Lecture 5: Calculus and Linear Algebra

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Chapter 10: Differential equations

In many economic applications, we have to deal with time as a continuous variable, and to model those situations one uses differential equations.

Examples: the most common models are exponential (savings account) or sub-exponential models (epidemic growth).

Read sections 24.1 and 24.2 of Simon-Blume

Exercises: 24.5 (Simon-Blume)

Ordinary differential equations

An ordinary first order differential equation is an equation of the form

$$\dot{y}(t) = F(y(t), t).$$

We usually write

$$\dot{y} = F(y, t).$$

If F only depends on y it is called autonomous: $\dot{y} = F(y)$.

Example 1: Exponential model $\dot{y} = ry$

Solution :

$$y(t) = y(0)e^{rt}.$$

Proof: integrating: $\int_0^t \frac{dy}{y} = \int_0^t ds$, gives $\ln y(t) - \ln y(0) = rt$. Then take exponentials.

Ordinary differential equations

Example 2: Exponential model $\dot{y} = r(t)y$

Solution :
$$y(t) = y(0)e^{\int_0^t r(s)ds}.$$

Proof: integrating: $\int_0^t \frac{dy}{y} = \int_0^t r(s)ds$, gives $\ln y(t) - \ln y(0) = \int_0^t r(s)ds$. Then take exponentials.

Example 3: Exponential model $\dot{y} = r(t)y + b(t)$,

Solution : $y(t) = \left(y(0) + \int_0^t b(s)e^{-\int_0^s r(u)du}ds\right) e^{\int_0^t r(s)ds}.$

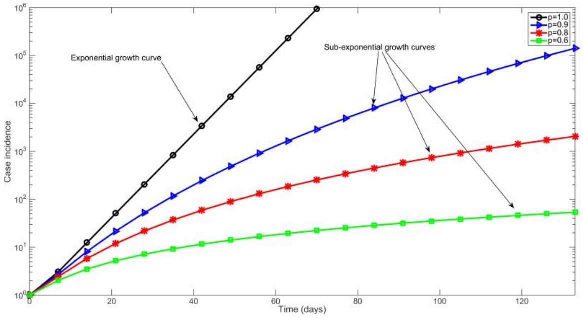
Proof: write $(\dot{y} - r(t)y = b(t))e^{-\int_0^t r(s)ds}$, and integrate $\int_0^t \frac{d}{ds}\left(y(s)e^{-\int_0^s r(u)du}\right) = \int_0^t b(s)e^{-\int_0^s r(u)du}ds$.

Example 4: Sub-Exponential model $\dot{y} = ry^p$, $p \in (0,1)$.

Solution :

$$y(t) = (y(0)^{1-p} + (1-p)rt)^{1/(1-p)}.$$

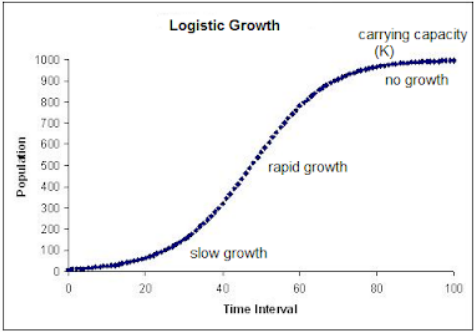
Proof: integrating: $\int_0^t \frac{dy}{y^p} = rt$, gives $y(t)^{1-p} = y(0)^{1-p} + (1-p)rt$.



Example 5: Logistic growth model

$$\dot{y} = y(1-y), \quad 0 < y(0) < 1, \quad 1 = \text{carrying capacity}$$

Solution: $y(t) = \frac{1}{1+ae^{-t}}, \quad a = \frac{1}{y(0)} - 1$ Proof: Use that $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$ and integrate



Cobb-Douglas production function

The solution to $\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$ with $K = K_0$ is

$$P(L, K_0) = C_1(K_0)L^{\alpha}.$$

Integrate $\int \frac{dP}{P} = \alpha \int \frac{dL}{L}$ so $\ln(P) = \alpha \ln(cL)$

The solution to $\frac{\partial P}{\partial K} = \beta \frac{P}{K}$ with $L = L_0$ is

$$P(L_0, K) = C_2(L_0)K^{\beta}.$$

Combining both equations we get

$$P(L, K) = bL^{\alpha}K^{\beta},$$

where b is a constant.

Assumption 1 shows that $\alpha, \beta > 0$.

Second order differential equation: an example

Let $x \to u(x)$ the utility function for wealth x.

The Relative Risk-Aversion is $-\frac{xu''(x)}{u'(x)}$

Assuming that the relative risk aversion is 1, we obtain the second order differential equation

$$u''(x) = -\frac{u'(x)}{x}.$$

Let $v(x) = u'(x)$. Then, the equation becomes a first order differential equation

$$\frac{dv}{dx} = -\frac{v}{x}.$$

The solution is $v(x) = k_1 x^{-1}$. Therefore $u(x) = k_2 + k_1 \ln x$.

Integrate $\int \frac{dv}{v} = -\int \frac{dx}{x}$.