Universitat Pompeu Fabra
Barcelona

Lesson 9 · Clustering and small world

Networks, Crowds and Markets

Today's Lecture

We continue our discussion of various phenomena observed in real networks. We study to what extend a basic model as the Erdős-Renyi model could explain it.

1. The clustering coefficient: definition, motivation, formulas.
2. Small World Phenomenon
3. Milgram's experiment
4. Power laws and hubs

Clustering

Clustering

Real networks are not tree-like. Friends of friends often know each other (and so triangles are common).

Examples:

- **Social networks:** If Alice knows Bob and Carol, it's likely Bob and Carol also know each other. $\rightarrow$ Social circles, community structure.
- **Trade networks:** Countries trading with the same partner often trade with each other. $\rightarrow$ Formation of regional trade blocs.
- **Financial networks:** Two banks lending to the same counterparties are likely connected through risk exposures. $\rightarrow$ Triangles increase contagion channels.
- **Citation or collaboration networks:** If researcher A collaborates with both B and C, B–C collaboration becomes more probable. $\rightarrow$ Knowledge diffusion through closed triads.

Clustering coefficient: definition

Definition

For node v with degree $\deg(v) = k_v$:

$$C_v = \frac{\#\text{ links among neighbors of } v}{\binom{k_v}{2}} \in [0, 1].$$

- Measures “friend-of-friend closure.”
- $C_v = 1$: neighbors form a clique; $C_v = 0$: none connected.
- Average clustering coefficient: $\overline{C} = \frac{1}{N} \sum_v C_v$.

Clustering in Erdős–Rényi networks

Suppose $\deg(v) = k_v$. Consider two neighbors u, w.

Each pair u, w gets connected (independently) with probability p.

The expected number of links among neighbors is $\mathbb{E}L_v = p\binom{k_v}{2}$.

Thus

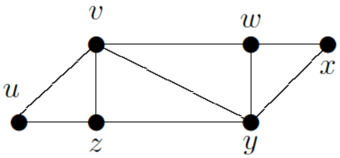
$$\mathbb{E}[C_v] \;=\; \mathbb{E}\left[\frac{L_v}{\binom{k_v}{2}}\right] \;=\; \frac{\mathbb{E}[L_v]}{\binom{k_v}{2}} \;=\; p.$$

Implications:

- In the sparse regime $p = c/N$: $\mathbb{E}[C_i] \approx c/N \to 0$.
- Prediction: clustering vanishes as N grows.
- Real networks (social, financial, trade) exhibit far higher clustering. $\Rightarrow$ Mismatch: motivates richer models leading so sparse networks with nontrivial clustering coefficients.

Exercise

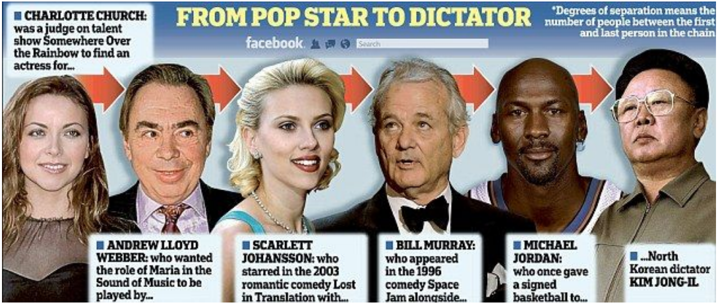
Determine the Clustering Coefficient for nodes w and y.



Small World Phenomenon

Six degrees of separation

- The idea: in social networks, any two people are separated by surprisingly few steps.
- Popularized as "six degrees of separation" in the 1960s.
- Relatively recent data (Facebook, 2016): average distance between users ≈ 3.5.
- LinkedIn advertises "3rd-degree connections" — the same principle in business networks.



Sandbox: Explore the Small World Yourself

The "six degrees" idea is easy to explore online and in data.

- **Wikipedia Game:** shortest path between two articles
  https://wiki.spaceface.dev/
- **Hollywood Game:** distance between any actor and Kevin Bacon
  https://oracleofbacon.org/
- **Mathematical Collaboration:** find distance between any two mathematicians (e.g., Paul Erdős and Piotr Zwiernik)
  https://mathscinet.ams.org/mathscinet/freetools/collab-dist
- **Social Graphs:** Facebook's *"Degrees of Separation"* study (avg. distance ≈ 3.5!) https://research.facebook.com/blog/2016/02/three-and-a-half-degrees-of-separation/

Even in enormous graphs, the average distance between two nodes often scales like $\log N$ — a hallmark of the Small World phenomenon.

Diameter in Erdős–Rényi Graphs

Let $c = \mathbb{E}(\overline{\deg}(G))$ in $G(N, p)$ with $p = c/N$ (sparse regime).

Branching-process heuristic:

- Running BFS, early layers are almost tree-like.
- Each node produces on average c new nodes.

Hence, the expected number of vertices within distance $\leq d$ from a node is approximately

$$N(d) \;\approx\; 1 + c + c^2 + \cdots + c^d \;=\; \frac{c^{d+1} - 1}{c - 1}.$$

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Assume $c \gg 1$. We estimate the graph diameter (or typical distance) by solving $N(d_{\max}) \approx N$:

$$N \;\approx\; c^{\,d_{\max}} \quad \Rightarrow \quad d_{\max} \;\approx\; \log_c N \;=\; \frac{\ln N}{\ln c}.$$

Diameter in Sparse Random Networks

- If the average degree is $c \gg 1$, then

$$\operatorname{diam}(G) \;\approx\; \frac{\ln N}{\ln c}.$$

- **Interpretation:** distances in random networks grow only logarithmically in N.
- Example: a network with $N = 1{,}000{,}000$ and $c = 10$ has diameter ≈ 6.
- This explains why even very large systems can feel “small”.

Diameter of Real Networks

Many real-world networks have very short path lengths. **Even shorter** than what $G(n, p)$ predicts for the same $d(G)$.

Network	N	L	‹k›	‹d›	d_{max}	ln*N*/ln‹*k*›
Internet	192,244	609,066	6.34	6.98	26	6.58
WWW	325,729	1,497,134	4.60	11.27	93	8.31
Power Grid	4,941	6,594	2.67	18.99	46	8.66
Mobile-Phone Calls	36,595	91,826	2.51	11.72	39	11.42
Email	57,194	103,731	1.81	5.88	18	18.4
Science Collaboration	23,133	93,437	8.08	5.35	15	4.81
Actor Network	702,388	29,397,908	83.71	3.91	14	3.04
Citation Network	449,673	4,707,958	10.43	11.21	42	5.55
E. Coli Metabolism	1,039	5,802	5.58	2.98	8	4.04
Protein Interactions	2,018	2,930	2.90	5.61	14	7.14

This is because real networks typically contain hubs and clustering (unlike ER).

Milgram's experiment

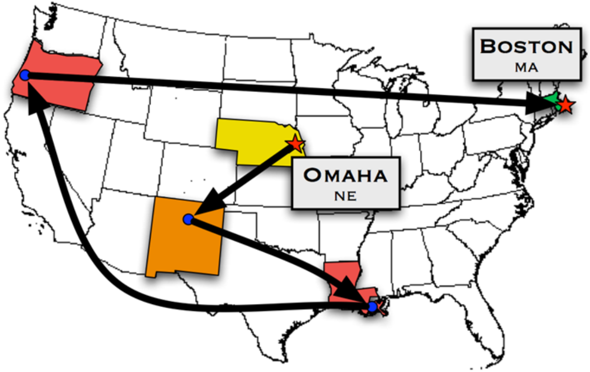

Subway
Subway

Milgram's Small-World Experiment (1967)

Psychologist Stanley Milgram wanted to measure the "social distance" between two random people in the US.

- He selected a target person living in Sharon, Massachusetts (a stockbroker).
- Random participants across the US received a letter with instructions:
  - If they knew the stockbroker the chain terminates.
  - Otherwise, they had to forward the letter to one acquaintance whom they believed was closer to the target (geographically, professionally, or socially).
  - Each acquaintance repeated the same rule, until (hopefully) the letter reached the target.

Output of Milgram's Experiment

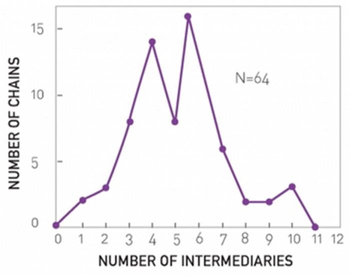


Results of Milgram's Experiment

- About 296 letters were originally sent out. Only 64 letters successfully reached the target.
- For those that did arrive, the average chain length was about

$$\text{Average path length} \approx 6.5.$$

- This gave rise to the popular phrase: **"six degrees of separation"**.



Why was this surprising?

- At the time, many expected paths to be much longer (tens or hundreds of steps).
- The experiment revealed that human social networks are extremely well connected.
- Even though only $\approx 20\%$ of letters arrived, the short paths were consistent.
- Interpretation: social networks have a **"small-world property"** — typical distances grow very slowly with population size.

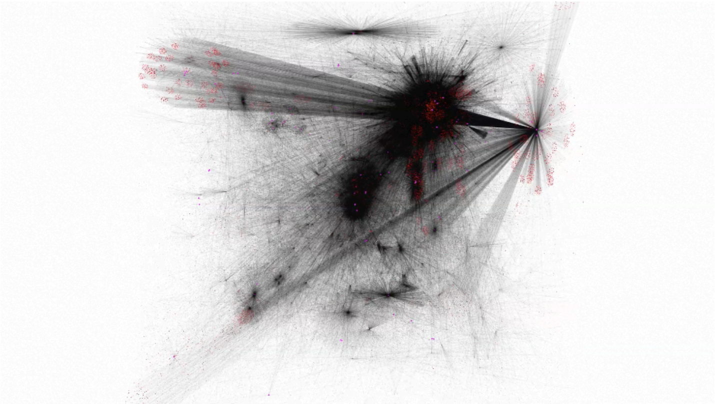
Power laws: motivating example

The World Wide Web

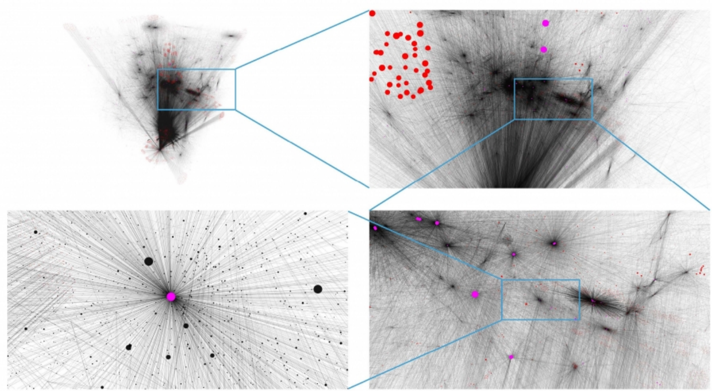
The visible Web today has roughly as many pages as the human brain has neurons — and far fewer links than the brain has synapses.

- The World Wide Web (WWW) is a network: nodes are web pages, edges are hyperlinks (URLs).
- The site WorldWideWebSize.com estimates that the indexed web (the portion that major search engines have indexed) contains at least 4 billion pages (as of Jan 15, 2025).
- The human brain has roughly 86 billion neurons.
- The structure can be mapped using a crawler that follows hyperlinks.
- The first large-scale map of the WWW for scientific purposes was created by Hawoong Jeong (Notre Dame, 1998).

Monitoring the map of WWW



Hubs in the WWW

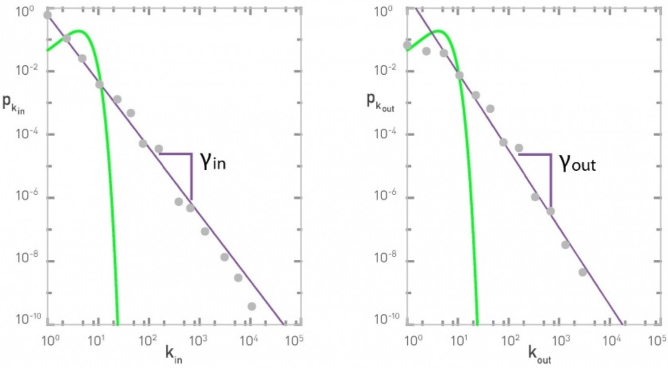


Close-ups reveal the presence of a few very highly connected nodes, often called *hubs*.

Degree distribution of the WWW

- Most pages have few links, but a few have extremely many.
- This is visible on a **log-log plot** of the degree distribution.

In-degree Out-degree



$$\log P(k) \sim -\gamma \log k \implies P(k) \sim k^{-\gamma}.$$

Power laws and hubs

Why "scale-free"?

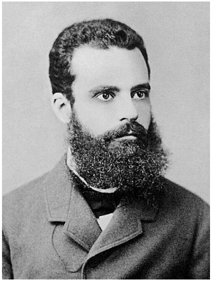
- In most distributions (e.g., Gaussian, Poisson) there is a *typical scale* — most values cluster around the mean.
- In a power law $p_k \propto k^{-\gamma}$ there is no typical degree.
  - Small degrees are common, but very large ones also appear with non-negligible probability

$$\frac{p_{ak}}{p_k} = \frac{(ak)^{-\gamma}}{k^{-\gamma}} = a^{-\gamma} \qquad \text{does not depend on k!.}$$

  - The same pattern repeats no matter how we "zoom" on the scale of k.
- Hence the term **scale-free**: there is no characteristic scale for degree.

Historical roots: Pareto and the 80/20 law

Vilfredo Pareto (1848–1923), Italian economist, observed that income distribution in society is very uneven.



- Incomes followed a distribution with a heavy tail: a small fraction of people held most of the wealth.
- This became the well-known "80/20 rule": e.g. 20% of people control 80% of wealth.
- Similar patterns appear in many domains:
  - 80% of web links point to about 20% of webpages.
  - A small number of firms or banks control a large share of markets.
  - A few researchers or papers receive most citations.

Connection: Pareto's law is an early example of a *power law* in economics, closely related to what we now see in network degree distributions.

Power law: Discrete formalism

We model the degree distribution of a scale–free network as

$$p_k \;=\; \frac{1}{\zeta(\gamma)}k^{-\gamma}, \quad k \geq 1,$$

where $\zeta(\gamma) = \sum_{k=1}^{\infty} k^{-\gamma}$ is the normalizing constant.

The function $\zeta(\gamma)$ is called the Riemann zeta function.

- The series converges only if $\gamma > 1$.
- In real networks, empirical exponents typically satisfy $2 < \gamma \leq 3$.

First two moments

If $Z \sim (p_k)$ then

$$\mathbb{E}Z = \sum_{k\geq 1} k p_k = \frac{1}{\zeta(\gamma)} \sum_{k\geq 1} k k^{-\gamma} = \frac{1}{\zeta(\gamma)} \sum_{k\geq 1} k^{-(\gamma-1)} = \frac{\zeta(\gamma-1)}{\zeta(\gamma)}.$$

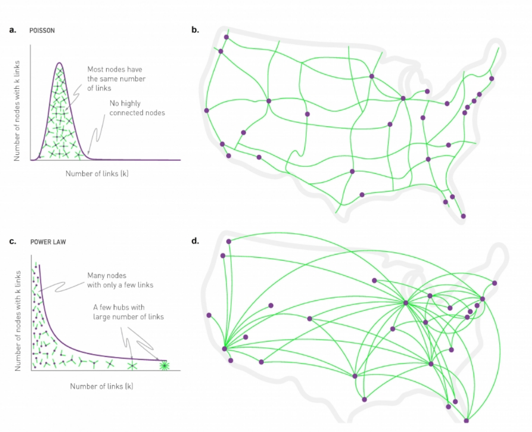
$$\mathbb{E}Z^2 = \frac{1}{\zeta(\gamma)} \sum_{k\geq 1} k^2 k^{-\gamma} = \frac{\zeta(\gamma-2)}{\zeta(\gamma)}$$

Recall, $\zeta(\gamma)$ only defined for $\gamma > 1$. The regime $2 < \gamma \leq 3$ is special:

- Since $\gamma - 1 > 1$, the mean exists.
- Since $\gamma - 2 \leq 1$, the variance does not!

(a very heavy-tailed distribution)

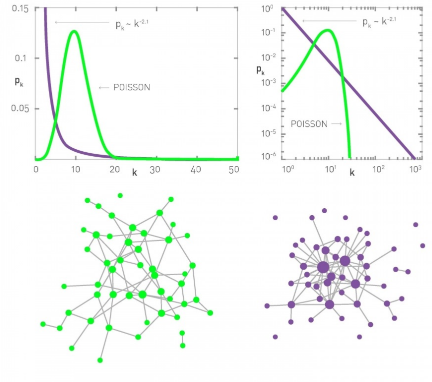
Scale-free vs Erdős–Rényi networks



Main differences in degree distribution

- The key difference lies in the **tail behavior**.
- For small k, power law predicts many low-degree nodes — ER does not.
- Around the mean degree $\mathbb{E}[\deg]$, ER has a sharp peak (Poisson), while scale-free is much flatter.
- For large k, scale-free has a heavy tail: a non-negligible number of very high degree nodes.

Implication: scale-free networks generate hubs, while ER almost never does.



ER Network

Scale-Free Network