Lecture 16 · Social Networks II, Market matching I

Networks, Crowds and Markets

First part of the lecture: Social networks

We started discussing mechanisms that drive social networks.

1. Triadic closure, strong triadic closure, local bridges.
2. Homophily and how it can be tested.
3. Social influence: after connecting nodes become more similar.

Today we continue this discussion in the context of affiliation networks:

4. Focal closure.
5. Membership closure.

Then we discuss social networks with signed links and polarization.

# Second part of the lecture: Matching markets

1. Perfect Matching

2. Valuations and optimal assignments

3. Prices and the Market-Clearing Property

4. Constructing a Set of Market-Clearing Prices

Affiliation networks

An **affiliation network** is a bipartite graph between people and foci/contexts (classes, clubs, workplaces, products).
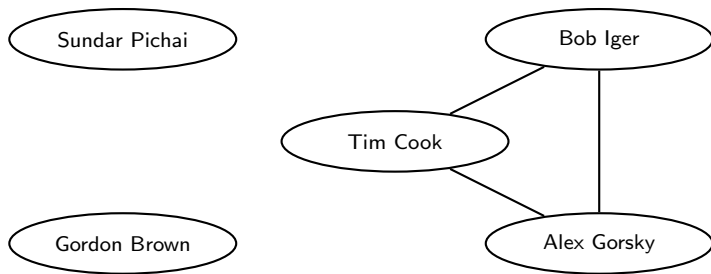
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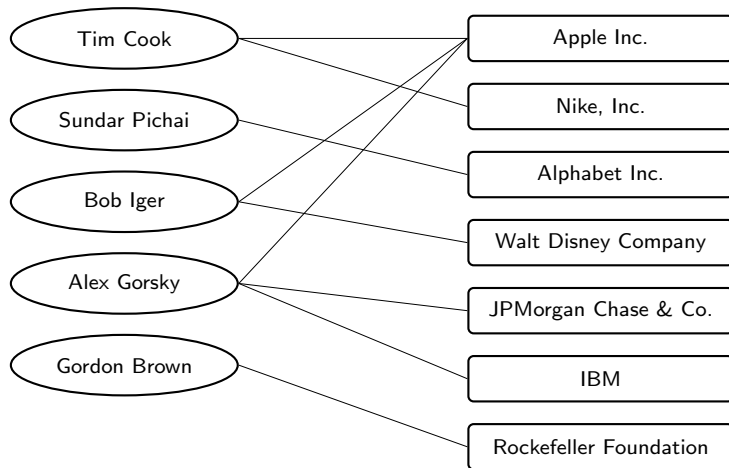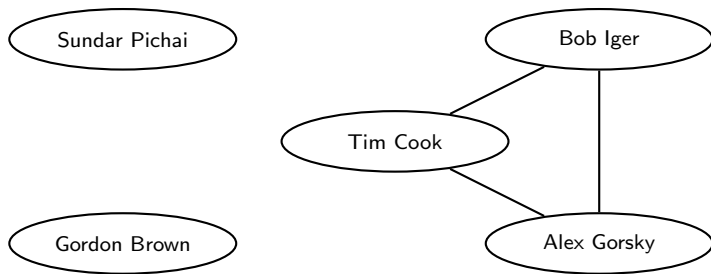
Social-affiliation networks

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A **social-affiliation network** is build on an affiliation network by allowing people to connect directly.

(4-5) Focal and membership closure

Mechanism that drive edge creation in social-affiliation networks.

Triadic closure: friends of a person tend to connect.
Focal closure: two people sharing a focus tend to connect.
Membership closure: a person joins a focus following a friend.
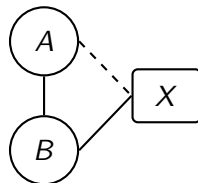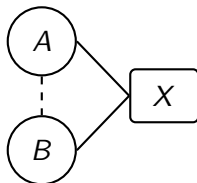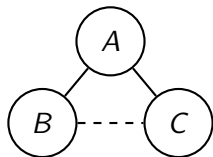
Positive and negative links

Positive and Negative Relationships

In Social Networks, ties may represent:

- Friendship

- Collaboration

- Sharing of information

- ...

Positive and Negative Relationships

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These relationships can either be:
- Positive relationships

- Negative relationships

The relationship in general have signed weights. We simplify here.

Structural balance; signed triangles
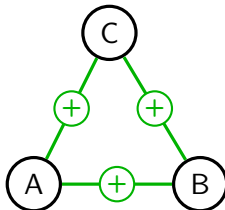
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Balanced

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Unbalanced

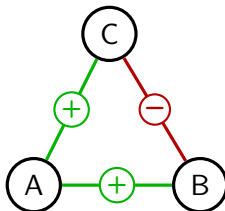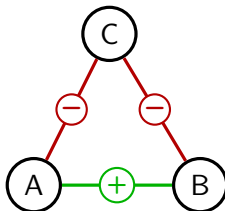We start by analyzing simple triangles with signed links.

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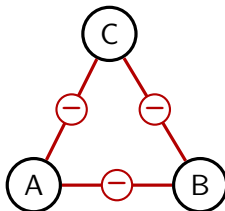
Unbalanced

Structural Balance Property and Polarization

Unbalanced triangles are sources of stress and psychological dissonance. Individuals strive to minimize them in their personal relationships.

A graph has structural balance property, if every triangle is balanced.

Balance Theorem: If G is a **complete** graph with structural balance property then either all edges are positive, or, there exists a split of the nodes into two sets C_1/C_2 such that the edges between C_1 and C_2 are negative and all the other edges are positive.

Proof. If all edges are positive, we are fine. Suppose that there is at least one negative edge. Take a vertex in G with a negative edge, call it a_1. Let $C_2 = \{b_1, \ldots, b_l\}$ be a set of nodes for which $a_1 - b_i$ is negative. If $l \geq 2$ consider the triangle $\{a_1, b_i, b_j\}$ to show that $b_i - b_j$ is positive for all i, j. Let $C_1 = \{a_1, \ldots, a_k\}$. If $k \geq 3$ consider the triangle $\{a_1, a_i, a_j\}$ to show that $a_i - a_j$ is positive for all i, j. Finally, consider the triangle $\{a_1, a_i, b_j\}$ to show that $a_i - b_j$ is negative for all i, j.

Summary

- Social ties arise via **triadic**, **focal**, **membership** closures, and **homophily/influence**.

- **Strong vs. weak ties:** STC implies local bridges tend to be weak; weak ties carry novel information.

- **Affiliation** structure explains high clustering via shared foci.

- These mechanisms motivate **community structure** and power many **link-prediction** heuristics.

The Bipartite Matching Problem

We start with the following example:

- The administrators of a college dormitory are assigning rooms to students for the new academic year.

- Each room is designed for a single student.

- Each student is asked to list several acceptable options for the room they would like to get.

This scenario can be modelled using a **bipartite graph**.
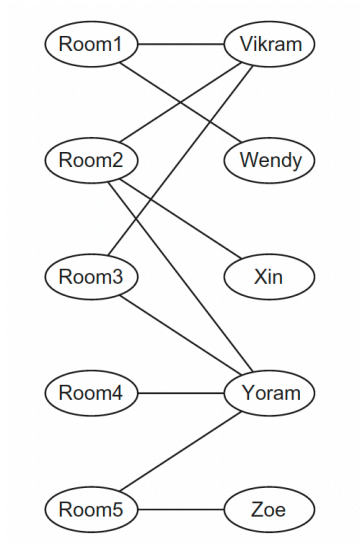
Perfect Matching

The aim of the matching process in our example is assigning each student a room that they would be happy to accept.

Considering we have the same number of nodes on each side, we will consider an assignment of rooms as a **perfect matching** if:

1. Each node is connected by an edge to the node it is assigned to.
2. No two nodes on the left are assigned to the same node on the right.
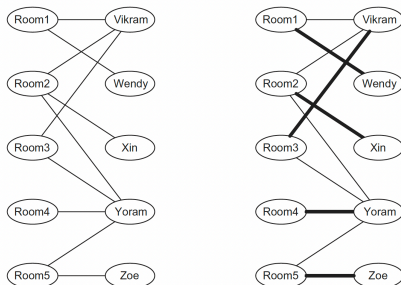
How to show there is no perfect matching?

Constricted sets in a bipartite graph is a subset S of vertices on the right such that their neighbours $N(S)$ satisfy $|S| > |N(S)|$.

Theorem: If a bipartite graph (with equal number of nodes on both sides) has no perfect matching, then it must contain a constricted set.
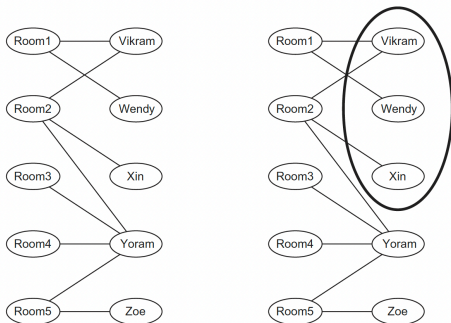
Valuations and Optimal Assignments

Change: Suppose now that students, instead of a binary list of preferences, they send a list of preferences with valuations.

Valuations are numbers that students will assign to each room according to their preferences regarding the quality of the room.
- think: the price they are willing to pay for the room.

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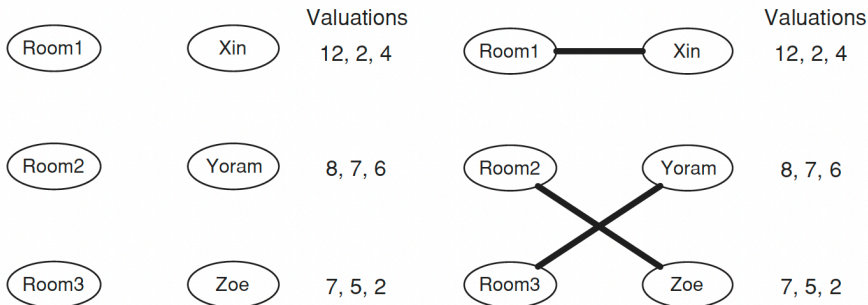
Optimal Assignment

Definition

The **Optimal Assignment** is the distribution of dorms that maximizes total evaluation.

This does not necessarily give everyone their favourite item.
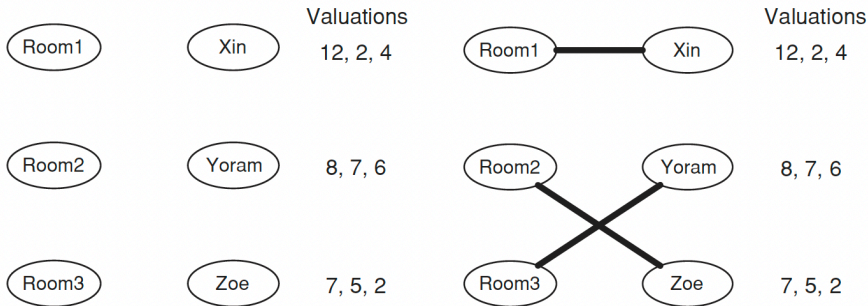
Prices and the Market-Clearing Property

Replace the **central administrator** by a market where individuals make decisions based on prices and their own evaluations.

- Suppose that we have a collection of sellers, each with a house for sale, and an equal-sized collection of buyers, each of whom wants a house.

- Each buyer has a non-negative valuation for each house.

- The valuation that a buyer j has for the house held by the seller i will be denoted v_{ij}.

Sellers	Buyers	Valuations
a	x	12, 4, 2
b	y	8, 7, 6
c	z	7, 5, 2

Prices and Payoffs

- Suppose that each seller i puts their house up for sale, offering to sell it for a price $p_i \geq 0$.

- If a buyer j buys the house from seller i at this price, their **payoff** is their valuation for this house, minus the price: $v_{ij} - p_i$.

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- If buyer j wants to maximize their payoff, they will buy from the seller i for which this quantity $v_{ij} - p_i$ is maximized.

  ▶ if this quantity is maximized in a tie between several sellers, then the buyer can maximize by choosing any of them.

  ▶ if their payoff $v_{ij} - p_i$ is negative for every choice of seller i, then the buyer would prefer not to buy any house, **with payoff zero**.
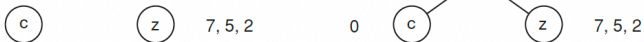
(a)
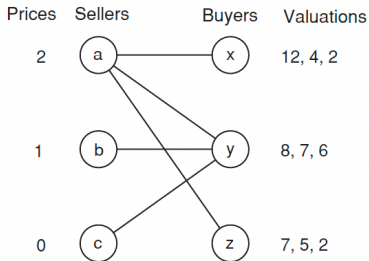
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(b)

Prices	Sellers	Buyers	Valuations
5	a — x		12, 4, 2
2	b	y	8, 7, 6
0	c	z	7, 5, 2

(c)

Prices	Sellers	Buyers	Valuations
2	a	x	12, 4, 2
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(d)

Prices	Sellers	Buyers	Valuations
3	a	x	12, 4, 2
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Market-Clearing Prices

The **preferred-seller graph**: the graph connecting buyers and sellers according to the preferred seller option of the buyers. (see Slide 20)

This graph clearly depends on the prices.

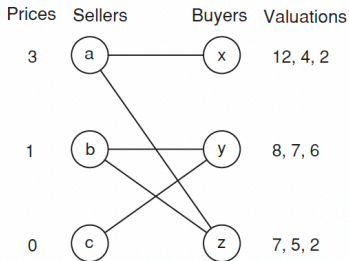The **market clearing prices** scenario is given when the preferred-seller graph has a perfect matching.

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Theorem (Two important properties)

Optimality of Market-Clearing Prices:
For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph gives the maximum total valuation.

Existence of Market-Clearing Prices:
For any set of valuations, there exist market-clearing prices .

Proof of optimality of Market-Clearing Prices

Suppose j-th buyer bought $b(j)$-th house.

The total payoff of buyers can be defined as:

$$M = \sum_{j=1}^{N}(v_{b(j)j} - p_j) = \sum_{j=1}^{N} v_{b(j)j} - \sum_{j=1}^{N} p_j$$

where only the highlighted term depends on the matching b.

Thus, maximizing the total payoff is equivalent to maximizing the total evaluation. Since the perfect matching maximizes the former, it maximizes the latter.

Sketch proof of Existence of Market-Clearing Prices

Why market-clearing prices must always exist?

For the formal proof see Section 10.6 in [EK].

Here we give an algorithmic proof.

Input: an arbitrary set of buyer valuations.

Output: market-clearing prices for these evaluations.

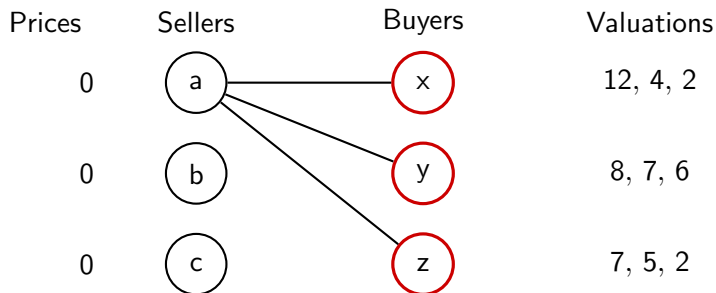The procedure will in fact be a kind of auction.

Part of the proof should be showing that this procedure always terminates but we omit the details.

Constructing a set of Market-Clearing Prices

A general round of the auction goes as follows:

1. At the start of each round, there is a current set of prices, with the smallest one equal to 0.
2. We construct the preferred-seller graph and check whether there is a perfect matching.
3. If there is, we're done: the current prices are market-clearing.
4. If not, we find a constricted set of buyers; $|C| > |N(C)|$.
5. Each seller in $N(C)$ (simultaneously) raises his price by one unit.
6. If necessary, we reduce the prices: The same amount is subtracted from each. price so that the smallest price becomes zero.
7. We now begin the next round of the auction, using these new prices.

Finding market-clearing prices
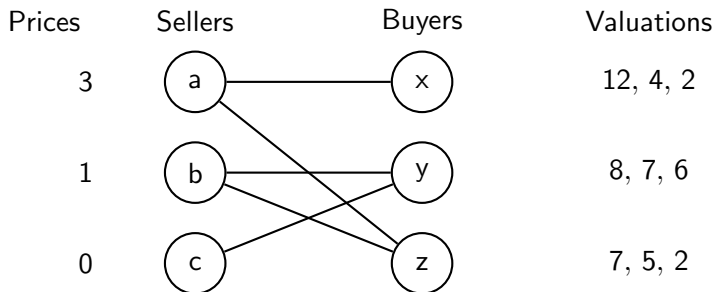
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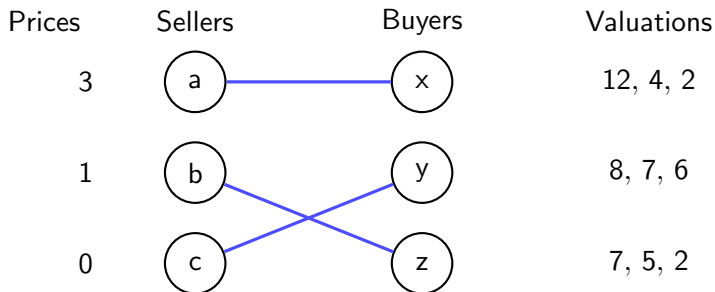
Does this always come to an end?

If we can prove that the auction must come to an end for any set of buyer valuations, then we've shown that market-clearing prices always exist.

Potential Energy of the Auction

Define the potential: $E_p = E_p^s + E_p^b$.

- $E_p^s = \sum_i p_i$ (total seller prices)
- $E_p^b = \sum_j \max_i(v_{ij} - p_i)$ (total best-response buyer utilities)

Basic facts (assuming $v_{ij} \geq 0$ not all zero, and prices $p_i \geq 0$):

- At $p = 0$: $E_0 = \sum_j \max_i v_{ij} > 0$.
- $E_p \geq 0$ always (seller prices ≥ 0, buyer best-response utilities ≥ 0).
- Decreasing all prices by 1 leaves E_p unchanged (seller part $-n$, buyer part $+n$).

One iteration: Let C be a constricted set.

- Increasing prices of $N(C)$ by 1 **increases** E_p^s by $|N(C)|$.
- Buyers in C lose 1 unit of best-response utility, so E_p^b **decreases** by $|C|$.
- Since $|C| > |N(C)|$, overall E_p strictly decreases in each step.

Why the Auction Always Terminates

Key fact: $E_p^b + E_p^s$ decreases by at least 1.

Consequences:

- E_p starts at a finite E_0.
- Each round reduces E_p by ≥ 1.
- E_p can never drop below 0.

Therefore:

> The auction must stop after at most E_0 rounds.