

Cobb-Douglas Production Function

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1 Introduction

In economics, the Cobb-Douglas functional form of production functions is widely used to represent the relationship of an output to inputs. It was proposed by Knut Wicksell (1851 - 1926), and tested against statistical evidence by Charles Cobb and Paul Douglas in 1928.

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899 - 1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested. While there are many other factors affecting economic performance, their model proved to be remarkably accurate.

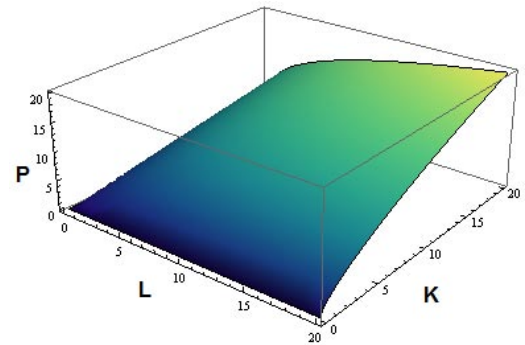


Figure 1: A two-input Cobb-Douglas production function

The function they used to model production was of the form:

$$P(L, K) = bL^\alpha K^\beta$$

where:

- P = total production (the monetary value of all goods produced in a year)
- L = labor input (the total number of person-hours worked in a year)
- K = capital input (the monetary worth of all machinery, equipment, and buildings)
- b = total factor productivity
- α and β are the output elasticities of labor and capital, respectively. These values are constants determined by available technology.

Output elasticity measures the responsiveness of output to a change in levels of either labor or capital used in production, *ceteris paribus*. For example if $\alpha = 0.15$, a 1% increase in labor would lead to approximately a 0.15% increase in output.

Further, if:

$$\alpha + \beta = 1,$$

the production function has constant returns to scale. That is, if L and K are each increased by 20%, then P increases by 20%.

Returns to scale refers to a technical property of production that examines changes in output subsequent to a proportional change in all inputs (where all inputs increase by a constant factor). If output increases by that same proportional change then there are constant returns to scale (CRTS), sometimes referred to simply as returns to scale. If output increases by less than that proportional change, there are decreasing returns to scale (DRS). If output increases by more than that proportion, there are increasing returns to scale (IRS)

However, if

$$\alpha + \beta < 1,$$

returns to scale are decreasing, and if

$$\alpha + \beta > 1,$$

returns to scale are increasing. Assuming perfect competition, α and β can be shown to be labor and capital's share of output.

2 Discovery

This section will discuss the discovery of the production formula and how partial derivatives are used in the Cobb-Douglas model.

2.1 Assumptions Made

If the production function is denoted by $P = P(L, K)$, then the partial derivative $\frac{\partial P}{\partial L}$ is the rate at which production changes with respect to the amount of labor. Economists call it the marginal production with respect to labor or the *marginal productivity of labor*. Likewise, the partial derivative $\frac{\partial P}{\partial K}$ is the rate of change of production with respect to capital and is called the *marginal productivity of capital*.

In these terms, the assumptions made by Cobb and Douglas can be stated as follows:

1. If either labor or capital vanishes, then so will production.
2. The marginal productivity of labor is proportional to the amount of production per unit of labor.
3. The marginal productivity of capital is proportional to the amount of production per unit of capital.

2.2 Solving

Because the production per unit of labor is $\frac{P}{L}$, assumption 2 says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α . If we keep K constant ($K = K_0$), then this partial differential equation becomes an ordinary differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L}$$

This separable differential equation can be solved by re-arranging the terms and integrating both sides:

$$\begin{aligned}\int \frac{1}{P} dP &= \alpha \int \frac{1}{L} dL \\ \ln(P) &= \alpha \ln(cL) \\ \ln(P) &= \ln(cL^\alpha)\end{aligned}$$

And finally,

$$P(L, K_0) = C_1(K_0)L^\alpha \quad (1)$$

where $C_1(K_0)$ is the constant of integration and we write it as a function of K_0 since it could depend on the value of K_0 .

Similarly, assumption 3 says that

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K}$$

Keeping L constant ($L = L_0$), this differential equation can be solved to get:

$$P(L_0, K) = C_2(L_0)K^\beta \quad (2)$$

And finally, combining equations (1) and (2):

$$P(L, K) = bL^\alpha K^\beta \quad (3)$$

where b is a constant that is independent of both L and K .

Assumption 1 shows that $\alpha > 0$ and $\beta > 0$.

Notice from equation (3) that if labor and capital are both increased by a factor m , then

$$\begin{aligned} P(mL, mK) &= b(mL)^\alpha (mK)^\beta \\ &= m^{\alpha+\beta} bL^\alpha K^\beta \\ &= m^{\alpha+\beta} P(L, K) \end{aligned}$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of m , as discussed earlier in Section 1.

3 Usage

This section will demonstrate the usage of the production formula using real world data.

3.1 An Example

Year	1899	1900	1901	1902	1903	1904	1905	...	1917	1918	1919	1920
P	100	101	112	122	124	122	143	...	227	223	218	231
L	100	105	110	117	122	121	125	...	198	201	196	194
K	100	107	114	122	131	138	149	...	335	366	387	407

Table 1: Economic data of the American economy during the period 1899 - 1920 [1]. Portions not shown for the sake of brevity

Using the economic data published by the government, Cobb and Douglas took the year 1899 as a baseline, and P , L , and K for 1899 were each assigned the value 100. The values for other years were expressed as percentages of the 1899 figures. The result is Table 1.

Next, Cobb and Douglas used the method of least squares to fit the data of Table 1 to the function:

$$P(L, K) = 1.01(L^{0.75})(K^{0.25}) \quad (4)$$

For example, if the values for the years 1904 and 1920 were plugged in:

$$\begin{aligned} P(121, 138) &= 1.01(121^{0.75})(138^{0.25}) \approx 126.3 \\ P(194, 407) &= 1.01(194^{0.75})(407^{0.25}) \approx 235.8 \end{aligned}$$

which are quite close to the actual values, 122 and 231 respectively.

The production function $P(L, K) = bL^\alpha K^\beta$ has subsequently been used in many settings, ranging from individual firms to global economic questions. It has become known as the **Cobb-Douglas production function**. Its domain is $\{(L, K) : L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

3.2 Difficulties

Even though the equation (4) derived earlier works for the period 1899 - 1922, there are currently various concerns over its accuracy in different industries and time periods.

Cobb and Douglas were influenced by statistical evidence that appeared to show that labor and capital shares of total output were constant over time in developed countries; they explained this

by statistical fitting least-squares regression of their production function. However, there is now doubt over whether constancy over time exists.

Neither Cobb nor Douglas provided any theoretical reason why the coefficients α and β should be constant over time or be the same between sectors of the economy. Remember that the nature of the machinery and other capital goods (the K) differs between time-periods and according to what is being produced. So do the skills of labor (the L).

The Cobb-Douglas production function was not developed on the basis of any knowledge of engineering, technology, or management of the production process. It was instead developed because it had attractive mathematical characteristics, such as diminishing marginal returns to either factor of production.

Crucially, there are no microfoundations for it. In the modern era, economists have insisted that the micro-logic of any larger-scale process should be explained. The C-D production function fails this test.

For example, consider the example of two sectors which have the exactly same Cobb-Douglas technologies:

if, for sector 1,

$$P_1 = b(L_1^\alpha)(K_1^\beta)$$

and, for sector 2,

$$P_2 = b(L_2^\alpha)(K_2^\beta),$$

that, in general, does not imply that

$$P_1 + P_2 = b(L_1 + L_2)^\alpha(K_1 + K_2)^\beta$$

This holds only if $\frac{L_1}{L_2} = \frac{K_1}{K_2}$ and $\alpha + \beta = 1$, i.e. for constant returns to scale technology.

It is thus a mathematical mistake to assume that just because the Cobb-Douglas function applies at the micro-level, it also applies at the macro-level. Similarly, there is no reason that a macro Cobb-Douglas applies at the disaggregated level.

This entire article was adapted from the following resources:

References

- [1] James Stewart, *Calculus: Early Transcendentals*. Thomson Brooks/Cole, 6th Edition, 2008, Pages 857 and 887
- [2] Wikipedia, *Cobb-Douglas*. http://en.wikipedia.org/wiki/Cobb_douglas