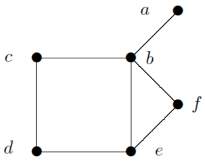
Universitat Pompeu Fabra Barcelona

Lecture 4 · Graph Theory III

Networks, Crowds and Markets

Warm Up

Given the following graph:



Determine:

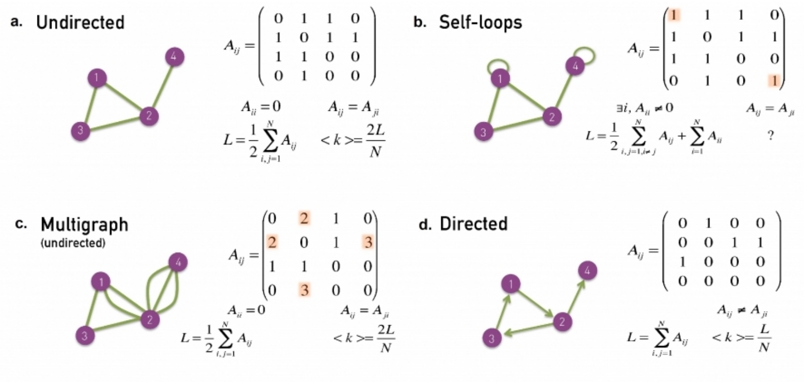
a) the average degree of the graph.

b) the degree distribution.

c) its adjacency matrix.

Average Degree

In a simple graph, the vector $A\mathbf{1}$ gives the degrees. Thus $\frac{1}{N}\mathbf{1}^\top A\mathbf{1} = \frac{1}{N}\sum_{i,j=1}^{N} A_{ij}$ gives the average degree. How about other cases?



Barabási denotes the average degree by $\langle k \rangle$. We do not follow this convention.

Today's Lecture

1. Distances in a graph
    1.1 Path / Shortest path / Distance
    1.2 Breadth First Search
    1.3 Diameter / Local Diameter
    1.4 Eccentricity
2. Connectivity
    2.1 Definition
    2.2 Bridge
3. Trees, Regular Graphs
4. Introduction to centrality measures
    - Degree centrality
    - Closeness centrality

Distances in graphs

Path

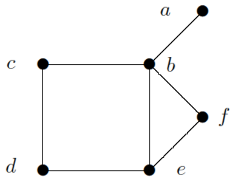
Definition

A **walk** in an undirected graph $G = (V, E)$ is a sequence of vertices $(v_0, \ldots, v_\ell)$ s.t. each consecutive pair $v_{i-1}v_i$ is an edge in E.

A **path** in an undirected graph $G = (V, E)$ is a sequence of distinct vertices $(v_0, \ldots, v_\ell)$ s.t. each consecutive pair $v_{i-1}v_i$ is an edge in E.

Length: the number of edges in the sequence, here ℓ.

In a weighted graph we could take the sum of the weights of the edges on the path; path weight.

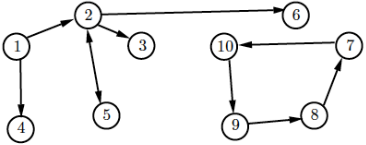


Examples: $p_1 = abfe$, $p_2 = cdef$, $p_3 = defba$

Directed Path

Definition

A **directed path** in a directed graph G is a finite sequence of edges $e_i = (x_i, y_i)$ such that for each $i \geq 1$, $x_{i+1} = y_i$.

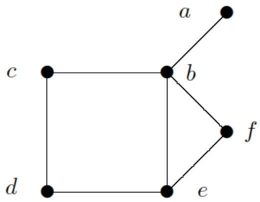


Examples:
$p_1 = \overline{126}$, $p_2 = \overline{523}$

Shortest Path

Definition

A **shortest path** in a graph G between two nodes u, v is any path that connects u with v having minimum length.



abe is the shortest path between a and e

abcd and *abed* are both shortest path between a and d

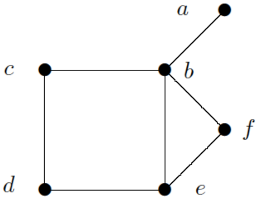
In `NetworkX`:

```
path = nx.shortest_path(G, source=1, target=3)
print("Shortest path:", path)
length = nx.shortest_path_length(G, source=1, target=3)
print("Path length:", length)
```

Distance

Definition

The **distance** $d_G(u, v)$ in a graph G between two nodes u, v is the length of any shortest path connecting u with v or infinite if there is no such path.



$$d(a, e) = 2$$

Properties of the graph distance

Let $G = (V, E)$ be a graph. The distance $d(u, v)$ between nodes u, v has the following properties:

- $d(u, v) \geq 0$, and $d(u, v) = 0 \iff u = v$.
- $d(u, v) = d(v, u)$ (symmetry).
- $d(u, v) \leq d(u, w) + d(w, v)$ (triangle inequality).
- If there is no path between u and v, then $d(u, v) = \infty$.

Connectedness: G is **connected** if $d(u, v) < \infty$ for every pair $u, v \in V$.

Extension: In weighted graphs with nonnegative edge weights, $d(u, v)$ is the minimum total weight of any path between u and v.

Diameter of a graph

Diameter and the Moore bound

Definition

The diameter $\mathrm{diam}(G)$ of a graph $G = (V, E)$ is the maximum distance between two vertices:

$$\mathrm{diam}(G) = \max_{u,v \in V} d(u, v).$$

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Theorem (Moore Bound)

$$N \;\leq\; 1 + \Delta \cdot \sum_{k=0}^{D-1} (\Delta - 1)^k.$$

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Why is this useful?

It shows that, if degrees are bounded, you cannot have both a very small diameter and a very large N.

Small-world networks keep diameters small using **hubs** or **shortcuts**.

Proof: Moore Bound

Fix a vertex v. Since the diameter is D, every vertex lies within distance D of v. To get the maximal N given the constraints:

- Distance 0: only v.
- Distance 1: at most Δ neighbors of v.
- Distance 2: each neighbor adds at most $\Delta - 1$ new vertices, giving at most $\Delta(\Delta - 1)$.
- Distance i: at most $\Delta(\Delta - 1)^{i-1}$ vertices.

Summing up to distance D,

$$N \leq 1 + \Delta \sum_{i=0}^{D-1} (\Delta - 1)^i.$$

When $\Delta = 2$, the graph is a path or cycle, giving $N \leq 2D + 1$. $\square$

Breath–First Search (BFS)

Definition

The Breadth–First Search explores a graph starting from a root node v:

- Discover all neighbors of v first.
- Then, in order, discover neighbors of those neighbors, and so on.

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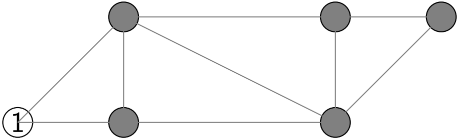

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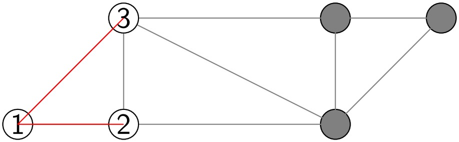

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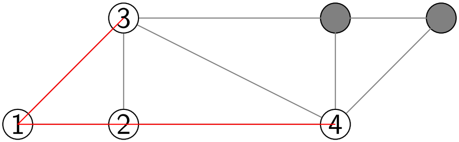

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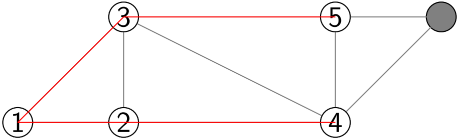

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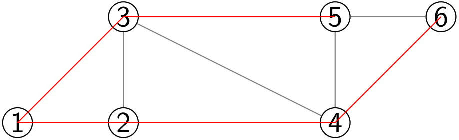

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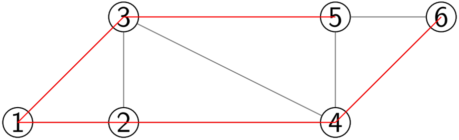
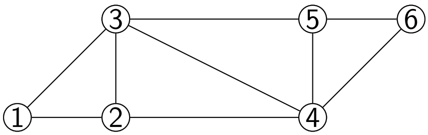


BFS naturally finds shortest paths from the root to all other vertices.

The marked path 1–2–4–6 is a shortest path from 1 to 6 (length 3). The path 1–3–2–4–6 is not shortest.

Diameter



$$\mathrm{diam}(G) = 3$$

Definition

The **diameter** $\mathrm{diam}(G)$ of a graph G is the max distance between two pair of nodes of G

$$\mathrm{diam}(G) = \max_{u,v \in V} d(u, v).$$

Graph Eccentricity

Definition

The **eccentricity of a vertex** u is: $\varepsilon(u) = \max_{v \in V} d(u, v)$.

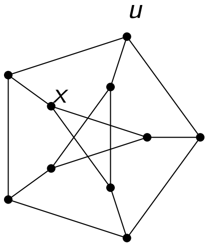
The **eccentricity of a graph** is: $\mathrm{ecc}(G) = \min_{u \in V} \varepsilon(u)$.

(every minimizer is in some sense central)

- Captures the distance from the "most central" vertex to the farthest node.
- **Applications:**
  - In communication networks: optimal placement of a hub or server.
  - In social networks: identifying the most central or influential actors.
  - In epidemics: best/worst nodes to start monitoring or intervention.

Exercise 1

Given the following graph:



a) Determine the shortest path and length between nodes u and x.

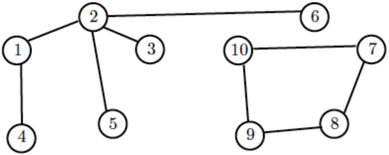
b) Determine its diameter.

c) Determine the local diameter of node z

d) Which is the minimum number of edges that we have to add to the graph so that its eccentricity is 1?

Connectivity

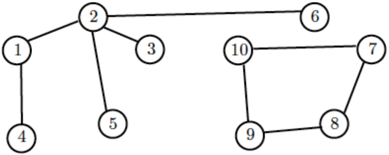
Recall: Connectivity



Definition

Two nodes u, v are connected in G if there exists a path connecting both nodes. A graph is connected if every two nodes are connected.

Recall: Connectivity



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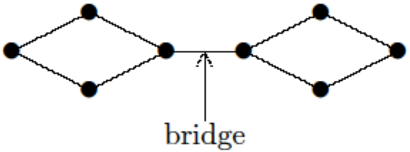
Definition

The connected components G are the maximal connected subgraphs.

Bridge

Definition

A **bridge** in a graph G is any edge such that when removing it from the graph, the number of connected components is increased.



(More) special graphs

Special Graphs: Trees

Definition

A tree is a connected and acyclic graph

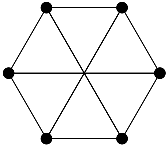
Paths P_n, Stars S_n are trees

- In a tree, every pair of vertices is joined by a unique path.
- The number of edges is one less than the number of vertices.
- Every connected graph contains a tree as a subgraph with the same vertex set (spanning tree).

Trees appear everywhere

Internet routing (spanning trees avoid cycles); **Data structures** in computer science (binary search trees, decision trees); **Phylogenetics** in biology (evolutionary trees); **Epidemics** and **rumor spreading** (branching processes); **Hierarchical clustering** (data science).

Special Graphs: Regular Graphs



Definition

A graph is called r-**regular** if every node has exactly degree r.

- Example: a cycle C_n is 2-regular.
- Example: a complete graph K_n is $(n-1)$-regular.

For an r-regular graph with N vertices:

$$L = \frac{Nr}{2}.$$

(try to prove it; see also Exercise C below)

Exercise A: Complements

a) Show that the complement of an r–regular graph on N nodes is $(N - r - 1)$–regular.

b) Is the complement of a bipartite graph always bipartite? Give a counterexample if not.

c) Give an example of a 4-node graph G such that both G and its complement are connected.

Exercise B: Connectivity

a) Show that if a graph G is not connected, then its complement $\overline{G}$ must be connected.

b) Consider a tree T (connected, acyclic). Is its complement always connected? Give an example.

Exercise C: Adjacency matrix

Let A be the adjacency matrix of a graph G.

a) Show that G is r–regular if and only if

$$A\mathbf{1}_N = r\mathbf{1}_N,$$

where $\mathbf{1}_N$ is the all-ones vector in $\mathbb{R}^N$.

b) Suppose

$$I + A + A^2 + A^3 = J,$$

where J is the all-ones matrix. What does this imply about the diameter of G?

Centrality: Motivation

Why centrality?

Which is the most important node in a network?

- In a social network: the most influential person.
  - Advertisers buy access to central nodes.
- In trade or finance: the most systemic firm or bank.
  - Lehman Brothers was "central" in interbank lending.
- In transport: the airport whose closure causes the largest disruption.

Challenge: "importance" is not unique. Different aspects motivate different measures:

- Many neighbours $\rightarrow$ **Degree centrality**.
- Close to everyone $\rightarrow$ **Closeness centrality**.
- On many shortest paths $\rightarrow$ **Betweenness centrality**.

Why centrality? Motivating examples

Different networks ⇒ different notions of importance

- **Social media (Twitter/X):** A user with millions of followers is central by *degree*. Another user might have fewer followers but be the main source of breaking news retweets ⇒ central by *betweenness*.
- **Transport networks:** Heathrow or Atlanta airports are central because they connect many international routes. Closure disrupts global traffic. ⇒ degree & betweenness both matter.
- **Electric power grids:** Centrality can mean carrying the largest electrical load (flow-based), or being geographically close to all others (closeness).
- **Economics and finance:** A highly connected bank in the interbank lending network may be systemically important (degree). A bank that connects otherwise disjoint clusters may trigger contagion (betweenness).

Centrality measures (overview)

- **Degree centrality:** $\deg(v)$.
- **Closeness centrality:** inverse average distance from v to all others.
- **Betweenness centrality:** share of shortest paths that go through v.
- **Eigenvector centrality:** v is important if its neighbours are important.
- **PageRank:** adapted to directed graphs (web, citation networks), weighting incoming links by their sources.

Degree centrality

Degree centrality

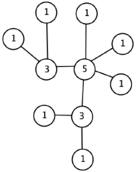
Definition

The degree centrality of a node v is its degree:

$$C_{\mathrm{deg}}(v) = \mathrm{deg}(v).$$

Interpretation:

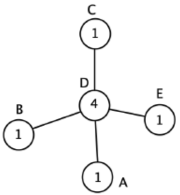
- High degree node can directly influence/reach many others.
- In undirected networks: count of adjacent edges.
- In directed networks: sometimes split into in-degree and out-degree centrality.



Degree via adjacency matrix

Let A_G be the $N \times N$ adjacency matrix of G. For $\mathbf{1} = (1, \ldots, 1)^T$,

$$w = A_G \mathbf{1} = \begin{pmatrix} \deg(1) \\ \deg(2) \\ \vdots \\ \deg(N) \end{pmatrix}.$$



Normalised degree centrality

To compare networks of different sizes, normalise by $N - 1$:

$$C'_{\mathrm{deg}}(v) = \frac{\deg(v)}{N-1} \in [0, 1].$$

Interpretation: fraction of all possible nodes to which v is directly connected.

Closeness centrality

Closeness Centrality

Definition

The closeness centrality of u is

$$C_{\mathrm{close}}(u) = \frac{N-1}{\sum_{v \neq u} d(u,v)},$$

where $d(u,v)$ is the distance between u and v.

- Large if u is on average close to everyone else.
- Small if many nodes are far from u.
- Values lie in $(0,1]$ after normalisation.

Distance matrix

Definition

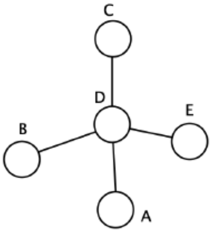
The distance matrix D_G has entries $D_G(i,j) = d(i,j)$.

Example:

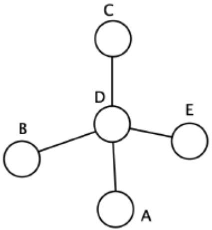
$$D_G = \begin{pmatrix} 0 & 2 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

Closeness centrality requires row sums of D_G.

Closeness Centrality



Closeness Centrality



$$\bar{c} = \frac{1}{N-1} D_G \cdot \mathbf{1} = \frac{1}{4} \begin{pmatrix} 0 & 2 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 7 \\ 7 \\ 7 \\ 4 \\ 7 \end{pmatrix} \rightarrow c = \begin{pmatrix} 4/7 \\ 4/7 \\ 4/7 \\ 1 \\ 4/7 \end{pmatrix}$$

Degree vs. Closeness Centrality

In many real-world networks, nodes with higher degree also tend to have higher closeness centrality.

- High-degree nodes are usually closer (on average) to all others, since they connect to many parts of the network.
- Empirically, this relationship often follows a **logarithmic law:**

$$\boxed{\frac{1}{C_{\mathrm{close}}(v)} \approx -\frac{1}{\ln(\alpha - 1)} \log(\deg(v)) + \beta}$$

where α, β are constants depending on the network's structure.

(Empirical relation observed, e.g., in scale-free and small-world networks.)