HOMEWORK_6

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Q2.9

设$Y_1, Y_2, \cdots, Y_n$ i.i.d $\sim U[0,t]$是随机变量序列，其次序统计量为$Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}$，则

$$(S_1, S_2, \cdots, S_k, \ldots, S_n) \overset{d}{=} (Y_{(1)}, Y_{(2)}, \cdots, Y_{(k)}, \cdots, Y_{(n)}) \tag{1}$$

从而取足够小的h，使得$x < S_k < x+h$，令事件$A = \{x < S_k < x+h | N_t = n\}$，则

$$\begin{aligned}
P(A) =& P(x < S_k < x+h | N_t = n) = P(x < Y_{(k)} < x+h) && (2)\\
=& P(Y_{(1)} \le x, \cdots, Y_{(k-1)} \le x, x < Y_{(k)} < x+h, Y_{(k+1)} \ge x+h, \cdots, Y_{(n)} \ge x+h) && (3)\\
=& P(Y_{(1)} \le x) \cdots P(Y_{(k-1)} \le x) P(x < Y_{(k)} < x+h) P(Y_{(k+1)} \ge x+h) \cdots P(Y_{(n)} \ge x+h) && (4)\\
=& C_n^{k-1} (\frac{x}{t})^{k-1} \cdot (n-k+1) \cdot \frac{h}{t} (\frac{t-x-h}{t})^{n-k} && (5)\\
=& k C_n^k (\frac{x}{t})^{k-1} \cdot \frac{h}{t} \cdot (\frac{t-x-h}{t})^{n-k} && (6)
\end{aligned}$$

从而S_k的概率密度函数

$$f(x) = \lim_{h \to 0} \frac{P(A)}{h} = k C_n^k \cdot \frac{x^{k-1}(t-x)^{n-k}}{t^n} \tag{7}$$

Q2.26

(1)

取足够小的h，使得

$$x < S_2 < x+h < y < S_5 < y+h \tag{8}$$

令事件$A = \{x < S_2 < x+h < y < S_5 < y+h\}$。则

$$\begin{aligned}
P(A) =& P(\{x < S_2 < x+h < y < S_5 < y+h\}) && (9)\\
=& P(N(x) = 1, N(x+h) = 2, N(y) = 4, N(y+h) = 5) && (10)\\
& (h\text{足够小，} h < \min\{S_2 - S_1, S_3 - S_2, S_5 - S_4, S_6 - S_5\}) && (11)\\
=& P(N(x) = 1), N(x+h) - N(x) = 1, N(y) - N(x+h) = 2, N(y+h) - N(y) = 1 && (12)\\
=& P(N(x) = 1) P(N(x+h) - N(x) = 1) P(N(y) - N(x+h) = 2) P(N(y+h) - N(y) = 1) && (13)\\
=& (\lambda x e^{-\lambda x}) \cdot (\lambda h e^{-\lambda h}) (\frac{(\lambda(y-x-h))^2}{2} e^{-\lambda(y-x-h)}) (\lambda h e^{-\lambda h}) && (14)
\end{aligned}$$

所以(S_2, S_5)的联合概率密度函数

$$f(x,y) = \lim_{h \to 0} \frac{P(A)}{h^2} = \begin{cases} \frac{\lambda^5 x (y-x)^2 e^{-\lambda y}}{2}, & 0 < x < y \\ 0, & otherwise \end{cases} \tag{15}$$

(2)

有

$$E(S_1) = E(S_1 | N_t = 0) P(N_t = 0) + E(S_1 | N_t \ge 1) P(N_t \ge 1) \tag{16}$$

而

$$\begin{cases} E(S_1) & = \frac{1}{\lambda} \\ E(S_1|N_t=0) & = t+\frac{1}{\lambda} \\ P(N_t=0) & = e^{-\lambda t} \\ P(N_t \geq 1) & = 1-e^{-\lambda t} \end{cases} \tag{17}$$

代入即得

$$E(S_1|N_t \geq 1) = \frac{E(S_1)-E(S_1|N_t=0)P(N_t=0)}{P(N_t \geq 1)} = \frac{\frac{1}{\lambda}-(t+\frac{1}{\lambda})e^{-\lambda t}}{1-e^{-\lambda t}} \tag{18}$$

(3)

取足够小的h，使得$x < S_1 < x+h < y < S_2 < y+h$，满足$0 < x < y$，即取

$$h = \min\{t-S_1, S_2-t\} \tag{19}$$

此时有

$$\begin{cases} x+h < S_1+h < t \\ y > S_2-h > t \end{cases} \tag{20}$$

令事件

$$A = \{x < S_1 < x+h < y < S_2 < y+h|N(t)=1\} \tag{21}$$

则

$$\begin{aligned} & P(A) \\ =& P(x < S_1 < x+h < y < S_2 < y+h|N(t)=1) \\ =& \frac{P(x < S_1 < x+h < y < S_2 < y+h, N(t)=1)}{P(N(t)=1)} \\ =& \frac{P(N(x)=0, N(x+h)=1, N(t)=1, N(y)=1, N(y+h)=2)}{P(N(t)=1)} \\ =& \frac{P(N(x)=0, N(x+h)-N(x)=1, N(t)-N(x+h)=0, N(y)-N(t)=0, N(y+h)-N(y)=1)}{P(N(t)=1)} \\ =& \frac{e^{-\lambda x}\lambda h e^{-\lambda h} e^{-\lambda(t-x-h)} e^{-\lambda(y-t)} \lambda h e^{-\lambda h}}{\lambda t e^{-\lambda t}} \\ =& \frac{\lambda h^2 e^{-\lambda(y+h-t)}}{t} \end{aligned}$$

从而(S_1, S_2)在$N(t)=1$下的条件概率密度函数

$$f(x,y) = \lim_{h \to 0} \frac{P(A)}{h^2} = \begin{cases} \frac{\lambda e^{-\lambda(y-t)}}{t}, & 0 < x < t < y \\ 0, & otherwise \end{cases} \tag{29}$$

Q2.28

由题，$X(t) = \sum_{i=1}^{N(t)} \rho_i$为平稳无后效流，其中$\{\rho_i : i \geq 1\}$独立同分布，取值为正整数，且与泊松过程$\{N(t) : t \geq 0\}$独立。设$X(t)$的母函数为$g_X(S)$，有

$$g_X(s) = Es^X \tag{30}$$

而

$$E(s^X|N_t=n) = E(s^{\sum_{i=1}^{N_t}\rho_i}|N_t=n) = E(s^{\sum_{i=1}^{n}\rho_i}) = (E(s^{\rho_1}))^n = (\frac{sp}{1-s+sp})^n = A^n, \quad s(1-p) < 1$$

其中$A = \frac{sp}{1-s+sp}$。所以母函数

$$
\begin{aligned}
g_X(s) =& E(s^X) = E(E(s^X|N_t)) && (32)\\
=& \sum_{n=0}^{\infty} E(s^X|N_t = n)P(N_t = n) && (33)\\
=& \sum_{n=0}^{\infty} A^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} && (34)\\
=& e^{\lambda t(A-1)} && (35)\\
=& e^{\lambda t \frac{s-1}{1-s+sp}} && (36)
\end{aligned}
$$

则

$$
g_X^{'}(s) = e^{\lambda t \frac{s-1}{1-s+sp}} \cdot \lambda t \cdot \frac{1-s+sp-(s-1)(p-1)}{(1-s+sp)^2} = e^{\lambda t \frac{s-1}{1-s+sp}} \cdot \lambda t \cdot \frac{p}{(1-s+sp)^2} \tag{37}
$$

所以

$$
EX(t) = g_X^{'}(1) = \frac{\lambda t}{p} \tag{38}
$$

Q2.14

有一更新过程，$P(X_n = 1) = \frac{1}{3}$，$P(X_n = 2) = \frac{2}{3}$。

1. 计算$P(N(1) = k)$

$$
P(N(1) = k) = P(S_k \le 1, S_{k+1} > 1) \tag{39}
$$

考虑到$S_k = \sum_{n=1}^{k} X_n \ge k$，因此有$k \le 1$。所以

$$
P(N(1) = k) = \begin{cases} P(S_0 \le 1, S_1 > 1) = P(X_1 > 1) = \frac{2}{3}, & k = 0 \\ P(S_1 \le 1, S_2 > 1) = P(X_1 \le 1) = \frac{1}{3}, & k = 1 \end{cases} \tag{40}
$$

2. 计算$P(N(2) = k)$

$$
P(N(2) = k) = P(S_k \le 2, S_{k+1} > 2) \tag{41}
$$

同样有$S_k \ge k$，因此$k \le 2$，所以

$$
P(N(2) = k) = \begin{cases} P(S_0 \le 2, S_1 > 2) & = P(X_1 > 2) = 0, & k = 0 \\ P(S_1 \le 2, S_2 > 2) & = P(X_1 + X_2 > 2) & \\ & = 1 - P(X_1 = 1, X_2 = 1) = \frac{8}{9}, & k = 1 \\ P(S_2 \le 2, S_3 > 2) & = P(S_2 \le 2) & \\ & = P(X_1 = 1, X_2 = 1) = \frac{1}{9}, & k = 2 \end{cases} \tag{42}
$$

3. 计算$P(N(3) = k)$

$$
P(N(3) = k) = P(S_k \le 3, S_{k+1} > 3) \tag{43}
$$

同样有$S_k \ge k$，因此$k \le 3$，所以

$$
P(N(3) = k) = \begin{cases} P(S_0 \le 3, S_1 > 3) & = P(X_1 > 3) = 0, & k = 0 \\ P(S_1 \le 3, S_2 > 3) & = P(X_1 + X_2 > 3) & \\ & = P(X_1 = 2, X_2 = 2) = \frac{4}{9}, & k = 1 \\ P(S_2 \le 3, S_3 > 3) & = P(X_1 + X_2 \le 3, \sum_{i=1}^{3} X_i > 3) & \\ & = P(X_1 + X_2 = 2, X_3 \ge 2) & \\ & + P(X_1 + X_2 = 3, X_3 \ge 1) & \\ & = \frac{14}{27}, & k = 2 \\ P(S_3 \le 3, S_4 > 3) & = P(X_1 + X_2 + X_3 \le 3) = \frac{1}{27}, & k = 3 \end{cases} \tag{44}
$$

Q2.15

考虑随机变量序列$\{X_n : n \geq 1\}$，以及汽车的间距$\{Y_n : n \geq 1\}$，其中$Y_1 = 0$表示第一辆车靠着大门停放，Y_i表示第$i-1$辆车和第i辆车之间的间距，$i \geq 1$。第n辆车的长度为X_n，$n \geq 1$。

令$Z_n = Y_n + X_n$。记$U_n = Z_{n+1}$，$n \geq 1$，则$\{U_n : n \geq 1\}$为一个独立同分布的随机变量序列，且

$$
\begin{aligned}
EX_n = \int_0^{\infty} xe^{-x}dx = -(x+1)e^{-x}\Big|_0^{\infty} = 1 \\
EU_n = EX_n + EY_n = 1 + \frac{1}{2} = \frac{3}{2}
\end{aligned}
\tag{45}
$$

由题，令$S_n = \sum_{k=1}^{n} U_n$，$n \geq 1$，当$x \geq X_1$时，有$N_x = 1 + N_{x-X_1} = 1 + V_y$。其中$y = x - X_1$，且

$$
V_y = N_{x-X_1} = \sup\{n \geq 1 : S_n \leq y, S_{n+1} > y\} \tag{46}
$$

由基本更新定理，有

$$
\lim_{y \to \infty} \frac{m_V(y)}{y} = \frac{1}{\mu} = \frac{2}{3} \tag{47}
$$

而当$x < X_1$时，$N_x = 0$。故

$$
EN_x = E(E(N_x|X_1)) = \int_0^{\infty} E(N_x|X_1 = u)f(u)du \tag{48}
$$

$$
= \int_0^{\infty} (1 + EV_{x-u})e^{-u}du = 1 + \int_0^{\infty} EV_{x-u}e^{-u}du \tag{49}
$$

所以

$$
\lim_{x \to \infty} \frac{EN_x}{x} = \lim_{x \to \infty}\left[\frac{1}{x} + \int_0^{\infty} \frac{EV_{x-u}}{x}e^{-u}du\right] \tag{50}
$$

$$
= \int_0^{\infty} \lim_{y \to \infty} \frac{EV_y}{y+u}e^{-u}du \tag{51}
$$

$$
= \int_0^{\infty} \lim_{y \to \infty} \frac{m_V(y)}{y}e^{-u}du \tag{52}
$$

$$
= \frac{2}{3}\int_0^{\infty} e^{-u}du = \frac{2}{3} \tag{53}
$$

Q2.17

有

$$
f(x) = \lambda^2 x e^{-\lambda x} \tag{54}
$$

则

$$
\tilde{F}(s) = \int_0^{\infty} e^{-st}f(t)dt = \int_0^{\infty} e^{-st}\lambda^2 t^2 e^{-\lambda t}dt = \frac{\lambda^2}{(s+\lambda)^2} \tag{55}
$$

从而

$$
\tilde{m}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)} = \frac{\lambda^2}{s(s+2\lambda)} = \frac{\lambda}{2}\cdot\left(\frac{1}{s} - \frac{1}{s+2\lambda}\right) \tag{56}
$$

考虑$\tilde{m}(s)$的反拉普拉斯变换，有

$$
m^{'}(t) = \frac{\lambda}{2}(1 - e^{-2\lambda t}) \tag{57}
$$

从而

$$m(t) = \int_0^t m^{'}(u)du + m(0) \tag{58}$$

$$= \frac{\lambda}{2}\int_0^t (1 - e^{-2\lambda u})du \tag{59}$$

$$= \frac{\lambda t}{2} + \frac{e^{-2\lambda t}}{4} - \frac{1}{4} \tag{60}$$