Homework_4

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Q3.12

先证明一个引理：

引理：$\forall i \in S$，i为非常返态，$G_{ii} = \frac{1}{1-f_{ii}}$

引理的证明如下：

首先，有$f_{ii} < 1$。令$P(\rho) = \sum_{n=0}^{\infty} p_{ii}^{(n)} \rho^n$，$F(\rho) = \sum_{n=1}^{\infty} f_{ii}^{(n)} \rho^n$，则

$$\begin{aligned}
P(\rho) &= 1 + \sum_{n=1}^{\infty} p_{ii}^{(n)} \rho^n && (1)\\
&= 1 + \sum_{n=1}^{\infty} \sum_{l=1}^{n} f_{ii}^{(l)} p_{ii}^{(n-l)} \rho^n && (2)\\
&= 1 + \sum_{l=1}^{\infty} f_{ii}^{(l)} \rho^l \sum_{n=l}^{\infty} p_{ii}^{(n-l)} \rho^{n-l} && (3)\\
&= 1 + (\sum_{l=1}^{\infty} f_{ii}^{(l)} \rho^l)(\sum_{n=0}^{\infty} p_{ii}^{n} \rho^n) = 1 + F(\rho)P(\rho) && (4)
\end{aligned}$$

令$\rho \to 1^-$，有

$$G_{ii} = 1 + f_{ii} G_{ii} \Longrightarrow G_{ii} = \frac{1}{1-f_{ii}} \tag{5}$$

引理得证。

回原题，有

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{ij}^{(n)} &= \sum_{n=1}^{\infty} \sum_{l=1}^{n} f_{ij}^{(l)} p_{jj}^{(n-l)} && (6)\\
&= \sum_{l=1}^{\infty} f_{ij}^{(l)} \sum_{n=l}^{\infty} p_{jj}^{(n-l)} && (7)\\
&= (\sum_{l=1}^{\infty} f_{ij}^{(l)})(\sum_{n=0}^{\infty} p_{jj}^{(n)}) && (8)\\
&= f_{ij} G_{jj} && (9)\\
&= \frac{f_{ij}}{1-f_{jj}} && (10)
\end{aligned}$$

原题得证。

Q3.15

(1)

下面我们证明一个更一般的命题：$\forall n \geq 0$，有

$$p_{0i}^{(n)} = \frac{b_i}{\sigma_n}, \quad 0 \leq i \leq n \tag{11}$$

我们用归纳法进行证明。

$n = 0$时，$p_{00}^{(0)} = 1$，成立。

$n = 1$时，$p_{00}^{(1)} = p_{00} = \frac{b_0(\beta_0-\beta_1)}{b_0} = \frac{1}{b_0+b_1} = \frac{b_0}{\sigma_1}$；$p_{00}^{(1)} = \frac{\beta_1}{\beta_0} = \frac{b_1}{\sigma_1}$，也成立。

下假设命题对$n = k-1$（$k \geq 2$）成立，考虑$n = k$的情形。

$\forall 1 \leq i \leq k$有

$$
\begin{aligned}
p_{0i}^{(k)} =& p_{0(i-1)}^{k-1} p_{(i-1)i} + \sum_{j=i}^{k-1} p_{0j}^{(k-1)} p_{ji} && (12)\\
=& \frac{b_{i-1}}{\sigma_{k-1}} \frac{\beta_i}{\beta_{i-1}} + \sum_{j=i}^{k-1} \frac{b_j}{\sigma_{k-1}} \cdot \frac{b_i(\beta_j - \beta_{j+1})}{b_j} && (13)\\
=& \frac{b_i \sigma_{i-1}}{\sigma_{k-1}\sigma_i} + \frac{b_i}{\sigma_{k-1}} \sum_{j=i}^{k-1} (\beta_j - \beta_{j+1}) && (14)\\
=& \frac{b_i}{\sigma_{k-1}} \left(\frac{\sigma_{i-1}}{\sigma_i} + \beta_i - \beta_n \right) && (15)\\
=& \frac{b_i}{\sigma_{k-1}} (1 - \beta_n) && (16)\\
=& \frac{b_i}{\sigma_{k-1}} \cdot \frac{\sigma_{k-1}}{\sigma_k} = \frac{b_i}{\sigma_k} && (17)
\end{aligned}
$$

且

$$
\begin{aligned}
p_{00}^{(k)} &= \sum_{j=0}^{k-1} p_{0j}^{(k-1)} p_{j0} && (18)\\
&= \sum_{j=0}^{k-1} \frac{b_j}{\sigma_{k-1}} \cdot \frac{b_0(\beta_j - \beta_{j+1})}{b_j} && (19)\\
&= \frac{b_0}{\sigma_{k-1}} \sum_{j=0}^{k-1} (\beta_j - \beta_{j+1}) && (20)\\
&= \frac{b_0}{\sigma_{k-1}} (\beta_0 - \beta_k) && (21)\\
&= \frac{b_0}{\sigma_{k-1}} \cdot \frac{\sigma_{k-1}}{\sigma_k} = \frac{b_0}{\sigma_k} && (22)
\end{aligned}
$$

因此命题对$n = k$也成立。从而由归纳法，知命题得证。

命题中令$i = 0$，即得

$$
p_{00}^{(n)} = \frac{b_0}{\sigma_n} = \sigma_n^{-1} \tag{23}
$$

(2)

首先，根据转移概率可知该马尔可夫链是不可约的。从而该马尔可夫链是非常返链$\Longleftrightarrow$链上所有状态都是非常返的$\Longleftrightarrow$状态0是非常返的$\Longleftrightarrow \sum_{n=0}^{\infty} p_{00}^{(n)} < +\infty$，也即

$$
\sum_{n=0}^{\infty} \frac{1}{\sigma_n} < \infty \tag{24}
$$

得证。

Q3.19

(1)

对$\{X_n, n \geq 0\}$，$\forall i, j \in \mathbb{Z}$有

$$
\begin{aligned}
p_{ij} &= P(X_1 = j | X_0 = i) && (25)\\
&= P(Y_0 + Y_1 = j | Y_0 = i) && (26)\\
&= \frac{P(Y_0 + Y_1 = j, Y_0 = i)}{P(Y_0 = i)} && (27)\\
&= P(Y_1 = j - i) = \begin{cases} p, & j = i+1 \\ 1, & j = i-1 \\ 0, & otherwise \end{cases} && (28)
\end{aligned}
$$

对$\{X_n^{'}, n \geq 0\}$，$\forall i, j \in \mathbb{Z}$，有转移概率$p_{ij}^{'}$：

$$
\begin{aligned}
p_{ij}^{'} &= P(X_{n+1}^{'} = j | X_n^{'} = i) && (29)\\
&= P(X_{T_{01}+n+1} = j | X_{T_{01}+n} = i) && (30)\\
&= P(X_1 = j | X_0 = i) = p_{ij} && (31)
\end{aligned}
$$

所以$\{X_n, n \geq 0\}$与$\{X_n^{'}, n \geq 0\}$具有相同$\mathbf{P}$。

(2)

证明：

$$
\begin{aligned}
P(T_{01}=5,T_{12}^{'}=3) =&P(X_i\neq 1,X_5=1,1\leq i\leq 4,X_j^{'}\neq 2,1\leq j\leq 2,X_3^{'}=2) &(32)\\
=&P(X_i\neq 1,X_5=1,1\leq i\leq 4,X_j\neq 2,6\leq j\leq 7,X_8=2) &(33)\\
=&P(Y_i=X_i-X_{i-1},Y_5=1-X-4,1\leq i\leq 4,X_i\neq 1 &(34)\\
&Y_j=X_j-X_{j-1},Y_8=2-X_7,6\leq j\leq 7,X_j\neq 2) &(35)\\
=&\sum P(Y_i=x_i-x_{i-1},Y_5=1-x_4,1\leq i\leq 4,x_i\neq 1 &(36)\\
&Y_j=x_j-x_{j-1},y_8=2-y_7,6\leq j\leq 7,x_j\neq 2) &(37)\\
=&\sum P(Y_i=x_i-x_{i-1},Y_5=1-x_4,1\leq i\leq 4,x_i\neq 1) &(38)\\
&\times\sum P(Y_j=x_j-x_{j-1},y_8=2-y_7,6\leq j\leq 7,x_j\neq 2) &(39)\\
=&P(T_{01}=5)\times\sum P(Y_j=x_j-x_{j-1},y_{T_{01}+3}=2-y_{T_{01}+2},T_{01}+1\leq j\leq T_{01}+2,x_j\neq 2) &(40)\\
=&P(T_{01}=5)P(T_{12}^{'}=3) &(41)
\end{aligned}
$$

所以$\{T_{01}=5\}$与$\{T_{12}^{'}=3\}$独立。

另一方面：

$$
\begin{aligned}
P(T_{12}^{'}=i) =&P(X_j^{'}\neq 2,X_i=2,1\leq j<i|X_0^{'}=1) &(42)\\
=&\frac{P(X_{T_{01}}=1,X_{T_{01}+j}\neq 2,X_{T_{01}+i}=2,1\leq j<i)}{P(X_{T_{01}}=1)} &(43)\\
=&\frac{P(X_0=1,X_j\neq 2,X_i=2,1\leq j<i)}{P(X_0=1)} &(44)\\
=&\frac{\sum P(Y_0=1,Y_j=x_j-x_{j-1},Y_i=2-x_{i-1},1\leq j<i,x_j\neq 2)}{P(Y_0=1)} &(45)\\
=&\sum P(Y_j=x_j-x_{j-1},Y_i=2-x_{i-1},1\leq j<i,x_j\neq 2) &(46)\\
&(\text{令}x_j=x_j^{'}+1,\text{则}x_j^{'}\neq 1,\text{再用}x_j\text{替换}x_j^{'}) &(47)\\
=&\sum P(Y_j=x_j-x_{j-1},Y_i=1-x_{i-1},1\leq j<i,x_j\neq 1) &(48)\\
=&\frac{\sum P(Y_0=0,Y_j=x_j-x_{j-1},Y_i=1-x_{i-1},1\leq j<i,x_j\neq 1)}{P(Y_0=0)} &(49)\\
=&P(X_j\neq 1,X_i=1,1\leq j<i|X_0=0)=P(T_{01}=i) &(50)
\end{aligned}
$$

故$\{T_{01}=5\}$与$\{T_{12}^{'}=3\}$独立同分布。

(3)

考虑T_{01}的母函数$g(s)$。有

$$
\begin{aligned}
g(s)\triangleq Es^X =&\sum_{k=1}^{\infty}s^kp_k &(51)\\
=&\sum_{k=1}^{\infty}s^kP(T_{01}=k) &(52)\\
=&sP(T_{01}=1)+\sum_{k=2}^{\infty}s^kP(T_{01}=k) &(53)\\
=&sp+\sum_{k=2}^{\infty}s^kqP(T_{-1\,1}=k-1) &(54)\\
=&sp+sq\sum_{k=1}^{\infty}s^kP(T_{-1\,1}=k) &(55)\\
=&sp+sq\cdot g_{-1}(s) &(56)
\end{aligned}
$$

其中$g_{-1}(s)$表示$T_{-1\,1}$的母函数。假设有一个起点在-1的随机游走，考虑其第一次走到位置0的时机：

$$
\begin{aligned}
g_{-1}(s) =& \sum_{k=1}^{\infty} s^k p_k = \sum_{k=2}^{\infty} s^k p_k && (57)\\
=& \sum_{k=2}^{\infty} s^k P(T_{-1\,1} = k) && (58)\\
=& \sum_{k=2}^{\infty} s^k \sum_{l=1}^{k-1} P(T_{-10} = l) P(T_{01} = k - l) && (59)\\
=& \sum_{k=2}^{\infty} s^k \sum_{l=1}^{k-1} P(T_{01} = l) P(T_{01} = k - l) && (60)\\
=& \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} s^l P(T_{01} = l) s^{k-l} P(T_{01} = k - l) && (61)\\
=& \sum_{l=1}^{\infty} s^l P(T_{01} = l) \sum_{k=l+1}^{\infty} s^{k-l} P(T_{01} = k - l) && (62)\\
=& \sum_{l=1}^{\infty} s^l P(T_{01} = l) \sum_{k=1}^{\infty} s^k P(T_{01} = k) && (63)\\
=& (g(s))^2 && (64)
\end{aligned}
$$

故

$$
g(s) = sp + sq(g(s))^2 \iff sqg^2(s) - g(s) + sp = 0 \tag{65}
$$

注意到这和Catalan数的母函数形式相近。直接求解这一方程，得到

$$
g(s) = \frac{1 \pm \sqrt{1 - 4pqs^2}}{2sq} \tag{66}
$$

注意到$g(0) = 0$，则

$$
\begin{aligned}
g(s) =& \frac{1 - \sqrt{1 - 4pqs^2}}{2sq} && (67)\\
=& \frac{1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4pqs^2)^n}{2sq} && (68)\\
=& \frac{-\sum_{n=1}^{\infty} \frac{(\frac{1}{2}) \cdot (\frac{1}{2} - 1) \cdot (\frac{1}{2} - n + 1)}{n!} (-4pqs^2)^n}{2sq} && (69)
\end{aligned}
$$

考虑其中s^n项的系数a_n。显然当$n = 2k$，$k \in \mathbb{N}^+$时，$a_n = 0$。下设$n = 2k - 1$，$k \in \mathbb{N}^+$。

则

$$
\begin{aligned}
a_{2k-1} =& -\frac{1}{2q} \cdot \left[\frac{(\frac{1}{2}) \cdot (\frac{1}{2} - 1) \cdot (\frac{1}{2} - k + 1)}{k!} \cdot (-4pq)^k \right] && (70)\\
=& -\frac{p^k q^{k-1}}{2} \cdot (-4)^k \cdot 2^{-k} \frac{1 \cdot (-1) \cdot (-3) \cdots (3 - 2k)}{k!} && (71)\\
=& \frac{p^k q^{k-1}}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2k - 3) \cdot 2^k k!}{(k!)^2} && (72)\\
=& \frac{p^k q^{k-1}}{2} \frac{(2k)!}{(k!)^2 (2k - 1)} = C_{2k}^k \frac{p^k q^{k-1}}{2(2k - 1)} && (73)
\end{aligned}
$$

而

$$
g(s) = \sum_{k=1}^{\infty} s^k P(T_{01} = k) \tag{74}
$$

从而

$$
P(T_{01} = n) = \begin{cases} 0, & n = 2k, k \in \mathbb{N}^+ \\ C_{2k}^k \frac{p^k q^{k-1}}{2(2k-1)}, & n = 2k - 1, k \in \mathbb{N}^+ \end{cases} \tag{75}
$$

进一步地，考虑$P(T_{01} = \infty)$。

有$\lim_{k \to \infty} P(T_{01} = 2k) = 0$，另一方面，

$$\lim_{k\to\infty} P(T_{01}=2k-1) = \lim_{k\to\infty} C_{2k}^k \frac{p^k q^{k-1}}{2(2k-1)} \tag{76}$$
$$= \lim_{k\to\infty} \frac{2^{2k}}{\sqrt{\pi k}} \cdot \frac{p^k q^{k-1}}{2(2k-1)} \quad \text{（斯特林公式）} \tag{77}$$
$$= \lim_{k\to\infty} \frac{(2p)^k (2q)^{k-1}}{\sqrt{\pi k}(2k-1)} \tag{78}$$

当$q=0$时，$\lim_{k\to\infty} P(T_{01}=2k-1)=0$。

当$q\neq 0$时，

$$\lim_{k\to\infty} P(T_{01}=2k-1) = \frac{1}{2q} \lim_{k\to\infty} \frac{(4pq)^k}{\sqrt{\pi k}(2k-1)} \tag{79}$$

注意到$pq \le (\frac{p+q}{2})^2 = \frac{1}{4}$，故$\lim_{k\to\infty} P(T_{01}=2k-1)=0$。

综上，$P(T_{01}=+\infty)=\lim_{k\to\infty}(T_{01}=k)=0$。

Q3.10

(1)

令$p_k = P(X_n = t, 某个n\ge 0 | X_0 = k)$，则有递推关系

$$p_k = \mu_k p_{k-1} + (1-\mu_k-\lambda_k)p_k + \lambda_k p_{k+1} \tag{80}$$
$$\Longleftrightarrow \lambda_k(p_{k+1}-p_k) = \mu_k(p_k - p_{k-1}) \tag{81}$$
$$\Longleftrightarrow \frac{p_{k+1}-p_k}{p_k-p_{k-1}} = \frac{\mu_k}{\lambda_k} \tag{82}$$

其中$t=0$或N，$1\le k\le N-1$。

令$q_k = p_k - p_{k-1}$，$1\le k\le N$，$r_s = \frac{\mu_s}{\lambda_s}, 1\le s< N$。则有

$$q_{k+1} = q_k \cdot r_k, 1\le k\le N-1 \tag{83}$$

则

$$q_k = \prod_{i=1}^{k-1} r_i q_1, 2\le k\le N \tag{84}$$

进而

$$p_N - p_0 = \sum_{k=1}^N q_k = q_1(1+\sum_{k=2}^N \prod_{i=1}^{k-1} r_i) \tag{85}$$

1. 当$t=0$时，也即$p_k = P(X_n=0, 某个n\ge 0|X_0=k)$，有$p_N=0$，$p_0=1$，从而

$$-1 = q_1(1+\sum_{k=2}^N \prod_{i=1}^{k-1} r_i) \tag{86}$$
$$\Longrightarrow q_1 = -\frac{1}{1+\sum_{k=2}^N \prod_{i=1}^{k-1} r_i} \tag{87}$$

   从而

$$p_k - p_0 = q_1(1+\sum_{j=2}^k \prod_{i=1}^{j-1} r_i) \tag{88}$$
$$\Longrightarrow p_k = 1 - \frac{1+\sum_{j=2}^k \prod_{i=1}^{j-1} r_i}{1+\sum_{j=2}^N \prod_{i=1}^{j-1} r_i} = \frac{\sum_{j=k+1}^N \prod_{i=1}^{j-1} r_i}{1+\sum_{j=2}^N \prod_{i=1}^{j-1} r_i} \overset{令r_0=1}{=} \frac{\sum_{j=k+1}^N \prod_{i=0}^{j-1} r_i}{\sum_{j=1}^N \prod_{i=0}^{j-1} r_i} \tag{89}$$

   也即

$$P(X_n=0, 某个n\ge 0|X_0=k) = \frac{\sum_{j=k+1}^N \prod_{i=0}^{j-1} r_i}{\sum_{j=1}^N \prod_{i=0}^{j-1} r_i} \tag{90}$$

2. $t=N$时，也即$p_k = P(X_n=N, 某个n\ge 0|X_0=k)$，有$p_N=1$，$p_0=0$，从而

$$
\begin{aligned}
1 =&q_1(1+\sum_{k=2}^{N}\prod_{i=1}^{k-1}r_i) && (91)\\
\Longrightarrow q_1 =&\frac{1}{1+\sum_{k=2}^{N}\prod_{i=1}^{k-1}r_i} && (92)
\end{aligned}
$$

从而

$$
\begin{aligned}
p_k-p_0 =&q_1(1+\sum_{j=2}^{k}\prod_{i=1}^{j-1}r_i) && (93)\\
\Longrightarrow p_k =&\frac{1+\sum_{j=2}^{k}\prod_{i=1}^{j-1}r_i}{1+\sum_{j=2}^{N}\prod_{i=1}^{j-1}r_i}\overset{令r_0=1}{=}\frac{\sum_{j=1}^{k}\prod_{i=0}^{j-1}r_i}{\sum_{j=1}^{N}\prod_{i=0}^{j-1}r_i} && (94)
\end{aligned}
$$

也即

$$
P(X_n=N,某个n\geq 0|X_0=k)=\frac{\sum_{j=1}^{k}\prod_{i=0}^{j-1}r_i}{\sum_{j=1}^{N}\prod_{i=0}^{j-1}r_i} \qquad (95)
$$

(2)

有递推关系

$$
\begin{aligned}
ET_{kt} =&\mu_k\sum_{i=0}^{\infty}(i+1)p_{k-1\,t}^{(i)}+(1-\mu_k-\lambda_k)\sum_{i=0}^{\infty}(i+1)p_{k\,t}^{(i)}+\lambda_k\sum_{i=0}^{\infty}(i+1)p_{k+1\,t}^{(i)} && (96)\\
=&\mu_k(ET_{k-1\,t}+p_{k-1\,t})+(1-\mu_k-\lambda_k)(ET_{kt}+p_{k\,t})+\lambda_k(ET_{k+1\,t}+p_{k+1\,t}) && (97)
\end{aligned}
$$

其中$t\in\{0,N\}$，$1\leq k\leq N-1$，$p_{k\,t}=P(X_n=t,某个n\geq 0|X_0=k)$。令$x_{kt}=ET_{kt}+p_{k\,t}$，$0\leq k\leq N$，$y_{st}=x_{st}-x_{s-1\,t}$，$1\leq s\leq N$。为了简洁性，下推导过程中省略 t，化简如下：

$$
\begin{aligned}
\mu_k(x_k-x_{k-1}) =&\lambda_k(x_{k+1}-x_k)+p_{k\,t} && (98)\\
\Longrightarrow y_{k+1} =&\frac{\mu_k}{\lambda_k}y_k-\frac{p_{k\,t}}{\lambda_k} && (99)
\end{aligned}
$$

其中$k\geq 1$，满足$\lambda_k>0$。依次递推，得

$$
\begin{aligned}
y_k =&\frac{\mu_{k-1}}{\lambda_{k-1}}y_{k-1}-\frac{p_{k-1\,t}}{\lambda_{k-1}} && (100)\\
=&\frac{\mu_{k-1}}{\lambda_{k-1}}(\frac{\mu_{k-2}}{\lambda_{k-2}}y_{k-2}-\frac{p_{k-2\,t}}{\lambda_{k-2}})-\frac{p_{k-1\,t}}{\lambda_{k-1}}=\frac{\mu_{k-1}\mu_{k-2}}{\lambda_{k-1}\lambda_{k-2}}y_{k-2}-(\frac{\mu_{k-1}p_{k-2\,t}}{\lambda_{k-1}\lambda_{k-2}}+\frac{p_{k-1\,t}}{\lambda_{k-1}}) && (101)\\
\cdots & && (102)\\
=&\frac{\prod_{i=1}^{k-1}\mu_i}{\prod_{i=1}^{k-1}\lambda_i}y_1-\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1}\mu_j)p_{i\,t}}{\prod_{j=i}^{k-1}\lambda_j}\quad(定义当m>n时，\prod_{s=m}^{n}f(s)=1) && (103)\\
=&\prod_{i=1}^{k-1}r_iy_1-\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1}r_j)p_{i\,t}}{\lambda_i}\quad(令r_j=\frac{\mu_j}{\lambda_j}，1\leq j<N) && (104)
\end{aligned}
$$

而

$$
\begin{aligned}
x_N-x_0 =&y_1+\sum_{k=2}^{N}y_k && (105)\\
=&y_1+\sum_{k=2}^{N}\left[\prod_{i=1}^{k-1}r_iy_1-\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1}r_j)p_{i\,t}}{\lambda_i}\right] && (106)\\
=&y_1+\sum_{k=2}^{N}\prod_{i=1}^{k-1}r_iy_1-\sum_{k=2}^{N}\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1}r_j)p_{i\,t}}{\lambda_i} && (107)
\end{aligned}
$$

1. $t=0$，即$x_k=ET_{k0}+p_{k0}$。此时$ET_{00}=ET_{N0}=0,p_{00}=1,p_{N0}=0$。从而$x_0=1$，$x_N=0$。进而$x_N-x_0=-1$，从而：

$$
y_1=\frac{\sum_{k=2}^{N}\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1}r_j)p_{i\,0}}{\lambda_i}-1}{1+\sum_{k=2}^{N}\prod_{i=1}^{k-1}r_i}=\frac{\sum_{k=1}^{N}\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1}r_j)p_{i0}}{\lambda_i}-1}{\sum_{k=1}^{N}\prod_{i=1}^{k-1}r_i}\quad(记\prod_{s=i}^{j}f(s)=1,当j<i时) \qquad (108)
$$

从而

$$\begin{aligned}
x_k - 1 = x_k - x_0 =& y_1 + \sum_{s=2}^{k} y_s \qquad (109)\\
=&(1+\sum_{s=2}^{k}\prod_{i=1}^{s-1} r_i)y_1 - \sum_{s=1}^{k}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i} \qquad (110)\\
=&\frac{(\sum_{s=1}^{k}\prod_{i=1}^{s-1} r_i)(\sum_{s=1}^{N}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i}-1)}{\sum_{s=1}^{N}\prod_{i=1}^{s-1} r_i} - \sum_{s=1}^{k}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i} \qquad (111)\\
=&\frac{(\sum_{s=1}^{k}\prod_{i=1}^{s-1} r_i)(\sum_{s=k+1}^{N}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i}-1)-(\sum_{s=k+1}^{N}\prod_{i=1}^{s-1} r_i)\cdot(\sum_{s=1}^{k}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i})}{\sum_{s=1}^{N}\prod_{i=1}^{s-1} r_i} \qquad (112)
\end{aligned}$$

所以

$$\begin{aligned}
ET_{k0} = x_k - p_{k0} =& \frac{(\sum_{s=1}^{k}\prod_{i=1}^{s-1} r_i)(\sum_{s=k+1}^{N}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i}-1)-(\sum_{s=k+1}^{N}\prod_{i=1}^{s-1} r_i)\cdot(\sum_{s=1}^{k}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{i0}}{\lambda_i})}{\sum_{s=1}^{N}\prod_{i=1}^{s-1} r_i} \qquad (113)\\
&+1-p_{k0} \qquad (114)
\end{aligned}$$

其中根据第（1）小题

$$p_{k0} = P(X_n = 0, 某个n \ge 0 | X_0 = k) = \frac{\sum_{j=k+1}^{N}\prod_{i=0}^{j-1} r_i}{\sum_{j=1}^{N}\prod_{i=0}^{j-1} r_i} \tag{115}$$

2. $t = N$，即$x_k = E_{T_{kN}} + p_{kN}$，此时边界条件$ET_{0N} = ET_{NN} = 0$，$p_{0N} = 0, p_{NN} = 1$。从而$x_0 = 0$，$x_N = 1$，进而$x_N - x_0 = 1$。从而：

$$y_1 = \frac{\sum_{k=2}^{N}\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{iN}}{\lambda_i}+1}{1+\sum_{k=2}^{N}\prod_{i=1}^{k-1} r_i} = \frac{\sum_{k=1}^{N}\sum_{i=1}^{k-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{iN}}{\lambda_i}+1}{\sum_{k=1}^{N}\prod_{i=1}^{k-1} r_i} \quad (记\prod_{s=i}^{j} f(s) = 1, 当j < i时) \tag{116}$$

从而

$$\begin{aligned}
x_k = x_k - x_0 =& y_1 + \sum_{k=2}^{k} y_s \qquad (117)\\
=&\frac{(\sum_{s=1}^{k}\prod_{i=1}^{s-1} r_i)(\sum_{s=k+1}^{N}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{iN}}{\lambda_i}+1)-(\sum_{s=k+1}^{N}\prod_{i=1}^{s-1} r_i)\cdot(\sum_{s=1}^{k}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{iN}}{\lambda_i})}{\sum_{s=1}^{N}\prod_{i=1}^{s-1} r_i} \qquad (118)
\end{aligned}$$

所以

$$\begin{aligned}
ET_{kN} = x_k - p_{kN} =& \frac{(\sum_{s=1}^{k}\prod_{i=1}^{s-1} r_i)(\sum_{s=k+1}^{N}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{iN}}{\lambda_i}+1)-(\sum_{s=k+1}^{N}\prod_{i=1}^{s-1} r_i)\cdot(\sum_{s=1}^{k}\sum_{i=1}^{s-1}\frac{(\prod_{j=i+1}^{k-1} r_j)p_{iN}}{\lambda_i})}{\sum_{s=1}^{N}\prod_{i=1}^{s-1} r_i} \qquad (119)\\
&-p_{kN} \qquad (120)
\end{aligned}$$

其中根据第（1）小题

$$p_{kN} = P(X_n = N, 某个n \ge 0 | X_0 = k) = \frac{\sum_{j=1}^{k}\prod_{i=0}^{j-1} r_i}{\sum_{j=1}^{N}\prod_{i=0}^{j-1} r_i} \tag{121}$$

Q3.18

考虑X的状态转移函数。

$$\begin{aligned}
p_{ij} =& P(X_n = j | X_{n-1} = i) = P(X_{n-1} + Y_n = j | X_{n-1} = i) \qquad (122)\\
=&\frac{P(X_{n-1} + Y_n = j, X_{n-1} = i)}{P(X_{n-1} = i)} \qquad (123)\\
=&\frac{P(X_{n-1} = j, Y_n = j - i)}{P(X_{n-1} = j)} \qquad (124)\\
=&P(Y_n = j - i) \qquad (125)\\
=&\begin{cases} p, & j = i+1 \\ q, & j = i-1 \\ 0, & otherwise \end{cases} \qquad (126)
\end{aligned}$$

也即状态i只能以概率p转移到$i+1$，或是以概率q转移到$i-1$。

(1)

$b=3$，有

$$\begin{aligned}P(T_1=k|X_0=1)=&P(X_i\neq 0,X_i\neq 3,1\le i<k,X_k=0\text{或}X_k=3|X_0=1) &(127)\\ =&P(X_i=1\text{或}X_i=2,1\le i<k,X_k=0\text{或}X_k=3|X_0=1) &(128)\\ =&\frac{P(X_0=1,X_i=1\text{或}X_i=2,1\le i<k,X_k=0\text{或}X_k=3)}{P(X_0=1)} &(129)\\ =&\frac{P(X_0=1,X_1=2,\cdots,X_{k-1}=(\frac{3}{2}+\frac{1}{2}\cdot(-1)^k),X_k=(\frac{3}{2}+\frac{3}{2}\cdot(-1)^k))}{P(X_0=1)} &(130)\\ =&P(X_1=2|X_0=1)P(X_2=1|X_1=2)\cdots &(131)\\ &P(X_k=(\frac{3}{2}+\frac{3}{2}\cdot(-1)^k)|X_{k-1}=(\frac{3}{2}+\frac{1}{2}\cdot(-1)^k)) &(132)\\ =&\begin{cases}p^tq^{t+1}, & k=2t+1,t\in\mathbb{N}\\ p^{t+1}q^{t-1}, & k=2t,t\in\mathbb{N}^+\end{cases} &(133)\end{aligned}$$

由上可知：当$2|k$时，$X_{T_1}=3$，否则$X_{T_1}=0$。

所以

$$\begin{aligned}P(X_{T_1}=3|X_0=1)=&\frac{\sum_{t=1}^{\infty}p^{t+1}q^{t-1}}{\sum_{t=1}^{\infty}p^{t+1}q^{t-1}+\sum_{t=0}^{\infty}p^tq^{t+1}} &(134)\\ =&\frac{p^2}{p^2+q} &(135)\\ =&\frac{p^2}{p^2-p+1} &(136)\end{aligned}$$

(2)

$b=5$，求$P(T_2=k|X_0=2)$。考虑到此时当$1\le i<k$时，均有$1\le X_i\le 4$，因此我们先求以下概率：

$$P_{kj}=P(X_k=j,1\le X_i\le 4,1\le i<k|X_0=2) \tag{137}$$

其中$j\in\{1,2,3,4\}$，$k\in\mathbb{N}$。显然，根据转移概率，我们有

$$\begin{cases}P_{k1}=qP_{k-1\,2}\\ P_{k2}=pP_{k-1\,1}+qP_{k-1\,3}\\ P_{k3}=pP_{k-1\,2}+qP_{k-1\,4}\\ P_{k4}=pP_{k-1\,3}\end{cases} \tag{138}$$

且有$P_{01}=0$，$P_{02}=1$，$P_{03}=0$，$P_{04}=0$。以及$P_{11}=q$，$P_{12}=0$，$P_{13}=p$，$P_{14=0}$。

下用归纳法证明以下命题：

令$A=\frac{5-\sqrt{5}}{10}$，$B=\frac{5+\sqrt{5}}{10}$，$a=\frac{1-\sqrt{5}}{2}$，$b=\frac{1+\sqrt{5}}{2}$，则

$$\begin{gathered}P_{k1}=\begin{cases}0, & k=2s,s\in\mathbb{N}\\ (Aa^{2s}+Bb^{2s})p^sq^{s+1}, & k=2s+1,s\in\mathbb{N}\end{cases}\\ P_{k2}=\begin{cases}(Aa^{2s}+Bb^{2s})p^sq^s, & k=2s,s\in\mathbb{N}\\ 0, & k=2s+1,s\in\mathbb{N}\end{cases}\\ P_{k3}=\begin{cases}0, & k=2s,s\in\mathbb{N}\\ (Aa^{2s+1}+Bb^{2s+1})p^{s+1}q^s, & k=2s+1,s\in\mathbb{N}\end{cases}\\ P_{k4}=\begin{cases}(Aa^{2s-1}+Bb^{2s-1})p^{s+1}q^{s-1}, & k=2s,s\in\mathbb{N}^+\\ 0, & k=2s+1\text{或}k=0,s\in\mathbb{N}\end{cases}\end{gathered} \tag{139}$$

对$k=0,1$，直接验证即知命题成立。

下假设命题对$k=2t,2t+1$成立，$t\ge 0$。考虑$k=2t+2,2t+3$的情形。

$k=2t+2$时，有

$$\begin{gathered}P_{2t+2\,1}=qP_{2t+1\,2}=0\\ P_{2t+2\,2}=pP_{2t+1\,1}+qP_{2t+1\,3}=p^{t+1}q^{t+1}(Aa^{2t}+Bb^{2t}+Aa^{2t+1}+Bb^{2t+1})=p^{t+1}q^{t+1}(Aa^{2t+2}+Bb^{2t+2})\\ (\because 1+a=\frac{3-\sqrt{5}}{2}=a^2,1+b=\frac{3+\sqrt{5}}{2}=b^2)\\ P_{2t+2\,3}=pP_{2t+1\,2}+qP_{2t+1\,4}=0\\ P_{2t+2\,4}=pP_{2t+1\,3}=(Aa^{2t+1}+Bb^{2t+1})p^{t+2}q^t\end{gathered} \tag{140}$$

进一步，$k=2t+3$时，有

$$
\begin{gathered}
P_{2t+3\,1}=qP_{2t+2\,2}=p^{t+1}q^{t+2}(Aa^{2t+2}+Bb^{2t+2})\\
P_{2t+3\,2}=pP_{2t+2\,1}+qP_{2t+2\,3}=0\\
P_{2t+3\,3}=pP_{2t+2\,2}+qP_{2t+2\,4}=p^{t+2}q^{t+1}(Aa^{2t+2}+Bb^{2t+2}+Aa^{2t+1}Bb^{2t+1})=p^{t+2}q^{t+1}(Aa^{2t+3}+Bb^{2t+3})\\
P_{2t+3\,4}=pP_{2t+2\,3}=0
\end{gathered}
\tag{141}
$$

都成立。故由归纳法，知命题得证。

回原题。考虑$P(T_2=k|X_0=2)$，对$k\geq 2$时，分析可知X_{k-1}一定为1或是4，则

$$
\begin{aligned}
P(T_2=k|X_0=2)=&P_{k-1\,1}\cdot q+P_{k-1\,4}\cdot p & (142)\\
=&\begin{cases}(Aa^{2s-2}+Bb^{2s-2})p^{s-1}q^{s+1}, & k=2s, s\in\mathbb{N}\\ (Aa^{2s-1}+Bb^{2s-1})p^{s+2}q^{s-1}, & k=2s+1, s\in\mathbb{N}\end{cases} & (143)\\
=&\begin{cases}\left[\frac{5-\sqrt{5}}{10}\cdot(\frac{1-\sqrt{5}}{2})^{2s-2}+\frac{5+\sqrt{5}}{10}\cdot(\frac{1+\sqrt{5}}{2})^{2s-2}\right]p^{s-1}q^{s+1}, & k=2s, s\in\mathbb{N}\\ \left[\frac{5-\sqrt{5}}{10}\cdot(\frac{1-\sqrt{5}}{2})^{2s-1}+\frac{5+\sqrt{5}}{10}\cdot(\frac{1+\sqrt{5}}{2})^{2s-1}\right]p^{s+2}q^{s-1}, & k=2s+1, s\in\mathbb{N}\end{cases} & (144)
\end{aligned}
$$

当$k=1$时，有$P(T_2=1|X_0=2)=0$。

考虑$P(X_{T_2}=5|X_0=2)$，有

$$
\begin{aligned}
P(X_{T_2}=5|X_0=2)=&\frac{\sum_{s=1}^{\infty}(Aa^{2s-1}+Bb^{2s-1})p^{s+2}q^{s-1}}{\sum_{s=1}^{\infty}(Aa^{2s-1}+Bb^{2s-1})p^{s+2}q^{s-1}+\sum_{s=1}^{\infty}(Aa^{2s-2}+Bb^{2s-2})p^{s-1}q^{s+1}} & (145)\\
=&\frac{\sum_{s=1}^{\infty}Aa^{2s-1}p^{s+2}q^{s-1}+\sum_{s=1}^{\infty}Bb^{2s-1}p^{s+2}q^{s-1}}{\sum_{s=1}^{\infty}Aa^{2s-1}p^{s+2}q^{s-1}+\sum_{s=1}^{\infty}Bb^{2s-1}p^{s+2}q^{s-1}+\sum_{s=1}^{\infty}Aa^{2s-2}p^{s-1}q^{s+1}+\sum_{s=1}^{\infty}Bb^{2s-2}p^{s-1}q^{s+1}} & (146)\\
=&\frac{\frac{Aap^3}{1-a^2pq}+\frac{Bbp^3}{1-b^2pq}}{\frac{Aap^3}{1-a^2pq}+\frac{Bbp^3}{1-b^2pq}+\frac{Aq^2}{1-a^2pq}+\frac{Bq^2}{1-b^2pq}} & (147)\\
=&\frac{p^3}{p^3+q^2-pq^3}=\frac{p^3}{p^4-2p^3+4p^2-3p+1} & (148)
\end{aligned}
$$