HOMEWORK_14

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Q7.10

有

$$dB_s = 0 \cdot dt + 1 \cdot dB_s \tag{1}$$

所以$b=0$，$\sigma=1$。取函数$f(t,x)=h(t)x$。则$\frac{\partial f}{\partial t}=h^{'}(t)x$，$\frac{\partial f}{\partial x}=h(t)$，$\frac{\partial^2 f}{\partial x^2}=0$。

令$Y_t=f(t,B_t)=h(t)B_t$，则由伊藤公式，有

$$\begin{aligned} dY_t =& \left(\frac{\partial f}{\partial t}+b\frac{\partial f}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)(t,B_t)dt+\left(\sigma\frac{\partial f}{\partial x}\right)(t,B_t)dB_t && (2)\\ =& h^{'}(t)B_t dt + h(t)dB_t && (3)\end{aligned}$$

两边积分，得

$$h(t)B_t = Y_t - Y_0 = \int_0^t h^{'}(s)B_s ds + \int_0^t h(s)dB_s = \int_0^t B_s dh(s) + \int_0^t h(s)dB_s \tag{4}$$

最后一个等号是因为$h(s)$是有限变差的，因此可以定义$h(s)$的R-S积分。

Q7.12（2）

有

$$dB_s = 0 \cdot dt + 1 \cdot dB_s \tag{5}$$

所以$b=0$，$\sigma=1$。取函数$f(t,x)=\frac{x^3}{3}$，则$\frac{\partial f}{\partial t}=0$，$\frac{\partial f}{\partial x}=x^2$，$\frac{\partial^2 f}{\partial x^2}=2x$。

令$Y(t)=f(t,B_t)=\frac{B_t^3}{3}$，则由伊藤公式，有

$$\begin{aligned} dY_t =& \left(\frac{\partial f}{\partial t}+b\frac{\partial f}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)(t,B_t)dt+\left(\sigma\frac{\partial f}{\partial x}\right)(t,B_t)dB_t && (6)\\ =& B_t dt + B_t^2 dB_t && (7)\end{aligned}$$

两边积分，得

$$\frac{B_t^3}{3} = Y_t - Y_0 = \int_0^t B_s ds + \int_0^t B_s^2 dB_s \tag{8}$$

也即

$$\int_0^t B_s^2 dB_s = \frac{B_t^3}{3} - \int_0^t B_s ds \tag{9}$$

得证。

Q7.14

(1)

有

$$dB_s = 0 \cdot dt + 1 \cdot dB_s \tag{10}$$

所以$b=0$，$\sigma=1$。取函数$f(t,x)=x^3$，则$\frac{\partial f}{\partial t}=0$，$\frac{\partial f}{\partial x}=3x^2$，$\frac{\partial^2 f}{\partial x^2}=6x$。

令$Y_t=f(t,B_t)=B_t^3$，则根据伊藤公式，有

$$\begin{aligned} dX_t =& d(B_t^3) = dY_t = \left(\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)(t,B_t)dt + \left(\sigma\frac{\partial f}{\partial x}\right)(t,B_t)dB_t & (11) \\ =& 3B_t dt + 3B_t^2 dB_t & (12) \end{aligned}$$

此即为X_t满足的伊藤随机微分方程。

(2)

$$X_t = \alpha + t + e^{B_t} \tag{13}$$

有

$$dB_s = 0 \cdot dt + 1 \cdot dB_s \tag{14}$$

所以$b=0$，$\sigma=1$。取函数$f(t,x)=\alpha+t+e^x$，则$\frac{\partial f}{\partial t}=1$，$\frac{\partial f}{\partial x}=e^x$，$\frac{\partial^2 f}{\partial x^2}=e^x$。

令$Y_t=f(t,B_t)=\alpha+t+e^{B_t}$，则根据伊藤公式，有

$$\begin{aligned} dX_t =& d(\alpha+t+e^{B_t}) = dY_t = \left(\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)(t,B_t)dt + \left(\sigma\frac{\partial f}{\partial x}\right)(t,B_t)dB_t & (15) \\ =& \left(1+\frac{1}{2}e^{B_t}\right)dt + e^{B_t}dB_t & (16) \end{aligned}$$

此即为X_t满足的伊藤随机微分方程。

(4)

$$X_t = e^{\frac{t}{2}}\cos B_t \tag{17}$$

有

$$dB_s = 0 \cdot dt + 1 \cdot dB_s \tag{18}$$

所以$b=0$，$\sigma=1$。取函数$f(t,x)=e^{\frac{t}{2}}\cos x$，则$\frac{\partial f}{\partial t}=\frac{1}{2}e^{\frac{t}{2}}\cos x$，$\frac{\partial f}{\partial x}=-e^{\frac{t}{2}}\sin x$，$\frac{\partial^2 f}{\partial x^2}=-e^{\frac{t}{2}}\cos x$
。

令$Y_t=f(t,B_t)=e^{\frac{t}{2}}\cos B_t$，则根据伊藤公式，有

$$
\begin{aligned}
dX_t =& d(e^{\frac{t}{2}}\cos B_t) = dY_t = \left(\frac{\partial f}{\partial t} + b\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)(t,B_t)dt + \left(\sigma\frac{\partial f}{\partial x}\right)(t,B_t)dB_t \quad (19)\\
=& \left(\frac{1}{2}e^{\frac{t}{2}}\cos B_t - \frac{1}{2}e^{\frac{t}{2}}\cos B_t\right)dt - e^{\frac{t}{2}}\sin B_t dB_t \quad (20)\\
=& -e^{\frac{t}{2}}\sin B_t dB_t \quad (21)
\end{aligned}
$$

此即为X_t满足的伊藤随机微分方程。

Q7.15

(1)

$$dX_t = uX_t dt + \sigma dB_t \quad (22)$$

考虑狭义的线性SDE的解，其中$a_1(t) = u, a_2(t) = 0, b_2(t) = \sigma$。求对应的特解（$dX_t = uX_t dt$）为

$$\rho_0(t) = X_0 \cdot e^{ut} \overset{X_0=1}{=} e^{ut} \quad (23)$$

则原随机微分方程的解为

$$
\begin{aligned}
X_t =& \rho_0(t)\left[X_0 + \int_0^t a_2(s)\rho_0^{-1}(s)ds + \int_0^t b_2(s)\rho_0^{-1}(s)dB_s\right] \quad (24)\\
=& e^{ut}[X_0 + \int_0^t \sigma e^{-us}dB_s] \quad (25)
\end{aligned}
$$

(3)

$$dX_t = \gamma dt + \alpha X_t dB_t \quad (26)$$

对应$a_1(t) = 0, a_2(t) = \gamma,\ b_1(t) = \alpha, b_2(t) = 0$。

其对应的齐次方程为

$$dX_t = 0 \cdot X_t dt + \alpha X_t dB_t \quad (27)$$

其基本解为

$$\rho_{t_0}(t) = \exp\left\{\int_{t_0}^t -\frac{\alpha^2}{2}ds + \int_{t_0}^t \alpha dB_s\right\} = e^{-\frac{\alpha^2}{2}(t-t_0)+\alpha(B_t - B_{t_0})} \quad (28)$$

所以原随机微分方程的一般解为

$$
\begin{aligned}
X_t =&\rho_{t_0}(t)\left[X_{t_0}+\int_{t_0}^t (a_2(s)-b_1(s)b_2(s))\rho_{t_0}^{-1}(s)ds+\int_{t_0}^t b_2(s)\rho_{t_0}^{-1}(s)dB_s\right] && (29)\\
=&\rho_{t_0}(t)\left[X_{t_0}+\int_{t_0}^t \gamma\rho_{t_0}^{-1}(s)ds\right] && (30)\\
=&e^{-\frac{\alpha^2}{2}(t-t_0)+\alpha(B_t-B_{t_0})}\left[X_{t_0}+\int_{t_0}^t \gamma e^{\frac{\alpha^2}{2}(s-t_0)-\alpha(B_s-B_{t_0})}ds\right] && (31)\\
=&X_{t_0}e^{-\frac{\alpha^2}{2}(t-t_0)+\alpha(B_t-B_{t_0})}+\gamma e^{-\frac{\alpha^2}{2}t+\alpha B_t}\int_{t_0}^t e^{\frac{\alpha^2}{2}s-\alpha B_s}ds && (32)
\end{aligned}
$$

(5)

$$
dX_t=(e^{-t}+X(t))dt+\sigma X_t dB_t \tag{33}
$$

对应$a_1(t)=1,a_2(t)=e^{-t},b_1(t)=\sigma,b_2(t)=0$。

考虑其对应的齐次方程

$$
dX_t=X(t)dt+\sigma X_t dB_t \tag{34}
$$

取$X_{t_0}=1$，其特解为

$$
\begin{aligned}
\rho_{t_0}(t) =&\exp\left\{\int_{t_0}^t (a_1(s)-\frac{1}{2}b_1^2(s))ds+\int_{t_0}^t b_1(s)dB_s\right\} && (35)\\
=&\exp\left\{\int_{t_0}^t (1-\frac{1}{2}\sigma^2)ds+\int_{t_0}^t \sigma dB_s\right\} && (36)\\
=&e^{(1-\frac{1}{2}\sigma^2)(t-t_0)+\sigma(B_t-B_{t_0})} && (37)
\end{aligned}
$$

所以原方程的一般解为

$$
\begin{aligned}
X_t =&\rho_{t_0}(t)\left[X_{t_0}+\int_{t_0}^t (a_2(s)-b_1(s)b_2(s))\rho_{t_0}^{-1}(s)ds+\int_{t_0}^t b_2(s)\rho_{t_0}^{-1}(s)dB_s\right] && (38)\\
=&\rho_{t_0}(t)\left[X_{t_0}+\int_{t_0}^t e^{-s}\rho_{t_0}^{-1}(s)ds\right] && (39)\\
=&e^{(1-\frac{1}{2}\sigma^2)(t-t_0)+\sigma(B_t-B_{t_0})}\left[X_{t_0}+\int_{t_0}^t e^{-s}e^{-(1-\frac{1}{2}\sigma^2)(s-t_0)-\sigma(B_s-B_{t_0})}ds\right] && (40)\\
=&X_{t_0}e^{(1-\frac{1}{2}\sigma^2)(t-t_0)+\sigma(B_t-B_{t_0})}+e^{(1-\frac{1}{2}\sigma^2)t+\sigma B_t}\int_{t_0}^t e^{(\frac{1}{2}\sigma^2-2)s-\sigma B_s}ds && (41)
\end{aligned}
$$