upf. Universitat Pompeu Fabra Barcelona

Seminar 5 · Networks, Crowds and Markets

Random graphs and communities

Configuration model and preferential attachment

In the first part we will experiment a bit with the two models.

See the corresponding colab.

Sampling the configuration model

We have N nodes and degree sequence $(k_1, \ldots, k_N)$. Let each node i own k_i *stubs* (half-edges).

Sampling algorithm (uniform over all pairings):

1. Make a list of all $2L = \sum_i k_i$ stubs. Label each stub by its node owner.
2. While unpaired stubs remain:
   2.1 Select a stub uniformly at random.
   2.2 Choose its partner uniformly at random among the remaining stubs.
   2.3 Connect their owners with an edge (i, j).
   2.4 Remove both stubs.
   2.5 Repeat.
3. The resulting multigraph is one sample from the configuration model.

Note: loops (i, i) and parallel edges may appear. They are rare for large sparse networks.

Exercise: Correctness of the algorithm

This is a reformulated version of this result.

Show that the algorithm described above generates a multigraph with the right degree sequence. Show that the probability of each pairing is uniform.

Solution: Why the pairing is uniform

Pairing of a set of $2L$ elements: a collection of 2-element subsets that are disjoint and whose union gives the whole set.

Let $2L = \sum_i k_i$ be the number of stubs. There are $(2L-1)!! = (2L-1)(2L-3)\cdots 1$ pairings between these stubs.

Proof of uniformity.
Record the sampled stubs as an ordered sequence $(s_1, s_2;\ s_3, s_4;\ \ldots;\ s_{2L-1}, s_{2L})$. Any fixed ordered sequence has probability

$$\frac{1}{2L} \cdot \frac{1}{2L-1} \cdot \frac{1}{2L-2} \cdots \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{(2L)!}.$$

A single unordered pairing can be realized by $L!$ orders of the L pairs and 2 orders inside each pair: $2^L L!$ sequences total. Hence

$$\Pr(\text{a given pairing}) = \frac{2^L L!}{(2L)!} = \frac{1}{(2L-1)!!},$$

the same for every pairing.

Pairings do not define the multigraph uniquely

As pointed out by Calixta in the first seminar group: pairings do not uniquely define multigraphs. For example, if we have two degree two vertices A, B with stubs A_1, A_2, B_1, B_2 then pairings $\{A_1, B_1\}, \{A_2, B_2\}$ and $\{A_1, B_2\}, \{A_2, B_1\}$ both encode the double edge between A and B.

The distribution over multigraphs is in general not uniform.

Depending on a multigraph, there may be a different number of pairings leading to it. On the next slide we discuss an example.

Counterexample

Take four nodes A, B, C, D with degrees $(1, 2, 2, 1)$. The stubs are $A_1, B_1, B_2, C_1, C_2, D_1$ with 15 pairings.

The simple graph $A - B - C - D$ corresponds to four pairings of the form $\{A_1, B_i\}, \{B_{3-i}, C_j\}, \{C_{3-j}, D_1\}$ for $i, j = 1, 2$.

The simple graph $A - C - B - D$ also corresponds to four pairings.

Double edge between B, C and an edge between A, D is defined by two pairings $\{B_i, C_j\}, \{B_{3-i}, C_{3-j}\}, \{A_1, D_1\}$ with $i = j$ or $i \neq j$.

The graph with B, C having loops and an edge between A, D is given by a single pairing.

The graphs with B having a loop and $A - C - D$ is given by two pairings.

The graphs with C having a loop and $A - B - D$ is given by two pairings.

$$\boxed{\text{The probability of each multigraph} = \frac{\#\text{pairings}}{15}.}$$

Exercise: Degree distribution in preferential attachment

Consider the **preferential attachment model** with $m = 1$.

Let d_t denote the degree of the initial vertex at time t.

(a) What is the distribution of d_5?

(b) What are $\Pr(d_t = 1)$ and $\Pr(d_t = t - 1)$?

(c) Find the exact expression for

$$\mathbb{E}[\, d_{t+1} - d_t \mid d_t = k \,].$$

(d) Show that

$$\mathbb{E}[\, d_t \,] = \frac{(2t-3)!!}{2^{t-2}(t-2)!} = \frac{(2t-3)!!}{(2t-4)!!}, \qquad \text{where } n!! = n(n-2)(n-4)\cdots .$$

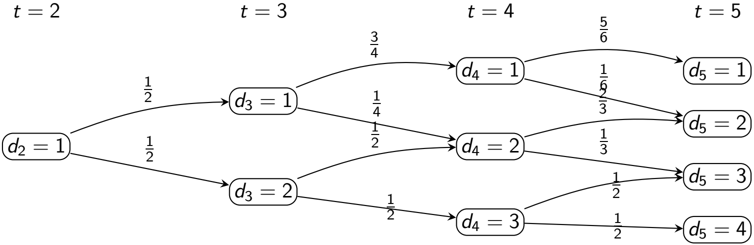
Hint: express the degree growth as a random multiplicative process and use the recurrence relation for $\mathbb{E}[d_t]$.

Solution sketch (a) – corrected transitions

Setup. At time t, total degree is $2(t-1)$. If $d_t = k$, then

$$\Pr(d_{t+1} = k+1 \mid d_t = k) = \frac{k}{2(t-1)}, \qquad \Pr(d_{t+1} = k \mid d_t = k) = 1 - \frac{k}{2(t-1)}.$$

Transition diagram (from $t = 2$ to $t = 5$).



Distribution at $t = 5$:

$\Pr(d_5 = 1) = \frac{5}{16}$, $\Pr(d_5 = 2) = \frac{5}{16}$, $\Pr(d_5 = 3) = \frac{1}{4}$, $\Pr(d_5 = 4) = \frac{1}{8}$.

Solution (b)–(d): Endpoints and mean

Endpoints (all $t \geq 2$):

$$\Pr(d_t = 1) = \prod_{s=2}^{t-1}\Big(1 - \frac{1}{2(s-1)}\Big) = \frac{(2t-4)!!}{(2t-3)!!},$$

$$\Pr(d_t = t-1) = \frac{(t-2)!}{(2t-4)!!} = \frac{1}{2^{t-2}}.$$

Drift and mean.

$$\mathbb{E}[d_{t+1} - d_t \mid d_t = k] = \frac{k}{2(t-1)} \quad \Rightarrow \quad \mathbb{E}[d_{t+1}] = \Big(1 + \frac{1}{2(t-1)}\Big)\mathbb{E}[d_t],$$

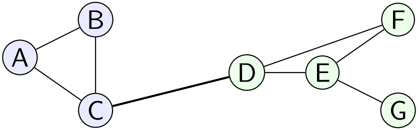
with $d_2 = 1$. Hence

$$\mathbb{E}[d_t] = \prod_{s=2}^{t-1}\Big(1 + \frac{1}{2(s-1)}\Big) = \frac{(2t-3)!!}{2^{t-2}(t-2)!} = \frac{(2t-3)!!}{(2t-4)!!}.$$

Check at $t = 5$: $\mathbb{E}[d_5] = \frac{7!!}{6!!} = \frac{105}{48} = \frac{35}{16} \approx 2.1875$, which matches the diagram.

Exercise: Detecting communities

Graph: two dense parts joined by a single bridge.



(a) Check if $\{A, B, C\}$ and $\{D, E, F, G\}$ form strong or weak communities.

(b) Compute the topological overlap similarity and perform one step of average linkage: which clusters merge first?

(c) Compare the modularity for two partitions:

$$\mathcal{P}_1 = \{\{A, B, C\}, \{D, E, F, G\}\}, \quad \mathcal{P}_2 = \{\{A, B, C\}, \{D, E, F\}, \{G\}\}.$$

For simplicity, on the next slide we give the similarity matrix.

Similarity matrix for the toy network (corrected)

$B_A = \{A, B, C\}$, $B_B = \{A, B, C\}$, $B_C = \{A, B, C, D\}$,
$B_D = \{D, E, F, C\}$, $B_E = \{E, D, F, G\}$, $B_F = \{F, D, E\}$, $B_G = \{G, E\}$.

	A	B	C	D	E	F	G
A	1.00	1.00	1.00	0.33	0	0	0
B	1.00	1.00	1.00	0.33	0	0	0
C	1.00	1.00	1.00	0.50	0.25	0.33	0
D	0.33	0.33	0.50	1.00	0.75	1.00	0.50
E	0	0	0.25	0.75	1.00	1.00	1.00
F	0	0	0.33	1.00	1.00	1.00	0.50
G	0	0	0	0.50	1.00	0.50	1.00

Note: bridge C-D yields $s_{CD} = 0.5$; triangle D-E-F gives $s_{DF} = s_{EF} = 1$; leaf G on E gives $s_{EG} = 1$ and $s_{DG} = s_{FG} = 0.5$.

Solution: Detecting communities

(a) Strong vs weak communities. Degrees:

$$\deg(A) = 2,\ \deg(B) = 2,\ \deg(C) = 3,\ \deg(D) = 3,\ \deg(E) = 3,\ \deg(F) = 2,\ \deg(G) = 1.$$

Group $S_1 = \{A, B, C\}$:

- A: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 0,$
- B: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 0,$
- C: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 1$ (edge to D).

So $\deg_{\text{in}}(v) > \deg_{\text{out}}(v)$ for all $v \in S_1$; S_1 is a strong community.
Group $S_2 = \{D, E, F, G\}$:

- D: $\deg_{\text{in}} = 2$ (edges to E, F), $\deg_{\text{out}} = 1$ (edge to C),
- E: $\deg_{\text{in}} = 3$, $\deg_{\text{out}} = 0$,
- F: $\deg_{\text{in}} = 2$, $\deg_{\text{out}} = 0$,
- G: $\deg_{\text{in}} = 1$, $\deg_{\text{out}} = 0$.

Again $\deg_{\text{in}}(v) > \deg_{\text{out}}(v)$ for all $v \in S_2$, so S_2 is also a strong community.

(b) One step of average linkage.
The largest similarities off the diagonal are 1.00, attained for the pairs (A, B), (A, C), (B, C) and (D, F), (E, F), (E, G). In a standard hierarchical clustering step, we first merge some pair with similarity 1 (for example A and B, or D and F); then the other pairs with similarity 1 quickly join the same two blocks $\{A, B, C\}$ and $\{D, E, F, G\}$.

(c) Modularity for $\mathcal{P}_1$ and $\mathcal{P}_2$.
Total number of edges is $m = 8$. A direct computation of the modularity

$$Q(\mathcal{P}) = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \mathbf{1}\{c_i = c_j\}$$

gives approximately

$$Q(\mathcal{P}_1) \approx 0.37, \qquad Q(\mathcal{P}_2) \approx 0.30.$$

So $\mathcal{P}_1$ has higher modularity; keeping G in the same community as D, E, F is preferred.