Lecture 5: Calculus and Linear Algebra

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Motivation: why separable ODEs?

Ordinary differential equations (ODEs) appear whenever we model how a quantity changes over time or another variable.

A simple but very important class: separable ODEs, where variables can be separated:

$$\frac{dy}{dx} = g(x)\,h(y).$$

- Widely used in economics, biology, and physics for growth/decay models.
- Method: rearrange into $\frac{dy}{h(y)} = g(x)\,dx$, then integrate both sides.
- Today: a very small illustrative example.

A toy separable ODE

Consider

$$v'(x) \;=\; \frac{\mathrm{d}v}{\mathrm{d}x} \;=\; -\frac{1}{x}\,v(x).$$

Step 1: Separate variables.

$$\frac{dv}{v} = -\frac{dx}{x}.$$

Step 2: Integrate both sides.

$$\int \frac{dv}{v} = -\int \frac{dx}{x} \quad \Rightarrow \quad \ln|v(x)| = -\ln|x| + C.$$

Hence

$$v(x) = C_1\,x^{-1}, \quad x \neq 0.$$

Recovering u from $v = u'$

Suppose $v(x) = u'(x)$. Then

$$u'(x) = C_1 \, x^{-1}.$$

Integrate again:

$$u(x) = C_1 \ln|x| + C_2.$$

Domain note: Because of $\ln|x|$, we must restrict to $x > 0$ or $x < 0$ separately.

Example with initial conditions

Suppose $u(1) = 0$ and $u'(1) = 2$.

$$u'(1) = 2 \;\Rightarrow\; C_1 = 2,$$
$$u(1) = 0 \;\Rightarrow\; 0 = 2\ln 1 + C_2 \;\Rightarrow\; C_2 = 0.$$

Therefore

$$u(x) = 2\ln x \quad (\text{valid on } x > 0).$$

Key takeaways

- A separable ODE can be solved by separating y-terms and x-terms, then integrating.
- Constants of integration appear at each step and are fixed by initial conditions.
- Always check the domain: functions like $\ln |x|$ force $x > 0$ or $x < 0$.
- Even this tiny example illustrates the general workflow.