Homework_3

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Q1.15

X在$X\geq 0$下的条件概率密度函数为：

$$f(x|X\geq 0)=\frac{f(x,X\geq 0)}{f(X\geq 0)}=\frac{\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}{\int_0^\infty\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\}dt}=\frac{\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}{\int_0^\infty\exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\}dt}\approx\frac{\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}}{2.44948},\forall x\geq 0 \tag{1}$$

当$\mu=2,\sigma=1$时，

$$\begin{aligned}
E(X|X\geq 0)&=\int_0^\infty xf(x|x\geq 0)dx &(2)\\
&=\int_0^\infty\frac{x\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx}{\int_0^\infty\exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\}dt} &(3)\\
&=\frac{\int_0^\infty x\exp\left\{-\frac{(x-2)^2}{2}\right\}dx}{\int_0^\infty\exp\left\{-\frac{(t-2)^2}{2}\right\}dt} &(4)\\
&=\frac{\int_{-2}^\infty(t+2)\exp\left\{-\frac{t^2}{2}\right\}dt}{\int_{-2}^\infty\exp\left\{-\frac{t^2}{2}\right\}dt} &(5)\\
&=\frac{\int_{-2}^\infty te^{-\frac{t^2}{2}}dt}{\int_{-2}^\infty\exp\left\{-\frac{t^2}{2}\right\}dt}+2 &(6)\\
&=-\frac{\int_{-2}^\infty de^{-\frac{t^2}{2}}}{\int_{-2}^\infty\exp\left\{-\frac{t^2}{2}\right\}dt}+2 &(7)\\
&=-\frac{e^{-\frac{t^2}{2}}\Big|_{-2}^\infty}{\int_{-2}^\infty\exp\left\{-\frac{t^2}{2}\right\}dt}+2 &(8)\\
&=\frac{1}{e^2\int_{-2}^\infty\exp\left\{-\frac{t^2}{2}\right\}dt}+2\approx 2.05525 &(9)
\end{aligned}$$

Q1.18

有：

$$\begin{aligned}
P(\xi=k)&=\sum_{n=0}^\infty P(N=n)P(\xi=k|N=n) &(10)\\
&=\sum_{n=k}^\infty\frac{\lambda^n}{n!}e^{-\lambda}C_n^kp^k(1-p)^{n-k} &(11)\\
&=p^k\sum_{n=k}^\infty\frac{\lambda^n}{n!}e^{-\lambda}\frac{n!}{k!(n-k)!}(1-p)^{n-k} &(12)\\
&=\frac{(\lambda p)^k}{k!}e^{-\lambda p}\sum_{n=k}^\infty\frac{(\lambda\cdot(1-p))^{n-k}}{(n-k)!}e^{-\lambda(1-p)}=\frac{(\lambda p)^k}{k!}e^{-\lambda p}\sim\mathcal{P}(\lambda p) &(13)
\end{aligned}$$

即ξ服从泊松分布$\mathcal{P}(\lambda p)$。故

$$\begin{aligned}E\xi=\lambda p\\ D\xi=\lambda p\end{aligned} \tag{14}$$

Q1.19

上周已经做过。

Q1.21

有

$$\begin{aligned}
D(X|Y) &= E[(X - E(X|Y))^2|Y] && (15)\\
&= E[(X - \sum_{i=0}^{n} x_i P(X = x_i|Y))^2|Y] && (16)\\
&= \sum_{i=0}^{n}[(x_i - \sum_{i=0}^{n} x_i P(X = x_i|Y))^2 P(X = x_i|Y)] && (17)\\
&= \sum_{i=0}^{n}(x_i^2 - 2x_i S + S^2)P(X = x_i|Y) \quad \text{设} S = \sum_{i=0}^{n} x_i P(X = x_i|Y) = EX && (18)\\
&= E(X^2|Y) - E^2(X|Y) && (19)
\end{aligned}$$

则

$$\begin{aligned}
E(D(X|Y)) + D(E(X|Y)) &= E[E(X^2|Y) - E^2(X|Y)] + [E(E^2(X|Y)) - E^2(E(X|Y))] && (20)\\
&= \sum_{j=0}^{m}[E(X^2|y_j) - E^2(X|y_j)]P(Y = y_j) + [E(E^2(X|Y)) - E^2(E(X|Y))] && (21)\\
&= \sum_{j=0}^{m} E(X^2|y_j)P(Y = y_j) - \sum_{j=0}^{m} E^2(X|y_j)P(Y = y_j) + [E(E^2(X|Y)) - E^2(E(X|Y))] && (22)\\
&= E(X^2) - E(E^2(X|Y)) + E(E^2(X|Y)) - E^2X && (23)\\
&= EX^2 - E^2X = DX && (24)
\end{aligned}$$

得证。

Q补充题1

假设连续型随机向量(X, Y)具有联合概率密度

$$f(x,y) = \begin{cases} 24(1-x)y, & 0 < y < x < 1 \quad (25)\\ 0, & otherwise \quad (26) \end{cases}$$

1. 求X概率密度$f_Y(x)$
2. 给定$x \in (0,1)$，求$X = x$条件下Y的条件期望
3. $E(X|Y)$

解：

1. 有

$$\begin{aligned}
f_Y(x) &= \int_{-\infty}^{\infty} f(x,y)dy && (27)\\
&= \begin{cases} 0, & x \le 0 \\ \int_0^x 24(1-x)ydy, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases} && (28)\\
&= \begin{cases} 12(1-x)x^2, & 0 < x < 1 \\ 0, & otherwise \end{cases} && (29)
\end{aligned}$$

2. $x \in (0,1)$，有

$$\begin{aligned}
E(Y|X=x) &= \int_0^x y \cdot P(Y = y|X = x)dy && (30)\\
&= \int_0^x y \cdot 24(1-x)ydy && (31)\\
&= 8(1-x)x^3 && (32)
\end{aligned}$$

3. 当$Y = y \in (0,1)$时，有

$$\begin{aligned}
E(X|Y=y) &= \int_{-\infty}^{\infty} xf(X = x|Y = y)dx && (33)\\
&= \int_y^1 x \cdot 24(1-x)ydx = (12x^2 - 8x^3)y\Big|_y^1 = 4y - 12y^3 + 8y^4 && (34)
\end{aligned}$$

   当$Y = y \notin (0,1)$时，有$f(x,y) = 0$，$E(X|Y = y) = 0$。

   综上，

$$E(X|Y) = \mathit{1}_{Y\in(0,1)} \cdot (4Y - 12Y^3 + 8Y^4) \tag{35}$$

其中$\mathit{1}_{Y\in(0,1)} = \begin{cases} 1, & Y \in (0,1) \\ 0, & Y \notin (0,1) \end{cases}$

Q3.1

(1)

有

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{36}$$

及$X_0 = 3$，$S = \{1,2,3\}$。

故

$$\begin{gathered} P(X_1 = 3) = P(X_1 = 3|X_0 = 3) = p_{33} = \frac{2}{3} \\ P(X_1 = 2) = P(X_1 = 2|X_0 = 3) = p_{32} = \frac{1}{3} \\ P(X_1 = 1) = P(X_1 = 1|X_0 = 3) = p_{31} = 0 \end{gathered} \tag{37}$$

所以$\pi(1) = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$。

则

$$\pi(2) = \pi(1)\mathbf{P}_1 = \begin{bmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{3} \end{bmatrix} \tag{38}$$

所以

$$E(X_2) = \sum_{i=1}^{3} i \cdot \pi_i(2) = \frac{1}{9} + 2 \cdot \frac{2}{9} + 3 \cdot \frac{2}{3} = \frac{23}{9} \tag{39}$$

$$E(X_2|X_1 = j) = \sum_{i=1}^{3} i \cdot P(X_2 = i|X_1 = j) = \sum_{i=1}^{3} i \cdot p_{ji} = \begin{cases} \sum_{i=1}^{3} i \cdot p_{1i} = 1, & j = 1 \\ \sum_{i=1}^{3} i \cdot p_{2i} = \frac{7}{3}, & j = 2 \\ \sum_{i=1}^{3} i \cdot p_{3i} = \frac{8}{3}, & j = 3 \end{cases} \tag{40}$$

故$E(X_2|X_1 = 1) = 1$，$E(X_2|X_1 = 2) = \frac{7}{3}$，$E(X_2|X_1 = 3) = \frac{8}{3}$。

$$E(X_3|X_2 = j) = \sum_{i=1}^{3} i \cdot P(X_3 = i|X_2 = j) = \sum_{i=1}^{3} i \cdot p_{ji} = \begin{cases} 1, & j = 1 \\ \frac{7}{3}, & j = 2 \\ \frac{8}{3}, & j = 3 \end{cases} \tag{41}$$

故$E(X_3|X_2 = 1) = 1$，$E(X_3|X_2 = 2) = \frac{7}{3}$，$E(X_3|X_2 = 3) = \frac{8}{3}$。

(2)

有

$$\mathbf{P}_2 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{42}$$

则

$$P(T = 1|X_0 = 3) = f_{31}^{(1)} = p_{31}^{(1)} = 0 \tag{43}$$

$$\begin{aligned} P(T = 2|X_0 = 3) = f_{31}^{(2)} &= P(X_2 = 1, X_1 \neq 1|X_0 = 3) && (44) \\ &= P(X_2 = 1, X_1 \neq 1, X_0 = 3) && (45) \\ &= p_{32}^{(1)} p_{21}^{(1)} + p_{33}^{(1)} p_{31}^{(1)} && (46) \\ &= \frac{1}{9} && (47) \end{aligned}$$

$$
\begin{aligned}
P(T=3|X_0=3)=&f_{31}^{(3)}=P(X_3=1,X_2\neq 1,X_1\neq 1|X_0=3) &(48)\\
=&P(X_3=1,X_2\neq 1,X_1\neq 1,X_0=3) &(49)\\
=&p_{32}^{(1)}(p_{22}^{(1)}p_{21}^{(1)}+p_{23}^{(1)}p_{31}^{(1)})+p_{33}^{(1)}(p_{33}^{(1)}p_{31}^{(1)}+p_{32}^{(1)}p_{21}^{(1)}) &(50)\\
=&\frac{2}{27} &(51)
\end{aligned}
$$

则$P(T\geq 4|X_0=3)=1-\sum_{i=1}^{3}P(T=i|X_0=3)=\frac{22}{27}$。

从而：

$$
\begin{aligned}
E(T\wedge 4|X_0=3)&=\sum_{i=1}^{3}i\cdot P(T=i|X_0=3)+4P(T\geq 4|X_0=3) &(52)\\
&=\frac{2}{9}+\frac{2}{9}+\frac{88}{27} &(53)\\
&=\frac{100}{27} &(54)
\end{aligned}
$$

(3)

令$v=\frac{2}{3}=p_{23}^{(1)}=p_{32}^{(1)}$有

$$
\begin{gathered}
P(T_{11}=1)=p_{11}^{(1)}=0\\
P(T_{11}=2)=p_{12}p_{21}+p_{13}p_{31}=\frac{1}{3}\\
P(T_{11}=3)=p_{12}^{(1)}p_{23}^{(1)}p_{31}^{(1)}+p_{13}^{(1)}p_{32}^{(1)}p_{21}^{(1)}=\frac{2}{9}\\
p(T_{11}=4)=p_{12}^{(1)}(p_{23}^{(1)}p_{32}^{(1)})p_{21}^{(2)}+p_{13}^{(1)}(p_{32}^{(1)}p_{23}^{(1)})p_{31}^{(1)}=p_{12}^{(1)}v^2p_{21}^{(2)}+p_{13}^{(1)}v^2p_{31}^{(1)}\\
P(T_{11}=5)=p_{12}^{(1)}(p_{23}^{(1)}p_{32}^{(1)}p_{23}^{(1)})p_{31}^{(1)}+p_{13}^{(1)}(p_{32}^{(1)}p_{23}^{(1)}p_{32}^{(1)})p_{21}^{(1)}=p_{12}^{(1)}v^3p_{21}^{(2)}+p_{13}^{(1)}v^3p_{31}^{(1)}
\end{gathered}
\tag{55}
$$

实际上，考虑$P(T_{11}=k)=P(X_k=1|X_0=1,X_l\neq 1,1\leq l<k)$，而$|S|=3$，因此实际上只有两条路径，也即$1\to(2\to 3)^t\to 1$以及$1\to(3\to 2)^t\to 1$。从而，$\forall k>1$，有

$$
P(T_{11}=k)=p_{12}^{(1)}v^{k-2}p_{21}^{(1)}+p_{13}^{(1)}v^{k-2}p_{31}^{(1)}=\frac{1}{3}(\frac{2}{3})^{k-2}\text{，当}k\text{为偶数} \tag{56}
$$

$$
P(T_{11}=k)=p_{12}^{(1)}v^{k-2}p_{31}^{(1)}+p_{13}^{(1)}v^{k-2}p_{21}^{(1)}=\frac{1}{3}(\frac{2}{3})^{k-2}\text{，当}k\text{为奇数} \tag{57}
$$

因此，当$k>1$时，均有$P(T_{11}=k)=\frac{1}{3}(\frac{2}{3})^{k-2}$。所以：

$$
\begin{aligned}
ET_{11}=&\sum_{k=1}^{\infty}kP(T_{11}=k) &(58)\\
=&\sum_{k=2}^{\infty}kP(T_{11}=k) &(59)\\
=&\frac{1}{3}\sum_{k=2}^{\infty}k\cdot(\frac{2}{3})^{k-2} &(60)
\end{aligned}
$$

令$S=\sum_{k=2}^{\infty}k\cdot(\frac{2}{3})^{k-2}$，则

$$
\frac{2}{3}S=\sum_{k=2}^{\infty}k\cdot(\frac{2}{3})^{k-1}=\sum_{k=3}^{\infty}(k-1)(\frac{2}{3})^{k-2} \tag{61}
$$

所以

$$
S-\frac{2}{3}S=2+\sum_{k=3}^{\infty}(\frac{2}{3})^{k-2}=4 \tag{62}
$$

从而$S=12$。因此

$$
ET_{11}=4 \tag{63}
$$

Q3.2

有

$$
\mathbf{P}=\begin{pmatrix}1-a & a\\ b & 1-b\end{pmatrix},\quad 0<a,b<1 \tag{64}
$$

下使用归纳法证明命题：$\forall n \geq 1$，有下等式成立：

$$\mathbf{P}^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix} \tag{65}$$

对$n=1$，

$$\begin{aligned}
\frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{1-a-b}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix} &= \frac{1}{a+b}\begin{pmatrix} b+(a-a^2-ab) & a+(-a+a^2+ab) \\ b+(-b+ab+b^2) & a+(b-ab-b^2) \end{pmatrix} && (66)\\
&= \frac{1}{a+b}\begin{pmatrix} (a+b)(1-a) & a(a+b) \\ b(a+b) & (a+b)(1-b) \end{pmatrix} && (67)\\
&= \begin{pmatrix} 1-a & a \\ b & a-b \end{pmatrix} = \mathbf{P} && (68)
\end{aligned}$$

成立。

假设命题对$n=k$，$k \geq 1$成立，下考虑$n=k+1$的情形。

有

$$\begin{aligned}
\mathbf{P}^{k+1} &= \mathbf{P}^k\mathbf{P} && (69)\\
&= \left[\frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^k}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}\right]\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} && (70)\\
&= \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^k}{a+b}\begin{pmatrix} a(1-a-b) & -a(1-a-b) \\ -b(1-a-b) & b(1-a-b) \end{pmatrix} && (71)\\
&= \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^{k+1}}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix} && (72)
\end{aligned}$$

即命题对$n=k+1$也成立。

故由归纳法，知原命题成立。

Q3.3

有$\phi_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n$。

且

$$P^n = (p_{ij}^{(n)})_{i,j\in S}, \quad n \geq 1 \tag{73}$$

当$n=0$时，

$$(p_{ij}^0)_{i,j\in S} = I \tag{74}$$

记$P^0 = I$故

$$\begin{aligned}
\Phi(z) = (\phi_{ij}(z)) = (\sum_{n=0}^{n} p_{ij}^{(n)} z^n) &= \sum_{n=0}^{n} P^n z^n && (75)\\
&= P^0 Z^0 + zP + z^2P^2 + \cdots + z^nP^n + \cdots && (76)\\
&= I + zP + z^2P^2 + \cdots + z^nP^n + \cdots = (I - zP)^{-1} && (77)
\end{aligned}$$

得证。

Q3.8

有转移矩阵：

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \tag{78}$$

状态3为吸收态。

(1)

有$P(T_{13}=1) = p_{13}^{(1)} = \frac{1}{4}$。当$k = T_{13} \geq 2$时：

考虑到$p_{21}^{(1)} = p_{31}^{(1)} = p_{32}^{(1)} = 0$，有

$$P(T_{13}=k)=(\frac{1}{2})^{k-1}\frac{1}{4}+\sum_{t=0}^{k-2}((\frac{1}{2})^t\cdot\frac{1}{4}\cdot(\frac{3}{4})^{k-2-t}\cdot\frac{1}{4}) \tag{79}$$
$$=\frac{1}{4}(\frac{1}{2})^{k-1}+\frac{1}{16}\sum_{t=0}^{k-2}(\frac{1}{2})^t(\frac{3}{4})^{k-2-t} \tag{80}$$
$$=\frac{1}{4}(\frac{1}{2})^{k-1}+\frac{1}{16}\frac{(\frac{3}{4})^{k-2}(1-(\frac{2}{3})^{k-1})}{1-\frac{2}{3}} \tag{81}$$
$$=\frac{3}{16}(\frac{3}{4})^{k-2} \tag{82}$$

当$T_{13}=1$时也成立。

从而

$$ET_{13}=\sum_{k=0}^{\infty}k\cdot\frac{3}{16}(\frac{3}{4})^{k-2}=\frac{1}{3}\sum_{k=0}^{\infty}k(\frac{3}{4})^k \tag{83}$$

令$S=\sum_{k=0}^{\infty}k(\frac{3}{4})^k$，则

$$\frac{3}{4}S=\sum_{k=0}^{\infty}k(\frac{3}{4})^{k+1}=\sum_{k=1}^{\infty}(k-1)(\frac{3}{4})^k \tag{84}$$

所以

$$\frac{1}{4}S=S-\frac{3}{4}S=\sum_{k=1}^{\infty}(\frac{3}{4})^k=3 \tag{85}$$

所以$S=12$，从而$ET_{13}=\frac{1}{3}S=4$。

(2)

对$i=1$，有

$$\begin{gathered}f_{11}^{(1)}=\frac{1}{2}\\ f_{11}^{(k)}=0,\quad k\geq 2\end{gathered} \tag{86}$$

则$f_{11}=\sum_{j=1}^{\infty}f_{11}^{(j)}=\frac{1}{2}$。

对$i=2$，有

$$\begin{gathered}f_{22}^{(1)}=\frac{3}{4}\\ f_{22}^{(k)}=0,\quad k\geq 2\end{gathered} \tag{87}$$

则$f_{22}=\sum_{j=1}^{\infty}f_{22}^{(j)}=\frac{3}{4}$。

对$i=3$，有$f_{33}^{(1)}=1$。所以$f_{33}=1$。

综上，$f_{11}=\frac{1}{2}$，$f_{22}=\frac{3}{4}$，$f_{33}=1$。

(3)

当$n\to\infty$，有

$$\begin{gathered}p_{11}^{(n)}=(\frac{1}{2})^n\to 0\\ p_{12}^{(n)}=\sum_{k=0}^{n-1}(\frac{1}{2})^k\cdot\frac{1}{4}\cdot(\frac{3}{4})^{n-1-k}=\frac{1}{4}\sum_{k=0}^{n-1}(\frac{3}{4})^{n-1}(\frac{2}{3})^k=\frac{1}{4}(\frac{3}{4})^{n-1}\sum_{k=0}^{n-1}(\frac{2}{3})^k\to 0\end{gathered} \tag{88}$$

从而$p_{13}^{(n)}=1-p_{12}^{(n)}-p_{11}^{(n)}\to 1$。

另一方面，

$$\begin{gathered}p_{22}^{(n)}=(\frac{3}{4})^n\to 0\\ p_{23}^{(n)}=1-p_{22}^{(n)}\to 1\\ p_{33}^{n}=1^n\to 1\end{gathered} \tag{89}$$

从而$n\to\infty$时，有

$$\mathbf{P}^n=\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix} \tag{90}$$

Q3.13

证明：

首先，记$\omega_{ij}=\begin{cases}\{i\}, & i>j\\ \{0,1,2,\cdots,i\}, & i=j\end{cases}$，则$P(R_n=j|R_{n-1}=i)=P(X_n\in\omega_{ij})=\sum_{k\in\omega_{ij}}a_k$，$\forall i,j,0\le j\le i$。

$$\begin{aligned}&P(R_{n+1}=j|R_n=i,R_{n-1}=i_{n-1},\cdots,R_1=i_1) &(91)\\ &=P(X_{n+1}\in\omega_{ij}|X_n\in\omega_{i_{n-1}i},,X_{n-1}\in\omega_{i_{n-2}i_{n-1}},\cdots,X_1=i_1) &(92)\\ &=P(X_{n+1}\in\omega_{ij}) &(93)\\ &=P(R_{n+1}=j|R_n=i) &(94)\end{aligned}$$

从而$\{R_i:i\ge1\}$是一个马尔可夫链。其转移概率

$$p_{ij}=P(R_n=j|R_{n-1}=i)=P(X_n\in\omega_{ij})=\sum_{k\in\omega_{ij}}a_k=\begin{cases}a_i, & i>j\\ \sum_{k=0}^i a_k, & i=j\end{cases} \tag{95}$$

其中$0\le j\le i$。

Q3.17

证明：有$\{X_n:n\ge0\}$是马尔可夫链。

我们首先使用归纳法证明

$$P(X_{n+1}=j|X_k\in B_k,0\le k\le h,X_n=i,X_{n-1}=i_{n-1},\cdots,X_{h+1}=i_{h+1})=P(X_{n+1}=j|X_n=i) \tag{96}$$

其中$0\le h\le n-1$。我们对h进行归纳证明。

首先，$h=0$时，命题化为

$$P(X_{n+1}=j|X_0\in B_0,X_n=i,X_{n-1}=i_{n-1},\cdots,X_1=i_1)=P(X_{n+1}=j|X_n=i) \tag{97}$$

而

$$\begin{aligned}&P(X_{n+1}=j|X_0\in B_0,X_n=i,X_{n-1}=i_{n-1},\cdots,X_1=i_1) &(98)\\ &=\sum_{t\in B_0}P(X_{n+1}=j|X_0=t,X_n=i,X_{n-1}=i_{n-1},\cdots,X_1=i_1)P(X_0=t|X_0\in B_0) &(99)\\ &=\sum_{t\in B_9}P(X_{n+1}=j|X_n=i)P(X_0=t|X_0\in B_0) &(100)\\ &=P(X_{n+1}=j|X_n=i)\sum_{t\in B_0}P(X_0=t|X_0\in B_0)=P(X_{n+1}=j|X_n=i) &(101)\end{aligned}$$

成立。

下设命题对$h=s$，$s\ge0$成立。考虑$h=s+1$的情形。有：

$$\begin{aligned}&P(X_{n+1}=j|X_n=i,X_{n-1}=i_{n-1},\cdots,X_{s+2}=i_{s+2},X_k\in B_k,0\le k\le s+1) &(102)\\ &=\sum_{t\in B_{s+1}}P(X_{n+1}=j|X_n=i,X_{n-1}=i_{n-1},\cdots,X_{s+2}=i_{s+2},X_{s+1}=t,X_k\in B_k,0\le k\le s)P(X_{s+1}=t|X_{s+1}\in B_{s+1}) &(103)\\ &=\sum_{t\in B_{s+1}}P(X_{n+1}=j|X_n=i)P(X_{s+1}=t|X_{s+1}\in B_{s+1}) &(104)\\ &=P(X_{n+1}=j|X_n=i)\sum_{t\in B_{s+1}}P(X_{s+1}=t|X_{s+1}\in B_{s+1})=P(X_{n+1}=j|X_n=i) &(105)\end{aligned}$$

也成立。故由归纳法，该命题成立。

特别地，在该命题中令$h=n-1$即知原命题得证。