Homework_8

潘子睿 2024310675

Q6.5(3)

由题，该生灭过程有$\lambda_n = n\lambda + a$，$\mu_n = n\mu$。设平稳分布为$\Pi = (\pi_i)_{i\in\mathbb{N}}$，则有以下等式成立：

$$\pi_i = \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}\pi_0 \tag{1}$$

当平稳分布存在时，有

$$\begin{aligned}
\pi_0 =& \left(1+\sum_{i=1}^{\infty}\frac{\prod_{j=0}^{i-1}\lambda_j}{\prod_{j=1}^{i}\mu_j}\right)^{-1} = \left(1+\sum_{i=1}^{\infty}\frac{\prod_{j=0}^{i-1}(j\lambda+a)}{i!\mu^i}\right)^{-1} &(2)\\
=&\left(1+\sum_{i=1}^{\infty}\frac{\lambda^i}{\mu^i}\cdot\frac{\prod_{j=0}^{i-1}(j+\frac{a}{\lambda})}{i!}\right)^{-1} &(3)\\
=&\left(1+\sum_{i=1}^{\infty}\left(-\frac{\lambda}{\mu}\right)^i\frac{\prod_{j=0}^{i-1}(\frac{-a}{\lambda}-j)}{i!}\right)^{-1} &(4)\\
=&\left((1-\frac{\lambda}{\mu})^{-\frac{a}{\lambda}}\right)^{-1} &(5)\\
=&\left(1-\frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}} &(6)
\end{aligned}$$

此时$\forall i > 0$,

$$\begin{aligned}
\pi_i =& \pi_0\cdot\frac{\prod_{j=0}^{i-1}\lambda_j}{\prod_{j=1}^{i}\mu_j} &(7)\\
=&\frac{\prod_{j=0}^{i-1}(\frac{a}{\lambda}+j)}{i!}\cdot(\frac{\lambda}{\mu})^i\cdot(1-\frac{\lambda}{\mu})^{\frac{a}{\lambda}} &(8)
\end{aligned}$$

有$\frac{\lambda+a}{\mu} < 1$，$a > 0$，则$\frac{\lambda}{\mu} < 1$，$\frac{j\lambda+a}{(j+1)\mu} < 1$，则以上平稳分布存在。

Q6.9

(1)

设系统的平稳分布为$\Pi = (\pi_i)_{i\in\mathbb{N}}$，则有

$$
\begin{aligned}
\pi_0^{-1} =& 1+\sum_{i=1}^{\infty}\frac{\prod_{j=0}^{i-1}\lambda_j}{\prod_{j=1}^{i}\mu_j} && (9)\\
=& 1+\sum_{i=1}^{s}\frac{\prod_{j=0}^{i-1}\lambda}{\prod_{j=1}^{i}j\mu}+\sum_{i=s+1}^{\infty}\frac{\prod_{j=0}^{i-1}\lambda}{\prod_{j=1}^{s}j\mu\prod_{j=s+1}^{i}s\mu} && (10)\\
=& 1+\sum_{i=1}^{s}\frac{\lambda^i}{\mu^i i!}+\sum_{i=s+1}^{\infty}\frac{\lambda^i}{s!s^{i-s}\mu^i} && (11)\\
=& \sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{s^s}{s!}\sum_{i=s+1}^{\infty}\rho^i && (12)\\
=& \sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{s^s}{s!}\frac{\rho^{s+1}}{1-\rho} && (13)\\
=& \sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{(s\rho)^s\rho}{s!(1-\rho)} && (14)
\end{aligned}
$$

因此当$\rho<1$时，π_0存在有限，此时

$$
\pi_0=\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1} \tag{15}
$$

$\forall 1\le k\le s$，有

$$
\pi_k=\pi_0\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{k}\mu_i}=\pi_0\frac{\lambda^k}{k!\mu^k}=\frac{\rho^k s^k}{k!\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{(s\rho)^s\rho}{s!(1-\rho)}\right)} \tag{16}
$$

$\forall k>s$，有

$$
\pi_k=\pi_0\frac{\prod_{i=0}^{k-1}\lambda_i}{\prod_{i=1}^{s}\mu_i\prod_{i=s+1}^{k}\mu_i}=\pi_0\frac{\lambda^k}{s!s^{k-s}\mu^k}=\frac{\rho^k s^s}{s!\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{(s\rho)^s\rho}{s!(1-\rho)}\right)} \tag{17}
$$

(2)

在稳态下，有

$$
\begin{aligned}
r =& P(Q(t)=0) && (18)\\
=& P(X(t)\le s) && (19)\\
=& \sum_{k=0}^{s}\pi_k && (20)\\
=& \sum_{k=0}^{s}\frac{\rho^k s^k}{k!}\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1} && (21)\\
=& \left(\sum_{k=0}^{s}\frac{\rho^k s^k}{k!}\right)\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!}+\frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1} && (22)
\end{aligned}
$$

及

$$E[Q(t)] = \sum_{k=0}^{\infty} kP(Q(t)=k) \tag{23}$$

$$= \sum_{k=0}^{\infty} kP(X(t)=s+k) = \sum_{k=1}^{\infty} k\pi_{s+k} \tag{24}$$

$$= \sum_{k=0}^{\infty} k \cdot \frac{\rho^{s+k}s^s}{s!}\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!} + \frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1} \tag{25}$$

$$= \left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!} + \frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1}\frac{s^s\rho^s}{s!}\sum_{k=0}^{\infty}k\rho^k \tag{26}$$

$$= \left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!} + \frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1}\frac{s^s\rho^s}{s!}\frac{\sum_{k=1}^{\infty}\rho^k}{1-\rho} \tag{27}$$

$$= \frac{1}{1-\rho}\sum_{k=1}^{\infty}\frac{s^s\rho^{k+s}}{s!}\left(\sum_{i=0}^{s}\frac{(\rho s)^i}{i!} + \frac{(s\rho)^s\rho}{s!(1-\rho)}\right)^{-1} \tag{28}$$

$$= \frac{1}{1-\rho}\sum_{k=1}^{\infty}\pi_{k+s} \tag{29}$$

$$= \frac{1}{1-\rho}(1-\sum_{k=0}^{s}\pi_k) = \frac{1-r}{1-\rho} \tag{30}$$

得证。

Q6.16

有$\{X(t): t \geq 0\}$是纯生过程，$\lambda_n = n\lambda + \delta$。求$P(t) = (p_{ij}(t))$。

对于该纯生过程，有

$$P_n^{'}(t) = -P_n(t)\lambda_n + P_{n-1}(t)\lambda_{n-1} = -(n\lambda+\delta)P_n(t) + ((n-1)\lambda+\delta)P_{n-1}(t) \tag{31}$$

分两种情况讨论。

若$X(0)=1$，即$P_1(0)=1$，$\forall n \geq 2$，$P_n(0)=0$。此时有$P_1^{'}(t) = -(\lambda+\delta)P_1(t)$。结合$P_1(0)=1$解得

$$P_1(t) = e^{-(\lambda+\delta)t} \tag{32}$$

进一步归纳递推，得

$$P_n(t) = \frac{\prod_{j=1}^{n-1}(\frac{\delta}{\lambda}+j)}{(n-1)!}e^{-(\lambda+\delta)t}[1-e^{-\lambda t}]^{n-1} \tag{33}$$

令

$$
\begin{aligned}
f_1(\rho,t)=&\sum_{n=1}^{\infty}P_n(t)\rho^n && (34)\\
=&e^{-(\lambda+\delta)t}\rho+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{n-1}(\frac{\delta}{\lambda}+j)}{(n-1)!}e^{-(\lambda+\delta)t}[1-e^{-\lambda t}]^{n-1}\rho^n && (35)\\
=&e^{-(\lambda+\delta)t}\rho+\rho\sum_{n=1}^{\infty}\frac{\prod_{j=1}^{n}(\frac{\delta}{\lambda}+j)}{n!}e^{-(\lambda+\delta)t}[1-e^{-\lambda t}]^{n}\rho^n && (36)\\
=&e^{-(\lambda+\delta)t}\rho+\rho e^{-(\lambda+\delta)t}\sum_{n=1}^{\infty}\frac{\prod_{j=1}^{n}(\frac{-\delta}{\lambda}-j)}{n!}(-\rho(1-e^{-\lambda t}))^n && (37)\\
=&\rho e^{-(\lambda+\delta)t}\sum_{n=0}^{\infty}C^n_{-\frac{\delta}{\lambda}-1}(-\rho(1-e^{-\lambda t}))^n && (38)\\
=&\rho e^{-(\lambda+\delta)t}(1-\rho(1-e^{-\lambda t}))^{-\frac{\delta}{\lambda}-1} && (39)
\end{aligned}
$$

从而考虑$X(0)=k$的情形，其可看成k个$X(0)=1$的纯生过程的相加，因此

$$
\begin{aligned}
f_k(\rho,t)=&f_1(\rho,t)^k && (40)\\
=&\rho^k e^{-(\lambda+\delta)kt}(1-\rho(1-e^{-\lambda t}))^{-\frac{k\delta}{\lambda}-k} && (41)\\
=&\rho^k e^{-(\lambda+\delta)kt}\sum_{m=0}^{\infty}C^m_{\frac{k\delta}{\lambda}+k+m-1}(1-e^{-\lambda t})^m\rho^m && (42)\\
=&\sum_{m=0}^{\infty}C^m_{\frac{k\delta}{\lambda}+k+m-1}e^{-(\lambda+\delta)kt}(1-e^{-\lambda t})^m\rho^{m+k} && (43)\\
=&\sum_{m=k}^{\infty}C^{m-k}_{\frac{k\delta}{\lambda}+m-1}e^{-(\lambda+\delta)kt}(1-e^{-\lambda t})^{m-k}\rho^{m} && (44)
\end{aligned}
$$

所以$\forall n\geq k$,

$$
p_{kn}(t)=C^{n-k}_{\frac{k\delta}{\lambda}+n-1}e^{-(\lambda+\delta)kt}(1-e^{-\lambda t})^{n-k} \tag{45}
$$

当$n<k$时，$p_{kn}(t)=0$。

Q6.17(3)

根据Q6.5（3）的结论，即有当平稳分布$\Pi=(\pi_i)$存在时，有

$$
\pi_0=(1-\frac{\lambda}{\mu})^{\frac{\delta}{\lambda}} \tag{46}
$$

及

$$
\begin{aligned}
\pi_n=&\frac{\prod_{j=0}^{n-1}(\frac{\delta}{\lambda}+j)}{n!}(\frac{\lambda}{\mu})^n(1-\frac{\lambda}{\mu})^{\frac{a}{\lambda}} && (47)\\
=&\frac{1}{n!}\frac{\delta}{\lambda}(\frac{\delta}{\lambda}+1)\cdots(\frac{\delta}{\lambda}+n-1)(\frac{\lambda}{\mu})^n(1-\frac{\lambda}{\mu})^{\frac{a}{\lambda}} && (48)
\end{aligned}
$$

而当$\lambda<\mu$，即$\frac{\lambda}{\mu}<1$，此时所有π_i，$i\geq 0$均存在有限。也即当$\lambda<\mu$时，平稳分布存在且唯一，因此此时链为正常返的，得证。

Q4.1

(1)

证明：

首先，$|X_n| = |\sum_{k=1}^n Y_k| \leq \sum_{k=1}^n |Y_k| < +\infty$，$|U_n| = |X_n - n(p-q)| \leq |X_n| + |n(p-q)| < +\infty$，$|V_n| = (\frac{p}{q})^{X_n} < +\infty$，$|W_n| = |U_n^2 - n[1-(p-q)^2]| \leq |U_n|^2 + |n[1-(p-1)^2]| < +\infty$。另一方面，

$$
\begin{aligned}
E(U_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0) =& E(\sum_{k=1}^{n+1} Y_k - (n+1)(p-q)|Y_n, Y_{n-1}, \cdots, Y_0) && (49)\\
=& \sum_{k=1}^{n} Y_k - (n+1)(p-q) + E(Y_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0) && (50)\\
=& \sum_{k=1}^{n} Y_k - (n+1)(p-q) + E(Y_{n+1}) && (51)\\
=& \sum_{k=1}^{n} Y_k - n(p-q) = U_n && (52)
\end{aligned}
$$

$$
\begin{aligned}
E(V_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0) =& E((\frac{q}{p})^{\sum_{k=1}^{n+1} Y_k}|Y_n, Y_{n-1}, \cdots, Y_0) && (53)\\
=& E(\prod_{k=1}^{n+1}(\frac{q}{p})^{Y_k}|Y_n, Y_{n-1}, \cdots, Y_0) && (54)\\
=& \prod_{k=1}^{n+1} E((\frac{q}{p})^{Y_k}|Y_n, Y_{n-1}, \cdots, Y_0) && (55)\\
=& \prod_{k=1}^{n}(\frac{q}{p})^{Y_k} E((\frac{q}{p})^{Y_{n+1}}|Y_n, Y_{n-1}, \cdots, Y_0) && (56)\\
=& (\frac{q}{p})^{X_n} E((\frac{q}{p})^{Y_{n+1}}) && (57)\\
=& V_n(p \cdot \frac{q}{p} + q \cdot \frac{p}{q}) = V_n && (58)
\end{aligned}
$$

$$
\begin{aligned}
& E(W_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0) && (59)\\
=& E(U_{n+1}^2 - (n+1)[1-(p-q)^2]|Y_n, Y_{n-1}, \cdots, Y_0) && (60)\\
=& E(U_{n+1}^2|Y_n, Y_{n-1}, \cdots, Y_0) - (n+1)[1-(p-q)^2] && (61)
\end{aligned}
$$

而

$$
\begin{aligned}
D(U_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0) =& E((U_{n+1} - U_n)^2|Y_n, Y_{n-1}, \cdots, Y_0) && (62)\\
=& E((Y_{n+1} - (p-q))^2|Y_n, Y_{n-1}, \cdots, Y_0) && (63)\\
=& E(Y_{n+1}^2) - 2(p-q)E(Y_{n+1}) + (p-q)^2 && (64)\\
=& 1 - (p-q)^2 && (65)
\end{aligned}
$$

所以

$$
\begin{aligned}
E(U_{n+1}^2|Y_n, Y_{n-1}, \cdots, Y_0) =& D(U_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0) + (E(U_{n+1}|Y_n, Y_{n-1}, \cdots, Y_0)^2 && (66)\\
=& U_n^2 + 1 - (p-q)^2 && (67)
\end{aligned}
$$

进而

$$
\begin{aligned}
E(W_{n+1}|Y_n,Y_{n-1},\cdots,Y_0) =& E(U_{n+1}^2|Y_n,Y_{n-1},\cdots,Y_0)-(n+1)[1-(p-q)^2] && (68)\\
=& U_n^2-n[1-(p-q)^2]=W_n && (69)
\end{aligned}
$$

综上，$\{U_n:n\geq 0\}$，$\{V_n:n\geq 0\}$，$\{W_n:n\geq 0\}$关于$\{Y_n:n\geq 0\}$是鞅。

(2)

证明：首先有$|X_n|<\infty$。且

$$
\begin{aligned}
E(X_{n+1}|Y_n,Y_{n-1},\cdots,Y_0) =& E(X_n+Y_{n+1}|Y_n,Y_{n-1},\cdots,Y_0) && (70)\\
=& X_n+E(Y_{n+1})=X_n+p-q>X_n && (71)
\end{aligned}
$$

所以$\{X_n:n\geq 0\}$关于$\{Y_n:n\geq 0\}$是下鞅。

(3)

有$EU_m=EX_m-m(p-q)=mEY-m(p-q)=0$。因此

$$
\begin{aligned}
cov(U_m,U_{m+n}) =& E((U_m-EU_m)(U_{m+n}-EU_{m+n}))=E(U_mU_{m+n}) && (72)\\
=& E(U_m(U_m+[\sum_{k=m+1}^{m+n}Y_k-n(p-q)])) && (73)\\
=& E(U_m^2)+E(U_m)E([\sum_{k=m+1}^{m+n}Y_k-n(p-q)]) && (74)\\
=& E(U_m^2) && (75)
\end{aligned}
$$

及$\sigma(U_m)=\sqrt{D(U_m)}=\sqrt{(E(U_m^2)-(EU_m)^2)}=\sqrt{E(U_m^2)}$，而

$$
E(U_m^2)=E((X_m-EX_m)^2)=DX_m=mDY \tag{76}
$$

所以相关系数

$$
\rho(U_m,U_{m+n})=\frac{cov(U_m,U_{m+n})}{\sigma U_m\cdot\sigma U_{m+n}}=\sqrt{\frac{E(U_m^2)}{E(U_{m+n}^2)}}=\sqrt{\frac{m}{m+n}} \tag{77}
$$

Q4.6

证明：因为$\{X_n:n\geq 0\}$是鞅，因此

$$
\begin{aligned}
X_n =& E(X_{n+1}|X_n,\cdots,X_0)=E(X_n+\xi_{n+1}|X_n,\cdots,X_0) && (78)\\
=& X_n+E(\xi_{n+1}|X_n,\cdots,X_0) && (79)\\
=& X_n+E(\xi_{n+1}|\xi_n,\cdots,\xi_0) && (80)
\end{aligned}
$$

因此$\forall n\geq 1$，$E(\xi_{n+1}|\xi_n,\cdots,\xi_0)=0$。考虑$\forall j>i>0$，有

而

$$
\begin{aligned}
&E(\xi_j\xi_i|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0) &(81)\\
=&\int_{a,b} abP(\xi_j=a,\xi_i=b|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0) &(82)\\
=&\int_{a,b} abP(\xi_i=b|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0)P(\xi_j=a|\xi_i=b,\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0) &(83)\\
=&\int_{b} bP(\xi_i=b|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0)\int_a aP(\xi_j=a|\xi_i=b,\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0) &(84)\\
=&\int_{b} bP(\xi_i=b|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0)E(\xi_j|\xi_i=b,\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0) &(85)\\
=&\int_{b} bP(\xi_i=b|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0)\cdot 0=0 &(86)
\end{aligned}
$$

所以

$$E(\xi_j\xi_i)=E(E(\xi_j\xi_i|\xi_{j-1},\cdots,\xi_{i+1},\xi_{i-1},\cdots,\xi_0))=0 \tag{87}$$

得证。

Q4.10(1)

证明：

有

$$E(X_{n+1}|X_n,X_{n-1},\cdots,X_0)=E(X_{n+1}|X_n) \tag{88}$$

而

$$E(X_{n+1}|X_n=i)=\sum_{j=0}^{\infty} jp_{ij}=\sum_{j=1}^{\infty}\frac{i^j}{(j-1)!}e^{-i}=i\sum_{j=0}^{\infty}\frac{i^j}{j!}e^{-i}=i \tag{89}$$

所以$E(X_{n+1}|X_n)=X_n$。进而有

$$E(X_{n+1}|X_n,X_{n-1},\cdots,X_0)=X_n \tag{90}$$

从而$E(X_{n+1})=E(E(X_{n+1}|X_n))=E(X_n)=\cdots=E(X_0)<+\infty$。因此$\{X_n,n\geq 0\}$是鞅。

Q4.23

证明：

(1)

首先，我们有 $E|X_n\vee c|<+\infty$，且由于X_n是$Y_0,Y_1,\cdots,Y_n$的函数，因此$X_n\vee C$也是$Y_0,Y_1,\cdots,Y_n$的函数。及

$$E(X_{n+1}\vee c|Y_n,Y_{n-1},\cdots,Y_0)\geq E(X_{n+1}|Y_n,Y_{n-1},\cdots,Y_0)=X_n \tag{91}$$

所以$\{X_n\vee c:n\geq 0\}$关于$\{Y_n:n\geq 0\}$是下鞅。

(2)

(1)中令$c=0$，即知$\{X_n^+ : n\geq 0\}$关于$\{Y_n : n\geq 0\}$是下鞅。