma_2+u^2\sigma_2^2du &(47)\\ =&\sigma_1^2t+\rho\sigma_1\sigma_2t^2+\frac{t^3}{3}\sigma_2^2 &(48)\end{aligned}$$

$$\begin{aligned}cov(Y(s),Y(t))=&EY(s)Y(t)-EY(s)EY(t) &(49)\\ =&EY(s)Y(t) &(50)\\ =&E\int_0^s X(u)du\int_0^t X(v)dv &(51)\\ =&E\int_0^s\int_0^t X(u)X(v)dvdu &(52)\\ =&\int_0^s\int_0^t EX(u)X(v)dvdu &(53)\\ =&\int_0^s\int_0^t(\sigma_1^2+(u+v)\rho\sigma_1\sigma_1+uv\sigma_2^2)dvdu &(54)\\ =&\int_0^s(\sigma_1^2t+ut\rho\sigma_1\sigma_2+\frac{t^2}{2}\rho\sigma_1\sigma_2+\frac{t^2}{2}u\sigma_2^2)du &(55)\\ =&\sigma_1^2st+\frac{s^2}{2}t\rho\sigma_1\sigma_2+\frac{t^2}{2}s\rho\sigma_1\sigma_2+\frac{s^2t^2}{4}\sigma_2^2 &(56)\end{aligned}$$

最后，考虑计算$cov(Z(s),Z(t))$。

有

$$
\begin{aligned}
EZ(s)Z(t) =&E\int_0^s X^2(u)du\int_0^t X^2(v)dv \\
=&E\int_0^s\int_0^t X^2(u)X^2(v)dvdu \\
=&\int_0^s\int_0^t E(X+uY)^2(X+vY)^2dvdu \\
=&\int_0^s\int_0^t E(X^2+2uXY+u^2Y^2)(X^2+2vXY+v^2Y^2)dvdu \\
=&\int_0^s\int_0^t E(X^4+u^2v^2Y^4+(2v+2u)X^3Y+(2uv^2+2u^2v)XY^3+(u^2+4uv+v^2)X^2Y^2)dvdu \\
=&\int_0^s\int_0^t (3\sigma_1^4+3\sigma_2^4u^2v^2+(2v+2u)3\rho\sigma_1^3\sigma_2+(2uv^2+2u^2v)3\rho\sigma_1\sigma_2^3+(u^2+4uv+v^2)(1+2\rho^2)\sigma_1^2 \\
=&\int_0^s [3\sigma_1^4t+\sigma_2^4u^2t^3+6ut\rho\sigma_1^3\sigma_2+3t^2\rho\sigma_1^3\sigma_2+2ut^3\rho\sigma_1\sigma_2^3+3u^2t^2\rho\sigma_1\sigma_2^3+(1+2\rho^2)\sigma_1^2\sigma_2^2(u^2t+2ut \\
=&3\sigma_1^4st+\frac{1}{3}\sigma_2^4s^3t^3+3s^2t\rho\sigma_1^3\sigma_2+3st^2\rho\sigma_1^3\sigma_2+s^2t^3\rho\sigma_1\sigma_2^3+s^3t^2\rho\sigma_1\sigma_2^3+(1+2\rho^2)\sigma_1^2\sigma_2^2(\frac{1}{3}s^3t+s^2
\end{aligned}
$$

> 其中，
>
> $$
> \begin{gathered}
> EX^3Y=EX^2EXY+EX^2EXY+EXYEX^2=3EX^2EXY=3\rho\sigma_1^3\sigma_2 \\
> EXY^3=3\rho\sigma_1\sigma_2^3 \\
> EX^2Y^2=EX^2EY^2+EXYEXY+EXYEXY=\sigma_1^2\sigma_2^2+2\rho^2\sigma_1^2\sigma_2^2 \\
> EX^4=3\sigma_1^4,\quad EY^4=3\sigma_2^4
> \end{gathered} \tag{65}
> $$

所以

$$
\begin{aligned}
&cov(Z(s),Z(t)) \\
=&EZ(s)Z(t)-EZ(s)EZ(t) \\
=&3\sigma_1^4st+\frac{1}{3}\sigma_2^4s^3t^3+3s^2t\rho\sigma_1^3\sigma_2+3st^2\rho\sigma_1^3\sigma_2+s^2t^3\rho\sigma_1\sigma_2^3+s^3t^2\rho\sigma_1\sigma_2^3+(1+2\rho^2)\sigma_1^2\sigma_2^2(\frac{1}{3}s^3t+s^2t^2+\frac{st^3}{3}) \\
&-(\sigma_1^2t+\rho\sigma_1\sigma_2t^2+\frac{t^3}{3}\sigma_2^2)(\sigma_1^2s+\rho\sigma_1\sigma_2s^2+\frac{s^3}{3}\sigma_2^2) \\
=&2\sigma_1^4st+\frac{2}{9}\sigma_2^4s^3t^3+2s^2t\rho\sigma_1^3\sigma_2+2st^2\rho\sigma_1^3\sigma_2+\frac{2}{3}s^2t^3\rho\sigma_1\sigma_2^3+\frac{2}{3}s^3t^2\rho\sigma_1\sigma_2^3+2\rho^2\sigma_1^2\sigma_2^2(\frac{1}{3}s^3t+\frac{1}{2}s^2t^2+\frac{st^3}{3})- \\
=&\frac{2}{9}\sigma_2^4s^3t^3+\frac{2}{3}\rho\sigma_1\sigma_2^3s^2t^3+\frac{2}{3}\rho\sigma_1\sigma_2^3s^3t^2+\frac{2}{3}\rho^2\sigma_1^2\sigma_2^2s^3t+\frac{2}{3}\rho^2\sigma_1^2\sigma_2^2st^3+(\rho^2+1)\sigma_1^2\sigma_2^2s^2t^2 \\
&+2\rho\sigma_1^3\sigma_2s^2t+2\rho\sigma_1^3\sigma_2st^2+2\sigma_1^4st
\end{aligned}
$$

(4)

首先，$\forall t\geq 0$，有

$$
\begin{aligned}
Y^{'}(t)=\lim_{h\to 0}\frac{Y(t+h)-Y(t)}{h}&=\lim_{h\to 0}\frac{\int_t^{t+h}X(u)du}{h} && (73)\\
&=\lim_{h\to 0}\frac{\int_t^{t+h}(X+uY)du}{h} && (74)\\
&=\lim_{h\to 0}\frac{Xh+\frac{h^2+2th}{2}Y}{h} && (75)\\
&=\lim_{h\to 0}(X+\frac{h+2t}{2}Y) && (76)\\
&=X+tY=X(t) && (77)
\end{aligned}
$$

另一方面

$$Z^{'}(t)=\lim_{h\to 0}\frac{Z(t+h)-Z(t)}{h}=\lim_{h\to 0}\frac{\int_t^{t+h}X^2(u)du}{h} \tag{78}$$
$$=\lim_{h\to 0}\frac{\int_t^{t+h}(X^2+2uXY+u^2Y^2)du}{h} \tag{79}$$
$$=\lim_{h\to 0}\frac{X^2h+(2th+h^2)XY+\frac{3t^2h+3th^2+h^3}{3}Y^2}{h} \tag{80}$$
$$=\lim_{h\to 0}X^2+(2t+h)XY+\frac{3t^2+3th+h^2}{3}Y^2 \tag{81}$$
$$=X^2+2tXY+t^2Y^2=X^2(t) \tag{82}$$

Q7.6

证明：

(1)

$$E\left(\frac{dX(t)}{dt}\right)=E\left(\lim_{h\to 0}\frac{X(t+h)-X(t)}{h}\right) \tag{83}$$
$$=\lim_{h\to 0}E\left(\frac{X(t+h)-X(t)}{h}\right)\quad (\because \{X(t):t\geq 0\}\text{的均方导数存在}) \tag{84}$$
$$=\lim_{h\to 0}\frac{EX(t+h)-EX(t)}{h} \tag{85}$$
$$=\frac{dEX(t)}{dt} \tag{86}$$

(2)

$$E\left(X(t)\frac{dX(t)}{dt}\right)=E\left(X(t)\lim_{h\to 0}\frac{X(t+h)-X(t)}{h}\right) \tag{87}$$
$$=E\left(\lim_{h\to 0}\frac{X(t)X(t+h)-X(t)X(t)}{h}\right) \tag{88}$$
$$=\lim_{h\to 0}E\left(\frac{X(t)X(t+h)-X(t)X(t)}{h}\right)\quad (\because \{X(t):t\geq 0\}\text{的均方导数存在}) \tag{89}$$
$$=\lim_{h\to 0}\frac{EX(t+h)X(t)-EX(t)X(t)}{h} \tag{90}$$
$$=\lim_{h\to 0}\frac{R(t+h,t)-R(t,t)}{h} \tag{91}$$
$$=\lim_{h\to 0}\left.\frac{\partial R(s,t)}{\partial s}\right|_{s=t} \tag{92}$$

得证。

Q7.8

(1)

证明：

首先，$\{X(t):t\in\mathbb{R}\}$是平稳过程，因此$\forall t_1,t_2,\cdots,t_n\in\mathbb{R}$，及$s>0$，有

$$(X(t_1),X(t_2),\cdots,X(t_n))\overset{d}{=}(X(t_1+s),X(t_2+s),\cdots,X(t_n+s)) \tag{93}$$

此外，有

$$R(\tau)=E[X(t)X(t+\tau)]=e^{-2|\tau|}\Rightarrow EX(t)^2=R(0)=1 \tag{94}$$

因此，$\forall t\geq 0$，$s>0$，有

$$
\begin{aligned}
E(X_{t+s}-X_t)^2 =& EX_{t+s}^2 - 2EX_{t+s}X_t + EX_t^2 && (95)\\
=& 2-2R(s) && (96)\\
=& 2-2e^{-2|s|} && (97)
\end{aligned}
$$

所以

$$
\lim_{s\to 0} E(X_{t+s}-X_t)^2 = \lim_{s\to 0}(2-2e^{-2|s|}) = 0 \tag{98}
$$

所以$\{X_t : t \geq 0\}$均方连续。

(3)

首先，因为$\{X_t : t \geq 0\}$均方连续，有

$$
EY(t) = E\int_0^t X(s)ds = \int_0^t EX(s)ds = 0 \tag{99}
$$

其次，不妨设$s \leq t$，

$$
\begin{aligned}
cov(Y(s), Y(t)) =& EY(s)Y(t) - EY(s)EY(t) && (100)\\
=& EY(s)Y(t) && (101)\\
=& E\int_0^s X(u)du \int_0^t X(v)dv && (102)\\
=& E\int_0^s \int_0^t X(u)X(v)dvdu && (103)\\
=& \int_0^s \int_0^t EX(u)X(v)dvdu \quad (\because X(t)\text{均方收敛}) && (104)\\
=& \int_0^s \int_0^t R(v-u)dvdu && (105)\\
=& \int_0^s \int_0^t e^{-2|v-u|}dvdu && (106)\\
=& \int_0^s \left(\int_0^u e^{-2|v-u|}dv + \int_u^t e^{-2|v-u|}dv\right)du && (107)\\
=& \int_0^s \left(\int_0^u e^{-2(u-v)}dv + \int_u^t e^{-2(v-u)}dv\right)du && (108)\\
=& \int_0^s \left(e^{-2u}\frac{e^{2v}}{2}\bigg|_0^u + e^{2u}\frac{e^{-2v}}{-2}\bigg|_u^t\right)du && (109)\\
=& \int_0^s [1-\frac{e^{-2u}}{2} - \frac{1}{2}e^{2(u-t)}]du && (110)\\
=& [u+\frac{e^{-2u}}{4} - e^{-2t}\frac{e^{2u}}{4}]\bigg|_0^s && (111)\\
=& s+\frac{1}{4}(e^{-2s}-1) - \frac{e^{-2t}}{4}(e^{2s}-1) && (112)\\
=& s+\frac{1}{4}e^{-2s} + \frac{1}{4}e^{-2t} - \frac{1}{4}e^{2(s-t)} - \frac{1}{4} && (113)
\end{aligned}
$$