Universitat Pompeu Fabra Barcelona

Lecture 2 · Graph Theory I

Networks, Crowds and Markets

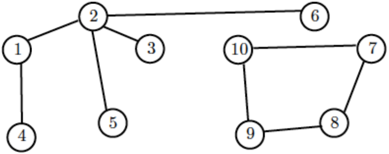
Today's Lecture

- Special graphs: complete, null, path, cycle.
- Degree. Handshaking Lemma. Degree sequence. Average degree. Degree distribution.
- Isomorphic graphs
- Subgraphs
- Adjacency matrix. Powers of adjacency matrix.
- Variations: Multigraphs, directed graphs, weighted graphs

Graph

Definition

A **graph** is a pair $G = (V, E)$ where $V = [N]$ is the set of N nodes (or vertices) and $E \subseteq \binom{V}{2}$ is a set of unordered pairs of distinct vertices, called edges (or links).



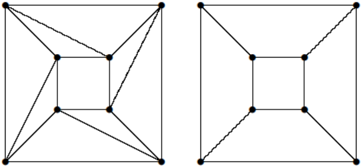
$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$E = \{\overline{12}, \overline{14}, \overline{23}, \overline{25}, \overline{26}, \overline{78}, \overline{89}, \overline{910}\}$$

We often write $\{i, j\}$ or simply ij for an edge rather than $\overline{ij}$.

Subgraphs

Definition

$H = (V', E')$ is a **subgraph** of $G = (V, E)$ if all the nodes and edges of H belong to G; $V' \subseteq V$, $E' \subseteq E$. We write $H \subseteq G$.



G $\quad$ H

Here G and H have the same vertex sets but this is not the case generally for a subgraph.

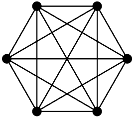
Special graphs

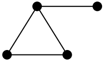
Special Graphs: The Complete Graph K_N

Definition

Graph is complete if every pair of vertices is joined by an edge.

A complete graph has $\binom{N}{2} = \frac{N(N-1)}{2}$ edges.





If $H \subseteq G$ and H is complete then H is called a clique of G.

In `NetworkX`:

`import networkx as nx` – load NetworkX
`import matplotlib.pyplot as plt`

`G = nx.complete_graph(N)` – the complete graph with N nodes.

`nx.draw(G, with_labels=True, node_color="lightblue", node_size=600)`
`plt.show()`.

Special Graphs: The Null Graph (aka the empty graph)

Definition

Graph is empty if it does not contain any edge, therefore $E = \emptyset$.

The complement $\overline{G}$ of a graph $G = (V, E)$ has the same set of vertices and an edge is in $\overline{G}$ if an only if it is not present in G.

The empty graph is the complement of the complete graph.

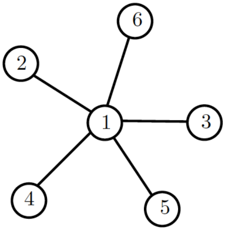
In `NetworkX`:

`G = nx.empty_graph(N)` – the empty graph with N nodes

Special Graphs: The Star Graph S_N

Definition

A **star graph** S_N has one central vertex connected to all other $N-1$ vertices, and no other edges. The central node has degree $N-1$, while all other nodes have degree 1.



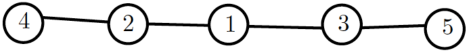
In `NetworkX`:

`G = nx.star_graph(N)` the star graph with $N+1$ nodes

The Path Graph P_N

Definition

A **Path Graph** is a connected graph $G = (V, E)$ whose nodes can be listed in order $(v_1, \dots, v_N)$ and the edges are $v_i v_{i+1}$. The central nodes have degree 2, the ones in the terminal vertices have degree 1.



Here $v_1 = 4$, $v_2 = 2$, $v_3 = 1$, $v_4 = 3$, $v_5 = 5$.

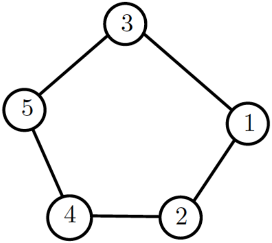
In `NetworkX`:

`G = nx.path_graph(N)` the path graph with N nodes

Special Graphs: The Cycle Graph C_N

Definition

A **cycle graph** C_N is obtained by connecting N vertices in a closed chain: $E = \{v_1 v_2, v_2 v_3, \ldots, v_{N-1} v_N, v_N v_1\}$.



In `NetworkX`:

`G = nx.cycle_graph(N)` the path graph with N nodes

Special Graphs: Bipartite graphs

Definition

A graph $G = (V, E)$ is bipartite if the set of vertices can be partitioned into two parts $V = V_1 \cup V_2$ such that all edges have one end in V_1 and one end in V_2

- The star S_N
- The full bipartite graph $K_{N,N}$
- Cycle C_N for even N.

Theorem

G is bipartite if and only G has no odd cycles as subgraphs.

In `NetworkX`:

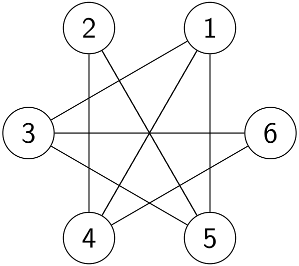
```
from networkx.algorithms import bipartite
nx.is_bipartite(G)
```

Degree and degree distribution

Degree

Definition

The **degree** of a node $v \in V$ in an undirected graph, represented by $\deg(v)$, is the number of edges incident to it.



$\deg(1) = 3$

$\deg(2) = 2$

$\deg(3) = 3$

$\deg(4) = 3$

$\deg(5) = 3$

$\deg(6) = 2$

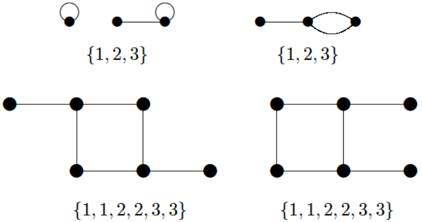
In `NetworkX`:

`print(G.degree(v))` – print the degree of node v

Degree sequence

Definition

The degree sequence of a graph is obtained by ordering, in an increasing way, the degrees of its nodes.



In `NetworkX`:

```
deg_seq = sorted([d for n, d in G.degree()])
print(deg_seq_sorted)
```
– print the degree sequence of G

Degrees

$G = (V, E)$ a graph, $N = |V|$, $L = |E|$

- $\delta(G) = \min_{v \in V} \deg(v)$ minimum degree of G
- $\Delta(G) = \max_{v \in V} \deg(v)$ maximum degree of G
- $\overline{\deg}(G) = \frac{1}{N} \sum_{v \in V} \deg(v)$ average degree of G. ([B] uses $\langle k \rangle$)

Theorem (Handshaking Lemma)

The sum of the degrees of the vertices of a graph equals twice the number of edges:

$$\sum_{v \in V} \deg(v) = 2L.$$

In particular, the number of vertices of odd degree is always even.

(each edge contributes $+1$ to the degree of two nodes)

Exercise

Let $G = (V, E)$ be a graph with $N = 10$ nodes with the following degree sequence $\{0, 0, 1, 1, 1, 2, 2, 2, 3, 6\}$.

a) Find the number of edges without drawing the graph.

b) Find the average degree of the nodes in G.

c) Draw a possible graph that satisfies the conditions.

Degree Distribution

Definition

The degree distribution, $p = (p_k)_{k \in \mathbb{N}}$, is a discrete function that provides the relative ratio of the vertices with a given degree k.

(In particular, $p_k = 0$ for $k \geq n$.)

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Note that:

- $p_k = \dfrac{N_k}{N}$, ($N_k =$ number of vertices of degree k)
- $\displaystyle\sum_{k=0}^{\infty} p_k = 1$ Normalization

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Note that:

- $p_k = \dfrac{N_k}{N}$, ($N_k =$ number of vertices of degree k)
- $\displaystyle\sum_{k=0}^{\infty} p_k = 1$ Normalization
- $\overline{\deg}(G) = \frac{1}{N} \sum_{v \in V} \deg(v) = \frac{1}{N} \sum_{k=0}^{\infty} k \cdot N_k = \sum_{k=0}^{\infty} k \cdot p_k$

Probabilistic viewpoint

This notation and terminology alludes to probability.

Consider a probability distribution $q = (q_k)$ on $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

If $X \sim q$ then $\mathbb{E}X = \sum_{k \geq 0} k \cdot q_k$.

Consider now a random sample $X_1, \ldots, X_N \sim q$.

The statistics N_k counts the number of times we observed $X_i = k$.

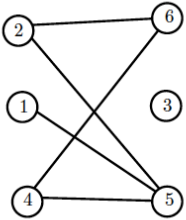
Sample distribution: $p = (p_k)$ with $p_k = \frac{1}{N} N_k$ for $k \geq 0$.

The sample average is

$$\overline{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i = \frac{1}{N} \sum_{k \geq 0} k \cdot N_k = \sum_{k \geq 0} k \cdot p_k.$$

If N is large $p \approx q$.

Example



$$p_k = \begin{cases} \frac{1}{6} & \text{if } k = 0 \\ \frac{1}{6} & \text{if } k = 1 \\ \frac{1}{2} & \text{if } k = 2 \\ \frac{1}{6} & \text{if } k = 3 \end{cases}$$

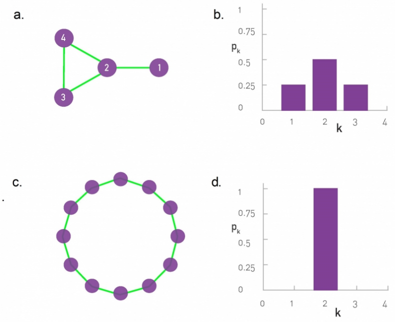
$$\overline{\deg}(G) = \sum_{k=0}^{\infty} k \cdot p_k = \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 1 + \frac{1}{2} \cdot 2 + \frac{1}{6} \cdot 3 = \frac{5}{3}$$

In `NetworkX`:

```
import numpy as np

values, counts = np.unique(deg_seq, return_counts=True)
distribution = counts / counts.sum()
```

Plotting the degree distribution

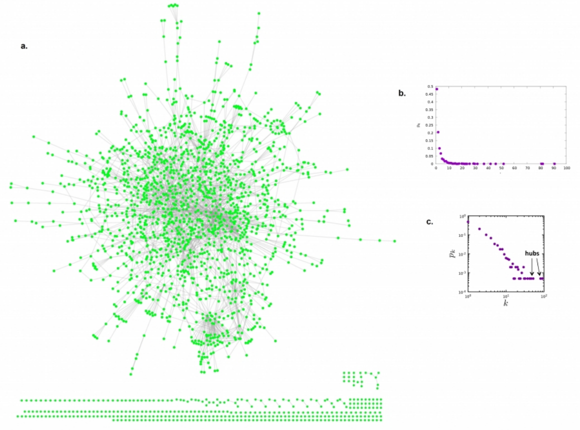


In NetworkX:

```
plt.hist(deg_seq, bins=range(max(deg_seq)+2), align='left', rwidth=0.8)
plt.xlabel("Degree")
plt.show()
```

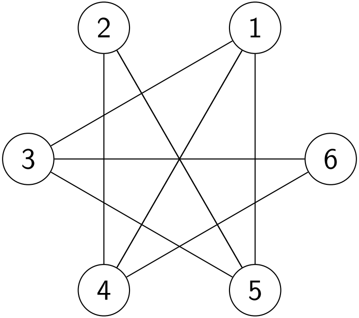
Power law

Many real networks have degree distributions that follow a **power law**, meaning the probability of high-degree nodes decays as a power of their degree, creating rare but influential hubs.



Exercise

Given the following graph:

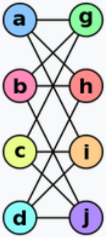
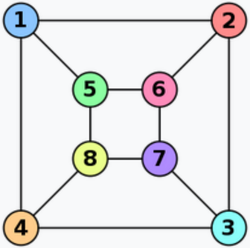


a) Determine the average degree of the graph.

b) Determine the degree distribution.

Isomorphic graphs

Isomorphic Graphs

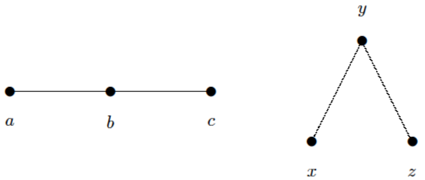
Graph G	Graph H	An isomorphism between G and H
		$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

Isomorphic Graphs

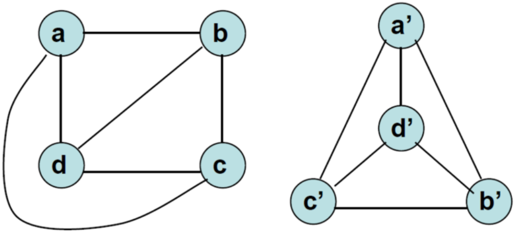
Definition

Two graphs G and H are **isomorphic** if there exists a bijection φ between their set of nodes that preserves the edges. Alternatively:

$$\exists \varphi : V_G \to V_H \ : \ \overline{xy} \in E_G \iff \overline{\varphi(x)\varphi(y)} \in E_H$$



Example: $V_H = \{a, b, c, d\}, \quad V_G = \{a', b', c', d'\}.$



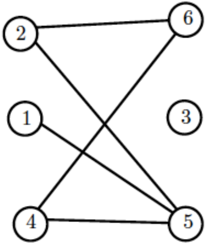
$\varphi(a) = a', \quad \varphi(b) = b', \quad \varphi(c) = c', \quad \varphi(d) = d'.$

Adjacency matrix

Adjacency Matrix

Definition

The **Adjacency Matrix** of an undirected graph is a square $N \times N$ matrix A in which the position (i, j) equals 1 if there is an edge between nodes i and j, otherwise, we type a 0.



$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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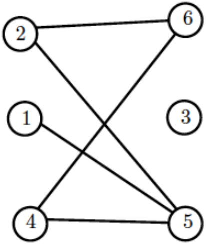


$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In `NetworkX`:

`nx.to_numpy_array(G, nodelist=sorted(G.nodes()))` – G to A

`G = nx.from_numpy_array(A)` – A to G

NumPy provides fast, memory-efficient arrays and mathematical tools for scientific computing.

Adjacency Matrix

Some elementary properties

- For an undirected graph, the adjacency matrix A is symmetric.
- The sum of entries in row i is $\deg(i)$.
- Since there are no self-loops, the diagonal entries are zero.
- The entry $(A^2)_{ij}$ counts the number of walks of length 2 from i to j. In particular, $\mathrm{tr}(A^2) = 2L$.
- More generally, $(A^m)_{ij}$ counts the walks of length m from i to j. In particular, $\mathrm{tr}(A^3) = 6\times$ (number of triangles).

Note

Matrix powers count walks — this will reappear later in centrality measures and diffusion.

More complicated types of graphs

Multigraphs and loops

Note

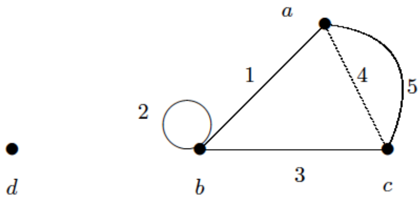
The graphs we have seen so far are called **simple** undirected graphs

- If two or more edges join the same pair of nodes, the graph is called a **multigraph**.
- If one edge joins a node with itself, we call it a **loop**.

In applications more complicated types of graphs may appear.

Undirected multigraphs

Example of a **multigraph with a loop**:



$$V = \{a, b, c, d\} \qquad E = \{1, 2, 3, 4, 5\}$$

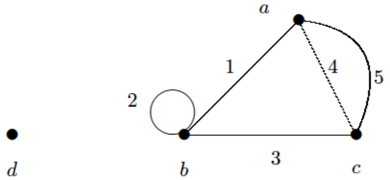
Remark: In this case, the edges have their own labels.

In `NetworkX`:

```
G = nx.MultiGraph()
G.add_nodes_from([1, 2, 3])
G.add_edge(1, 2)
G.add_edge(1, 2)   # parallel edge between 1 and 2
G.add_edge(2, 3)
```

Adjacency matrix of a multigraph

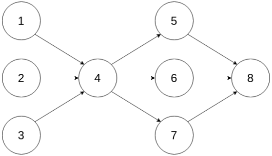
The entry ij is the number of edges joining i and j



$$A = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- **Remark:** Adjacency matrices are symmetric ($a_{ij} = a_{ji}$) as they are undirected graphs.

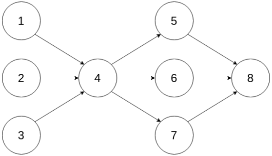
Directed Graph



Definition

A **directed graph** (digraph) is a pair $G = (V, E)$ where $V = [N]$ is the set of vertices and $E \subseteq V \times V$ is a set of ordered pairs of distinct vertices, called *directed edges* (or arcs).

Directed Graph



Definition

A **directed graph** (digraph) is a pair $G = (V, E)$ where $V = [N]$ is the set of vertices and $E \subseteq V \times V$ is a set of ordered pairs of distinct vertices, called *directed edges* (or arcs).

```
G = nx.DiGraph()
G.add_nodes_from(range(1, 9))
edges = [(1,4),(2,4),(3,4),(4,5),(4,6),(4,7),(5,8),(6,8),(2,8)]
G.add_edges_from(edges)
```

Examples of Directed Graphs

- **World Wide Web (WWW)** Vertices are webpages, edges are hyperlinks. Directed edges capture the one-way nature of links.

- **X/Twitter** Vertices are accounts, edges are "follow" relationships. Edges are directed (A follows B does not imply B follows A).

- **Academic Citation Network** Vertices are papers, edges are citations. Naturally directed in time: newer papers cite older ones.

- **SWIFT Network** Vertices are banks or institutions, edges represent international money transfers. Direction indicates sender $\rightarrow$ receiver of funds.

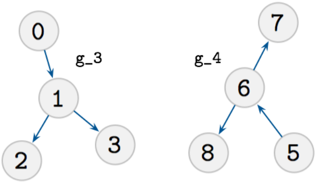
- **Credit Networks** Vertices are individuals or institutions, edges indicate lending/borrowing. Direction shows the obligation flow (debtor $\rightarrow$ creditor).

Isomorphism of directed graphs

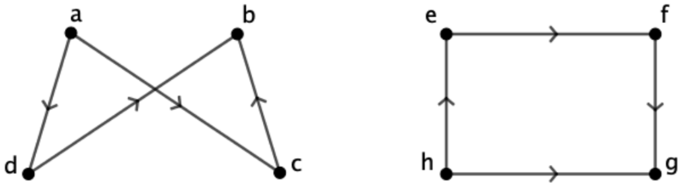
Definition

Two directed graphs G and H are **isomorphic** if there exists a bijection φ between their set of nodes that preserves the edges with its direction.

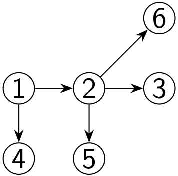
$$\exists \varphi : V_G \to V_H \text{ such that } (x, y) \in E_G \iff (\varphi(x), \varphi(y)) \in E_H$$



Are they isomorphic?



In-degree and out-degree



Definition

For a directed graph $G = (V, E)$ and $v \in V$:

- The **in-degree** $d^-(v)$ is the number of edges arriving at v.
- The **out-degree** $d^+(v)$ is the number of edges leaving v.

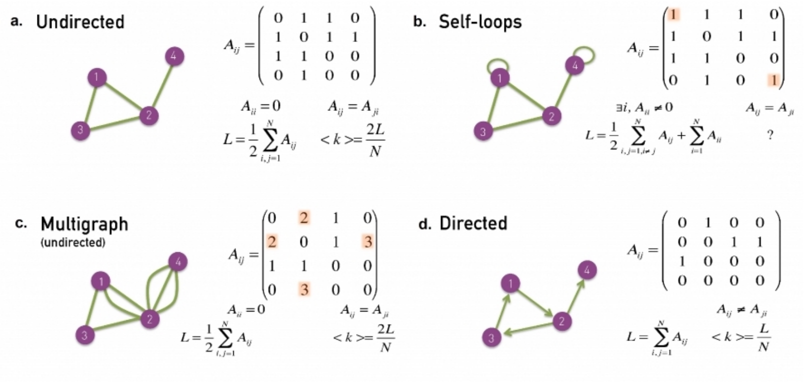
Theorem (Handshaking Lemma for directed graphs)

$$\sum_{v \in V} d^+(v) \;=\; \sum_{v \in V} d^-(v) \;=\; |E|.$$

Average Degree

If A is the adjacency matrix, then the vector $A\mathbf{1}$ gives the degrees.

Thus $\frac{1}{N}\mathbf{1}^\top A\mathbf{1}$ gives the average degree.

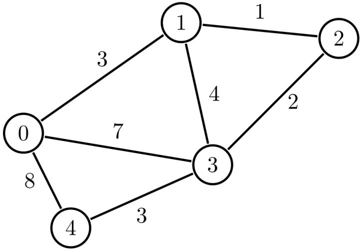


Barabási denotes the average degree by $\langle k \rangle$. We do not follow this convention.

Weighted Graphs

Definition

A **weighted graph** is a pair (G, w) where $G = (V, E)$ is a simple graph and $w : E \to \mathbb{R}$ assigns a weight to each edge. Weights may represent distance, travel time, cost, or interaction strength (often nonnegative).



In `NetworkX` we can create a weighted graph from the (weighted) adjacency matrix – replace each 1 with the corresponding weight.

Examples of Weighted Graphs

Road Map V = intersections or cities, E = roads. *Weights:* distance, travel time, traffic congestion, or cost of tolls. Used in shortest–path algorithms (GPS navigation).

Social Networks V = individuals, E = relationships. *Weights:* frequency of interaction, strength of friendship, number of shared posts/messages. Captures tie strength rather than just "yes/no" connection.

E-mail Network V = people, E = e-mail communication. *Weights:* number of messages exchanged, total size of correspondence, or recency-weighted activity. Useful to detect communities or hubs of communication.

Food Chain Network V = species, E = "who eats whom". *Weights:* proportion of diet, biomass transfer, or energy flow between species. Central in ecology to understand stability of ecosystems.

Exercise

Suppose $G = (V, E)$ s.t. $L = 800$ edges and $N = 1000$ nodes. The average degree is

$$\overline{\deg}(G) \;=\; \frac{2L}{N} \;=\; 1.6.$$

For large graphs it is often reasonable to approximate the degree of a randomly chosen node by a Poisson distribution with mean $\lambda := \overline{\deg}(G)$:

$$\mathbb{P}(\deg(v) = k) \;\approx\; \frac{\lambda^k}{k!}\, e^{-\lambda}, \qquad k = 0, 1, 2, \ldots$$

a) Determine the average degree of the graph.

b) What is the probability that a node has no edges attached?

c) How many nodes of degree 3 do we expect to find?

(later we discuss more in detail when such Poisson approximations make sense)

Exercise

Let B be the adjacency matrix of a directed graph G with no loops and A be the adjacency matrix of the undirected version of G. Which of the following properties are true? Give evidence of your answer.

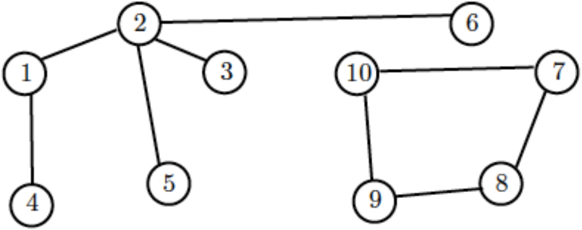
a) $a_{ij} + a_{ji} \geq 1$

b) $b_{ij} \leq a_{ij}$

c) $b_{ij} + b_{ji} \leq 1$

Exercise

Given the following graph:



a) List the set of vertices and edges

b) Find the degree for the 3 first nodes

c) Find the average degree of all the nodes in the graph