HOMEWORK_10

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Q4.7

首先，因为$\{S_n : n \geq 1\}$是鞅，因此$nEX_n = ES_n = ES_1 = EX_n$，所以$EX_n = 0$。因此由最大值不等式，有

$$P\left(\left|\frac{S_n}{n}\right| > \epsilon\right) \leq \frac{n\sigma^2_{X_n/n}}{\epsilon^2} \leq \frac{k}{n\epsilon^2} \tag{1}$$

令$n \to \infty$，即得

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n}\right| > \epsilon\right) = 0 \tag{2}$$

Q4.15

证明：

首先，有

$$\begin{aligned}
E\left[\max_{0\leq k\leq n} |X_k|\right] &= \int_0^\infty P\left(\max_{0\leq k\leq n} |X_k| > t\right)dt \qquad (3)\\
&= \int_0^\infty P\left(\max_{0\leq k\leq n} |X_k|^\alpha > t^\alpha\right)dt \qquad (4)\\
&= \int_0^{(E|X_k|^\alpha)^{\frac{1}{\alpha}}} P\left(\max_{0\leq k\leq n} |X_k|^\alpha > t^\alpha\right)dt + \int_{(E|X_k|^\alpha)^{\frac{1}{\alpha}}}^\infty P\left(\max_{0\leq k\leq n} |X_k|^\alpha > t^\alpha\right)dt \qquad (5)\\
&\leq \int_0^{(E|X_k|^\alpha)^{\frac{1}{\alpha}}} 1dt + \int_{(E|X_k|^\alpha)^{\frac{1}{\alpha}}}^\infty \frac{E|X_k|^\alpha}{t^\alpha}dt \quad (\because Doob\text{最大值不等式}) \qquad (6)\\
&= (E|X_k|^\alpha)^{\frac{1}{\alpha}} + E|X_k|^\alpha \cdot \left(-\frac{1}{\alpha-1}t^{1-\alpha}\right)\Big|_{(E|X_k|^\alpha)^{\frac{1}{\alpha}}}^\infty \qquad (7)\\
&= \frac{\alpha}{\alpha-1}(E|X_k|^\alpha)^{\frac{1}{\alpha}} \qquad (8)
\end{aligned}$$

得证。

Q课堂留题：证明Y和U关于N是鞅

> 设$N = \{N_t : t \geq 0\} \sim PP(\lambda)$，$\lambda > 0$。$\forall t \geq 0$，令$X_t = N_t - \lambda t$，$Y_t = X_t^2 - \lambda t$。
> $U_t = \exp\{-\theta N_t + \lambda t(1 - e^{-\theta})\}$（$\theta \in \mathbb{R}$）。

证明：

首先，$E|Y_t| \leq EX_t^2 + \lambda t = EN_t^2 - \lambda t EN_t + \lambda^2 t^2 + \lambda t < +\infty$。另一方面，显然$\forall t \geq 0$，$Y_t$由$\{N_s : 0 \leq s \leq t\}$决定，且$\forall 0 \leq t_0 < t_1 < \cdots < t_n < t_{n+1}$，有

$$E(Y_{t_{n+1}}|N_{t_n},\cdots,N_{t_0}) \tag{9}$$
$$=E(Y_{t_{n+1}}|N_{t_n}) \tag{10}$$
$$=E(X_{t_{n+1}}^2-\lambda t_{n+1}|N_{t_n}) \tag{11}$$
$$=E((N_{t_{n+1}}-\lambda t_{n+1})^2-\lambda t_{n+1}|N_{t_n}) \tag{12}$$
$$=E(N_{t_{n+1}}^2-2\lambda N_{t_{n+1}}t_{n+1}+\lambda^2 t_{n+1}^2-\lambda t_{n+1}|N_{t_n}) \tag{13}$$
$$=E(N_{t_n}^2+2N_{t_n}(N_{t_{n+1}}-N_{t_n})+(N_{t_{n+1}}-N_{t_n})^2-2\lambda(N_{t_n}+(N_{t_{n+1}}-N_{t_n}))t_{n+1}+\lambda^2t_{n+1}^2-\lambda t_{n+1}|N_{t_n}) \tag{14}$$
$$=N_{t_n}^2+2N_{t_n}\lambda(t_{n+1}-t_n)+\lambda^2(t_{n+1}-t_n)^2+\lambda(t_{n+1}-t_n)-2\lambda t_{n+1}N_{t_n}-2\lambda^2(t_{n+1}-t_n)t_{n+1} \tag{15}$$
$$+\lambda^2t_{n+1}^2-\lambda t_{n+1} \tag{16}$$
$$=N_{t_n}^2-2\lambda t_nN_{t_n}+\lambda^2t_n^2-\lambda t_n \tag{17}$$
$$=(N_{t_n}-\lambda t_n)^2-\lambda t_n=Y_{t_n} \tag{18}$$

所以$\{Y_t:t\geq 0\}$关于$\{N_t:t\geq 0\}$是鞅。

其次，考虑$\{U_t:t\geq 0\}$，有$E|U_t|=Ee^{-\theta N_t}e^{\lambda t(1-e^{-\theta})}=e^{\lambda t(1-e^{-\theta})}Ee^{-\theta N_t}<+\infty$。另一方面，显然$\forall t\geq 0$，$U_y$由$\{N_s:0\leq s\leq t\}$决定，且$\forall 0\leq t_0<t_1<\cdots<t_n<t_{n+1}$，有

$$E(U_{t_{n+1}}|N_{t_n},\cdots,N_{t_0}) \tag{19}$$
$$=E(U_{t_{n+1}}|N_{t_n}) \tag{20}$$
$$=E(e^{-\theta N_{t_{n+1}}+\lambda t_{n+1}(1-e^{-\theta})}|N_{t_n}) \tag{21}$$
$$=E(e^{-\theta N_{t_n}-\theta(N_{t_{n+1}}-N_{t_n})+\lambda t_{n+1}(1-e^{-\theta})}|N_{t_n}) \tag{22}$$
$$=e^{-\theta N_{t_n}+\lambda t_{n+1}(1-e^{-\theta})}E(e^{-\theta(N_{t_{n+1}}-N_{t_n})}|N_{t_n}) \tag{23}$$
$$=e^{-\theta N_{t_n}+\lambda t_{n+1}(1-e^{-\theta})}E(e^{-\theta(N_{t_{n+1}}-N_{t_n})}) \tag{24}$$
$$=e^{-\theta N_{t_n}+\lambda t_{n+1}(1-e^{-\theta})}e^{-\lambda(t_{n+1}-t_n)(1-e^{-\theta})} \tag{25}$$
$$=e^{-\theta N_{t_n}+\lambda t_n(1-e^{-\theta})}=U_{t_n} \tag{26}$$

所以$\{U_t:t\geq 0\}$关于$\{N_t:t\geq 0\}$是鞅。

Q5.1

(3)

证明:

$$\frac{\partial P}{\partial t}=-\frac{1}{2}(2\pi t)^{-\frac{3}{2}}\cdot 2\pi\exp(-\frac{x^2}{2t})+(2\pi t)^{-\frac{1}{2}}\exp(-\frac{x^2}{2t})\cdot(-\frac{x^2}{2})\cdot(-\frac{1}{t^2}) \tag{27}$$
$$=\exp(-\frac{x^2}{2t})(2\pi t)^{-\frac{1}{2}}\left[-\frac{1}{2t}+\frac{x^2}{2t^2}\right] \tag{28}$$
$$=\frac{1}{2}\exp(-\frac{x^2}{2t})(2\pi t)^{-\frac{1}{2}}\left[-\frac{1}{t}+\frac{x^2}{t^2}\right] \tag{29}$$

另一方面，

$$\frac{\partial^2 P}{\partial x^2}=\frac{\partial\left((2\pi t)^{-\frac{1}{2}}(-\frac{x}{t})\exp(-\frac{x^2}{2t})\right)}{\partial x} \tag{30}$$
$$=(2\pi t)^{-\frac{1}{2}}\left[\exp(-\frac{x^2}{2t})\frac{x^2}{t^2}-\frac{1}{t}\exp(-\frac{x^2}{2t})\right] \tag{31}$$
$$=\exp(-\frac{x^2}{2t})(2\pi t)^{-\frac{1}{2}}[-\frac{1}{t}+\frac{x^2}{t^2}]=2\frac{\partial P}{\partial t} \tag{32}$$

得证。

(5)

给定$B(t_i)=x_i$，$1\le i\le n$时，$B(t_{n+1})$的条件概率密度为

$$\begin{aligned}
&f_{B(t_{n+1})}(x|B(t_n)=x_n,\cdots,B(t_1)=x_1) &(33)\\
=&f_{B(t_{n+1})}(x|B(t_n)=x_n) &(34)\\
=&\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}\exp\left(-\frac{(x-x_n)^2}{2(t_{n+1}-t_n)}\right)\sim\mathcal{N}(x_n,t_{n+1}-t_n) &(35)
\end{aligned}$$

则

$$\begin{aligned}
&P(B(t_{n+1})\le x|B(t_1)=x_1,\cdots,B(t_n)=x_n) &(36)\\
=&\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}\exp\left(-\frac{(x-x_n)^2}{2(t_{n+1}-t_n)}\right)dx &(37)\\
=&\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}\int_{-\infty}^{x}\exp\left(-\frac{(x-x_n)^2}{2(t_{n+1}-t_n)}\right)dx &(38)
\end{aligned}$$

有

$$E(B(t_{n+1})|B(t_1),B(t_2),\cdots,B(t_n)=x_n)=x_n \tag{39}$$

所以

$$E(B(t_{n+1})|B(t_1),B(t_2),\cdots,B(t_n))=B(t_n) \tag{40}$$

$$Var(B(t_{n+1})|B(t_1)=x_1,B(t_2)=x_2,\cdots,B(t_n)=x_n)=t_{n+1}-t_n \tag{41}$$

(6)

给定$B(t_i)=x_i$，$1\le i\le n$时，$B(t_{n+1})$的条件概率密度

$$\begin{aligned}
&f_{B(t_{n+1})}(x|B(t_n)=x_n,B(t_{n-1})=x_{n-1},\cdots,B(t_1)=x_1) &(42)\\
=&\frac{f_{B(t_{n+1}),B(t_n),\cdots,B(t_1)}(x,x_n,x_{n-1},\cdots,x_1)}{f_{B(t_n),B(t_{n-1}),\cdots,B(t_1)}(x_n,x_{n-1},\cdots,x_1)} &(43)\\
=&\frac{1}{\sqrt{2\pi}}\frac{\frac{1}{\sqrt{t_{n+1}\prod_{i=1}^{n}(t_i-t_{i+1})}}\cdot\exp(-\frac{x^2}{2t_{n+1}}-\sum_{i=1}^{n}\frac{(x_i-x_{i+1})^2}{2(t_i-t_{i+1})})}{\frac{1}{\sqrt{t_n\prod_{i=1}^{n-1}(t_i-t_{i+1})}}\cdot\exp(-\frac{x_n^2}{2t_n}-\sum_{i=1}^{n-1}\frac{(x_i-x_{i+1})^2}{2(t_i-t_{i+1})})} &(44)\\
=&\frac{1}{\sqrt{2\pi}}\sqrt{\frac{t_n}{t_{n+1}(t_n-t_{n+1})}}\cdot\exp(-\frac{x^2}{2t_{n+1}}+\frac{x_n^2}{2t_n}-\frac{(x_n-x_{n+1})^2}{2(t_n-t_{n+1})}) &(45)\\
=&\frac{1}{\sqrt{2\pi\frac{t_{n+1}(t_n-t_{n+1})}{t_n}}}\exp\left(-\frac{(x-\frac{t_{n+1}}{t_n}x_n)^2}{2\frac{t_{n+1}(t_n-t_{n+1})}{t_n}}\right)\sim\mathcal{N}(\frac{t_{n+1}}{t_n}x_n,\frac{t_{n+1}(t_n-t_{n+1})}{t_n}) &(46)
\end{aligned}$$

可以看到，这与$B(t_i)$，$1\le i\le n-1$无关。实际上由上面的过程可以得到

$$\begin{aligned}
&f_{B(t_{n+1})}(x|B(t_n)=x_n,B(t_{n-1})=x_{n-1},\cdots,B(t_1)=x_1) &(47)\\
=&f_{B(t_{n+1})}(x|B(t_n)=x_n) &(48)
\end{aligned}$$

因此$\{B(t):t\ge 0\}$从逆向时间看也是马尔可夫过程。

Q5.2

(1)

有$Y(t)=tB(\frac{1}{t})$，$Y(0)\triangleq 0$。首先，$B(t)\sim\mathcal{N}(0,t)$，因此$Y(t)=tB(\frac{1}{t})\sim\mathcal{N}(0,t)$也是正态过程。并且当$t>0$时，$Y(t)$的轨道连续。而根据大数定律，有

$$\lim_{t\to 0^+}Y(t)=\lim_{t\to 0^+}\frac{B(\frac{1}{t})}{\frac{1}{t}}=\lim_{t\to\infty}\frac{B(t)}{t}=0=Y(0) \tag{49}$$

因此$Y(t)$在$t=0$处的轨道右连续。另一方面，$EY(t)=tEB(\frac{1}{t})=0$，及设$s<t$,

$$EY(t)Y(s)=E(stB(\frac{1}{t})B(\frac{1}{s}))=stEB(\frac{1}{t})B(\frac{1}{s})=st\cdot\frac{1}{t}=s \tag{50}$$

因此由定理5.1.3，知$Y(t)$是一个布朗运动。

(2)

有$W(t)=\frac{1}{a}B(a^2t)$，$a>0$。首先，$B(t)\sim\mathcal{N}(0,t)$，因此$W(t)=\frac{1}{a}B(a^2t)\sim\mathcal{N}(0,t)$也是正态过程，且$EW(t)=0$，轨道连续。有

$$W(0)=\frac{1}{a}B(0)=0 \tag{51}$$

设$0\le s<t$,

$$EW(s)W(t)=E\frac{1}{a^2}B(a^2s)B(a^2t)=\frac{1}{a^2}a^2s=s \tag{52}$$

因此由定理5.1.3，知$W(t)$也是一个布朗运动。

Q5.22

(2)

记

$$X=\{X(t)=e^{\lambda B(t)-\frac{1}{2}\lambda^2t},t\ge 0\} \tag{53}$$

首先，$|EX(t)|\le e^{\lambda|B(t)|}e^{-\frac{1}{2}\lambda^2t}<+\infty$。其次，$\forall 0\le t_0<t_1<\cdots<t_n<t_{n+1}$，有

$$\begin{aligned}
&E(X(t_{n+1})|X(t_n),\cdots,X(t_0)) &(54)\\
=&E(X(t_{n+1})|X(t_n)) &(55)\\
=&E(e^{\lambda B(t_{n+1})-\frac{1}{2}\lambda^2t_{n+1}}|X(t_n)) &(56)\\
=&E(X(t_n)e^{\lambda(B(t_{n+1})-B(t_n))-\frac{1}{2}\lambda^2(t_{n+1}-t_n)}|X(t_n)) &(57)\\
=&X(t_n)e^{-\frac{1}{2}\lambda^2(t_{n+1}-t_n)}Ee^{\lambda(B(t_{n+1})-B(t_n))} &(58)
\end{aligned}$$

而

$$
\begin{aligned}
&Ee^{\lambda(B(t_{n+1})-B(t_n))} && (59)\\
=&\int_{-\infty}^{\infty} e^{\lambda s}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}e^{-\frac{s^2}{2(t_{n+1}-t_n)}}ds && (60)\\
=&\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}e^{-\frac{(s-\lambda(t_{n+1}-t_n))^2}{2(t_{n+1}-t_n)}+\frac{\lambda^2(t_{n+1}-t_n)}{2}}ds && (61)\\
=&e^{\frac{\lambda^2(t_{n+1}-t_n)}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi(t_{n+1}-t_n)}}e^{-\frac{(s-\lambda(t_{n+1}-t_n))^2}{2(t_{n+1}-t_n)}}ds && (62)\\
=&e^{\frac{1}{2}\lambda^2(t_{n+1}-t_n)} && (63)
\end{aligned}
$$

所以

$$E(X(t_{n+1})|X(t_n),\cdots,X(t_0))=X(t_n) \tag{64}$$

因此X是鞅。

(3)

记

$$Y=\{Y(t)=B^2(t)-t,t\geq 0\} \tag{65}$$

首先，$|EY(t)|\leq|B(t)|^2+|t|<+\infty$。其次，$\forall 0\leq t_0<t_1<\cdots<t_n<t_{n+1}$，有

$$
\begin{aligned}
&E(Y(t_{n+1})|Y(t_n),Y(t_{n-1}),\cdots,Y(t_0)) && (66)\\
=&E(Y(t_{n+1})|Y(t_n)) && (67)\\
=&E(B^2(t_{n+1})-t_{n+1}|Y(t_n)) && (68)\\
=&E(B^2(t_n)+2B(t_n)(B(t_{n+1})-B(t_n))+(B(t_{n+1})-B(t_n))^2-t_{n+1}|Y(t_n)) && (69)\\
=&Y(t_n)+2B(t_n)E(B(t_{n+1})-B(t_n))+E(B(t_{n+1})-B(t_n))^2+t_n-t_{n+1} && (70)\\
=&Y(t_n)+0+(t_{n+1}-t_n)+t_n-t_{n+1} && (71)\\
=&Y(t_n) && (72)
\end{aligned}
$$

所以Y也是鞅。