Lecture 1 : Calculus

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Mathematics Brush-up

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Goal of the course

Review basic tools that are useful in different areas of economics and finance.

These tools will be crucial for all the courses of your program.

Warm up before the Master starts.

Main references:

1. *Mathematics for Economists* by C. P. Simon and L. Blume
2. *Mathematics of Economics and Business* by F. Werner and Y. N. Sotskov

Cultivate a Critical Mindset

Technical knowledge requires effort.

- **If a step feels "magical", stop.**
  Ask *why* it works, not just *how*.

- **Challenge every answer - especially given by LLMs.**
  Large-language models can sound confident yet be wrong;
  always verify with first principles or a textbook.

- **Use multiple lenses.**
  Check special cases, draw a picture, test with code, or look for a counter-example.

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1. Sequences and series
2. Functions of one variable
3. Differentiation
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Sequences

Sequence: an infinite set of numbers arranged in a definite order.

Notation: $\{a_n\}_{n=1,2,3,\ldots}$ where each a_n is a real number.

A sequence may be defined by a formula or by a recursion.

Examples:

1. $a_n = \frac{n-1}{2n+1}$ $\quad (a_1 = 0, a_2 = \frac{1}{5}, a_3 = \frac{2}{7}, \ldots)$
2. $a_1 = 2,\ a_{n+1} = 2a_n - 1$ $\quad (a_1 = 2, a_2 = 3, a_3 = 5, \ldots)$
3. $a_1 = 1, a_2 = 1,\ a_{n+1} = a_n + a_{n-1}$ $\quad (a_1 = 1, a_2 = 1, a_3 = 2, \ldots)$
4. Arithmetic sequence: $a_{n+1} = a_n + d$ or $a_n = a_1 + (n-1)d$
   a_1 is given, $d =$difference
5. Geometric sequence: $a_{n+1} = a_n r$ or $a_n = a_1 r^{n-1}$
   a_1 is given, $r =$ratio

Sequences arise in many economic applications, e.g. in economics of finance and investment.

Sequences help to understand the concept of a limit of a function and the significance of the number e.

They are also important when solving difference equations.

Read Chapter 2 of Werner-Sotskov

Exercises: 2.4, 2.5, 2.7 (Werner-Sotskov)

Economic examples

1. Invest an amount P(principal) at an annual interest rate R. After n years the total amount is the geometric sequence

$$a_n = P(1+R)^n \qquad (\text{ratio} = 1+R, a_1 = P(1+R))$$

   Example: $P = 1000$ euros, $R = 8\%$, $a_1 = 1080$, $a_{50} \approx 46901$.
   The annual interest rate R is also called annual percentage rate (APR).

2. Compounded interest: If instead the bank pays interests m times a year, after n years, we obtain

$$a_n = P\left(1+\frac{R}{m}\right)^{nm} \qquad (\text{ratio} = (1+\frac{R}{m})^m)$$

   Example: $P = 1000$ euros, $R = 8\%$, $m = 4$, $a_1 = 1082.43$.
   The investment grows 8.243% in one year, which is called the equivalent annual rate (EAR), that is, the rate that compounded annually, gives the same yield.

Remark: EAR does not depend on P or n. In fact, it is equal to

$$\left(1+\frac{R}{m}\right)^m - 1.$$

Example: if the bank pays an interest monthly ($m = 12$) at an APR of 6%, then the EAR is 6.17%

3. The Present Discounted Value of an amount A to be received in n years at an annual interest rate R is

$$a_n = \frac{A}{(1+R)^n}$$

Example: would you prefer 1000 euros now or 1050 in one year? It depends on R!

Properties of a sequence

A sequence is increasing if for every n, $a_{n+1} \geq a_n$, and strictly increasing if $a_{n+1} > a_n$.
(which is equivalent to $a_{n+1} - a_n > 0$, or $\frac{a_{n+1}}{a_n} > 1$ if $a_n > 0$)

A sequence is decreasing if for every n, $a_{n+1} \leq a_n$, and strictly decreasing if $a_{n+1} < a_n$.

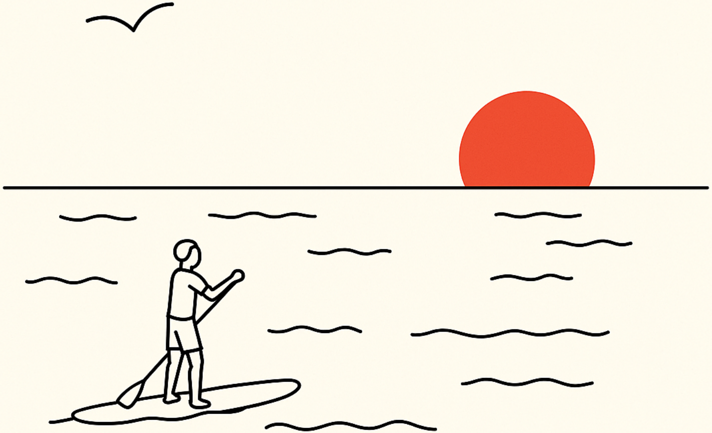
A sequence is (strictly) monotone if it is (strictly) increasing or (strictly) decreasing.

It is bounded if there exists a constant $C > 0$ such that, for all n, $a_n \in [-C, C]$.

Examples:

1. $a_n = 2(n-1)^2 - n$ is strictly increasing (since $a_{n+1} - a_n = 4n - 3 > 0$)
2. $a_n = \frac{1}{n^2}$ is bounded (since $a_n \in [-1, 1]$)

Can a strictly increasing sequence be bounded?



Limit of a sequence

A finite number a is called the limit of a sequence $\{a_n\}$ if as n gets larger, a_n gets closer to a. Notation: $\lim_{n\to\infty} a_n = a$.

A sequence is said to be convergent if it has a limit. Otherwise, we say that it has no limit or diverges (when the limit is ∞ or $-\infty$).

Examples

1. $a_n = \frac{1}{n^2}$ is convergent to 0
2. $a_n = (-1)^n$ has no limit
3. $a_n = n^2$ diverges
4. $a_n = r^n$ converges to 0 if $|r| < 1$, diverges if $r > 1$ and has no limit if $r \leq -1$.

Limit of a sequence

Definition: a is a limit of the sequence $\{a_n\}$ $\quad (\lim_{n\to\infty} a_n = a)$

$$\forall \epsilon > 0 \quad \exists N_\epsilon > 0 \quad s.t \quad \forall n > N_\epsilon \qquad |a_n - a| < \epsilon.$$

Example: the sequence $a_n = \frac{1}{n}$ is convergent to 0.

Indeed. For each $\epsilon > 0$ take any $N_\epsilon > \frac{1}{\epsilon}$.

Check from definition:

- $\lim_{n\to\infty} a_n = a$ is the same as $\lim_{n\to\infty} |a_n - a| = 0$
- If $a_n = a$ for all n then $\lim_{n\to\infty} a_n = a$.

If formal definition hard to work with, try to use bounds instead, e.g.

$$0 < \frac{1}{2n^2-1} \leq \frac{1}{n}, \quad n \geq 1 \quad \text{(and so the middle sequence converges to zero)} .$$

Limit of a sequence

More Examples

1. The sequence $\left(1+\frac{1}{n}\right)^n$ converges to the number e (≈ 2.71828)
2. The sequence $\left(1+\frac{x}{n}\right)^n$ converges to e^x, for any x. We will see a proof of these facts using integrals.

Economic example: if the interest is compounded continuously at an APR of R, the return after n years on an initial amount P is

$$\lim_{m\to\infty} P\left(1+\frac{R}{m}\right)^{nm} = Pe^{Rn}$$

and the present discounted value of an amount A received in n years is Ae^{-Rn}.

Example: if the interest is compounded continuously at an APR of 8%, then the EAR is 8.329% (since $e^{0.08} - 1 = 0.08329$)

Limit of a sequence

Theorem: Every bounded and monotone sequence is convergent.

(useful to show convergence but not to compute limits)

Remark: Observe that only bounded (for example $a_n = (-1)^n$) or only monotone (for example $a_n = n$) is not sufficient for being convergent.

Example: the sequence $a_1 = 1,\ a_{n+1} = \sqrt{3a_n}$ is bounded by 3 (proof by induction) and strictly increasing (since $\frac{a_{n+1}}{a_n} > 1$), thus convergent.

Write a couple of first entries of the sequence. Using geometric series show that this sequence converges to 3.

Theorem: If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then

$$\lim_{n\to\infty}(a_n + b_n) = a + b, \quad \lim_{n\to\infty} a_n b_n = ab, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b},$$

the last equality being true only if $b_n, b \neq 0$.

Partial sums

The nth partial sum s_n of a sequence $\{a_n\}$ is defined as the sum of the first n terms:

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Examples

1. If $a_n = 3 + (-1)^n \frac{2}{n}$ then $s_1 = 1, s_2 = 5, s_3 = \frac{22}{3}, \ldots$

   $(a_1 = 1, a_2 = 4, a_3 = \frac{7}{3}, \ldots)$
2. The nth partial sum of an arithmetic sequence is

$$s_n = na_1 + \frac{n(n-1)}{2} d.$$

3. The nth partial sum of a geometric sequence with ratio $r \neq 1$ is

$$s_n = a_1 \frac{1 - r^n}{1 - r}.$$

These formulas can help to compute the limit of a sequence

Economic example

Regular savings: We invest an amount A at the beginning of every year at an annual interest rate R. After n years the total amount is given by the geometric partial sum

$$\begin{aligned} s_n &= A(1+R)^n + A(1+R)^{n-1} + \cdots + A(1+R) \\ &= A(1+R)\frac{(1-(1+R)^n)}{1-(1+R)} \\ &= \frac{A(1+R)}{R}((1+R)^n - 1) \end{aligned}$$

Here $a_1 = A(1+R)$ and $r = 1+R$

Series and convergence of series

The sequence $\{s_n\}$ of partial sums of a sequence $\{a_n\}$ is called a series.

A series $\{s_n\}$ is said to converge if it has a limit s. In this case, the value s is called the sum of the series:

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k.$$

Example: The series associated to a geometric sequence is called a geometric series which converges if and only if $|r| < 1$, and in this case,

$$s = \sum_{k=1}^{\infty} a_1 r^{k-1} = \lim_{n\to\infty} a_1 \frac{1-r^n}{1-r} = \frac{a_1}{1-r}.$$

Chapter 2: Functions of one variable

A one variable function is simply a mapping that assigns a unique number y in $\mathbb{R}$ to each number x in $\mathbb{R}$. We write $f(x) = y$.

x is called the independent variable, or in economic applications, exogenous variable, and y is called the dependent variable or endogenous variable.

Economic examples: endogenous variables: cost, revenue, profit, demand, production, utility, etc. and exogenous variables: time, labor, capital, price, etc.

Read Chapter 3 of Werner-Sotskov and Sections 2.1, 2.2, 5.1, 5.2, 5.3 of Simon-Blume

Graph and domain

Definition

The graph of a functions consists of all points (x, y) in $\mathbb{R}^2$ such that $y = f(x)$.

The domain D of a function are the numbers x at which $f(x)$ is defined.

Notation Write $f : D \to \mathbb{R}$.

Example: The domain of the function $f(x) = \frac{1}{x}$ is $D = \mathbb{R} \setminus \{0\}$.

Sometimes it is interesting to consider a function in a restricted domain.

Properties of functions

A function $f : D \to \mathbb{R}$ is called increasing if

$$f(x_1) \leq f(x_2) \qquad \text{for any } x_1 < x_2, \quad x_1, x_2 \in D,$$

and strictly increasing if the sign is $<$. An increasing function is also called non-decreasing.

A function $f : D \to \mathbb{R}$ is called decreasing if

$$f(x_1) \geq f(x_2) \qquad \text{for any } x_1 < x_2, x_1, x_2 \in D,$$

and strictly decreasing if the sign is $>$. A decreasing function is also called non-increasing.

A function $f : D \to \mathbb{R}$ is called bounded if there exists a constant $C > 0$ such that

$$|f(x)| \leq C, \quad \text{for any } x \in D.$$

Convex/concave functions

If $x_1 < x_2$ are two points in $\mathbb{R}$, the points of the form

$$tx_1 + (1-t)x_2,$$

where $t \in [0,1]$ are called the convex combination of x_1 and x_2.
(They give all the points lying in the interval $[x_1, x_2]$)

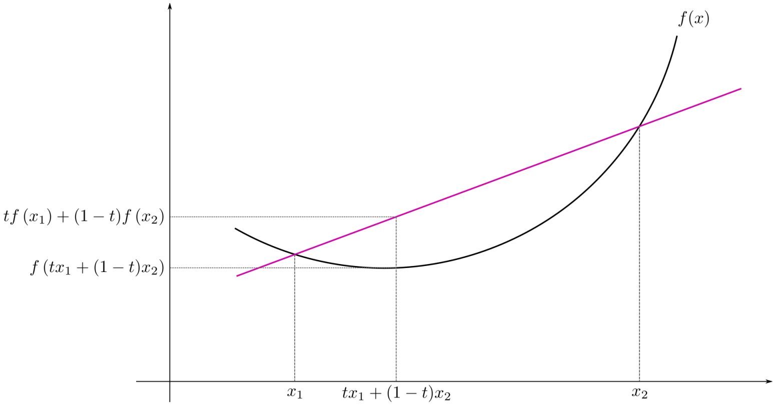
A set $D \subset \mathbb{R}$ is convex if the convex combination of any $x_1, x_2 \in D$ lies in D.

A function $f : D \to \mathbb{R}$, where D is convex is called convex (resp. concave) if for every $x_1, x_2 \in D$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq t\, f(x_1) + (1-t) f(x_2)$$

(resp. $\geq$). If for $t \in (0,1)$ and $x_1 \neq x_2$ the sign $<$ (resp. $>$) holds then it is strictly convex (resp. strictly concave).

Convex function



Quasi-convex/concave functions

A function $f : D \to \mathbb{R}$, where D is convex is called quasi convex if for every $x_1, x_2 \in D$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq \max(f(x_1), f(x_2)).$$

Similarly, quasi concave means that

$$f(tx_1 + (1-t)x_2) \geq \min(f(x_1), f(x_2)).$$

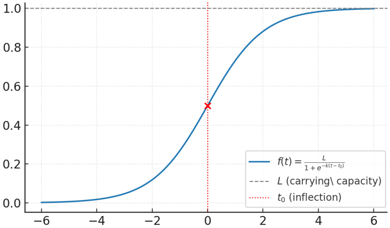
Remark: every convex function is quasi convex and every concave function is quasi concave.

Theorem: any strictly increasing function is quasiconvex and quasiconcave.

Proof: if $x_1 < x_2$ then $f(x_1) \leq f(tx_1 + (1-t)x_2) \leq f(x_2)$.

Economic example: logistic growth function

The standard logistic growth function is: $f(t) = \frac{L}{1+e^{-k(t-t_0)}}$.



It satisfies:

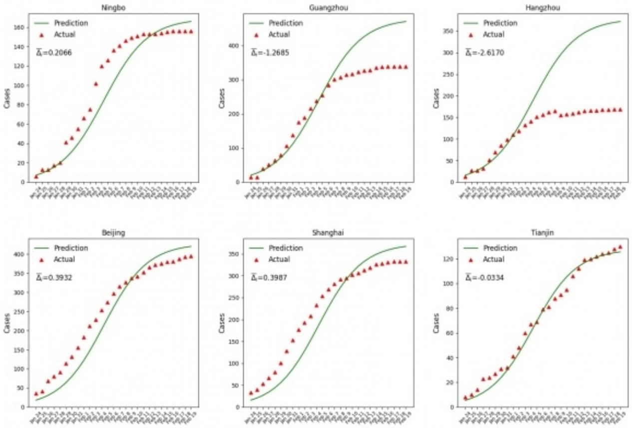
- Initial phase: roughly exp. growth when $x \ll x_0$,
- Midpoint: $f(t_0) = L/2$,
- Saturation: approaches L as $t \to \infty$.

(in the plot $L = 1, k = 1, and t_0 = 0$)

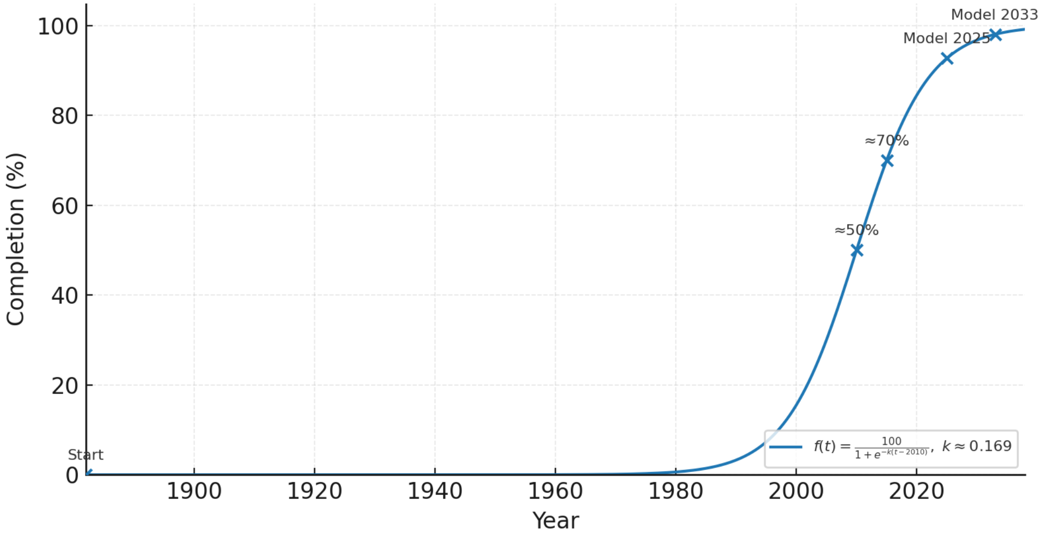
Interpretation: At first, the rate of increase increases, but after some point it slows down, and eventually decreases until it reaches some level.

Economic example

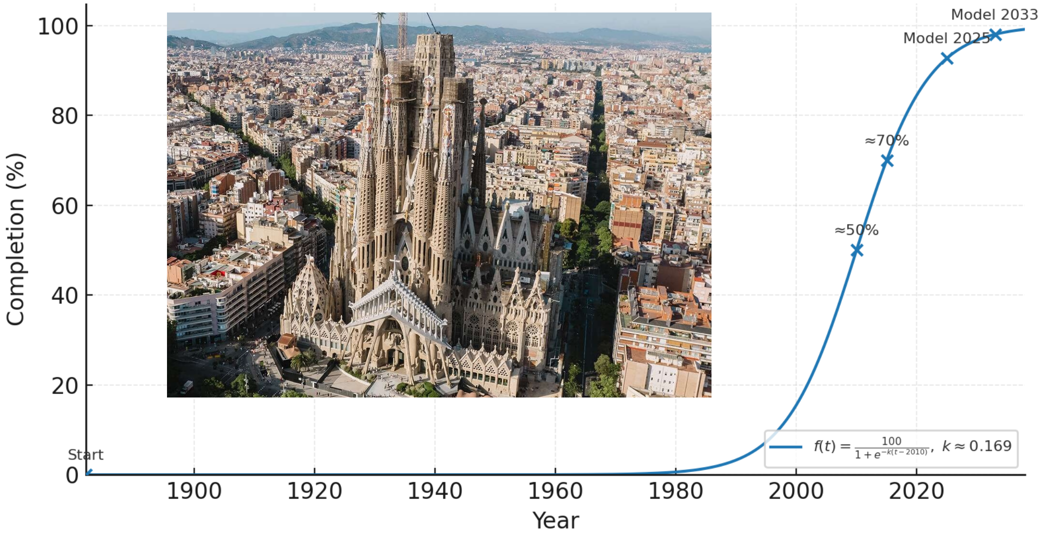
The logistic growth function is used to describe epidemic growth models. Check the following prediction versus epidemic curves over time of the COVID-19 in China.



Do you recognize this function?



Do you recognize this function?



Examples of functions

1. monomials: $f(x) = ax^k$, k natural number.
2. polynomials: sums of monomials
3. rational functions: ratios of polynomials
4. exponent functions: $f(x) = a^x$
5. trigonometric functions: $\sin(x)$, $\cos(x)$, etc.
6. linear functions: $f(x) = mx + b$, (m=slope and $(0, b)$=intercept)

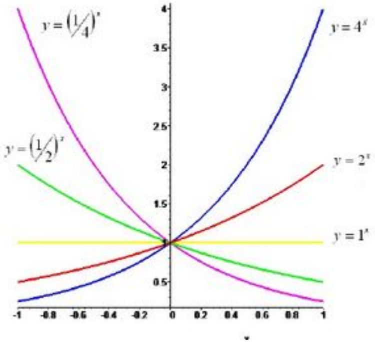
   (linear functions also write as $y - f(x_0) = m(x - x_0)$)
7. exponential function: $f(x) = e^x$ (recall this is the limit of the sequence $\left(1 + \frac{x}{n}\right)^n$)
8. base e logarithm function: $f(x) = \log x$ (defined as the unique function such that $e^{\log(x)} = x$ and $\log(e^x) = x$)

   recall: $e^{x+y} = e^x e^y$, $e^0 = 1$,

   $\log(xy) = \log(x) + \log(y)$, $\log(1) = 0$, $\log(a^x) = x\log(a)$.

Exponential functions

$f(x) = 1^x$		Horizontal line with y-intercept 1
$f(x) = a^x$,	$0 < a < 1$	Exponential decay
$f(x) = a^x$,	$a > 1$	Exponential growth



Chapter 3: Differentiation

Differentiation gives information on how the function changes with respect to small changes of the independent variable.

Example: how does the change of the price of some product affect the amount of product customers will buy ?

Differential calculus gives also information about extreme points, monotonicity, and convexity.

Read Chapter 4 of Werner-Sotskov, Chapters 2 to 5 of Simon-Blume

Exercises: 4.6(c), 4.10, 4.14, 4.15(a)-(c), 4.16(d), 4.17(b) (Werner-Sotskov), 5.5 (b)-(c)-(e) (Simon-Blume)

Limit of a function

Let $f : D \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. We say that L is the limit of f as x tends to x_0 if for any sequence $\{x_n\} \in D$ convergent to x_0, the sequence $\{f(x_n)\}$ converges to L.

Notation: $\lim_{x \to x_0} f(x) = L$. Remark: we do note need that $x_0 \in D$.

Example: $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ since $0 \leq |x \sin \frac{1}{x}| \leq |x|$.

Theorem

If $\lim_{x \to x_0} f(x) = y_1$ and $\lim_{x \to x_0} g(x) = y_2$, then

$$\lim_{x \to x_0} (f(x) + g(x)) = y_1 + y_2, \quad \lim_{x \to x_0} f(x)g(x) = y_1 y_2, \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{y_1}{y_2},$$

the last equality being true only if $y_2 \neq 0$ and $g(x) \neq 0$ around x_0.

Continuity of a function

Definition

We say that a function f is continuous at $x_0 \in D$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If f is continuous at every point of a set $B \subset D$, then we say that f is continuous at B.

Theorem: If f, g are continuous at x_0 then so are $f + g$ and fg. If moreover, $g(x) \neq 0$ around x_0, then $\frac{f}{g}$ is continuous at x_0.

Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous then it is bounded.

Continuity of a function

The composition of two functions f and g is the function defined as $h(x) = f(g(x))$, for all $x \in D_g$. Notation: $h = f \circ g$.

If f and g are continuous, so is $h = f \circ g$.

Moreover, $\lim_{x \to x_0} f(g(x)) = f(\lim_{x \to x_0} g(x))$.

Let $f : [a, b] \to \mathbb{R}$ be a strictly monotone continuous function with $f(a) = c$ and $f(b) = d$. We define the inverse function of f as the function $f^{-1} : [c, d] \to [a, b]$ such that $f \circ f^{-1} = \mathrm{id}$ and $f^{-1} \circ f = \mathrm{id}$.

If f is strictly monotone continuous function, so is f^{-1}.

(For example, $e^{\log(x)} = x$ and $\log(e^x) = x$.)

Derivative

Let $f : (a, b) \to \mathbb{R}$. The function f is said to be differentiable at $x \in (a, b)$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

exists. In this case, it denoted $f'(x)$ and called the derivative of f at x.

Theorem: If f is differentiable at x then f is continuous at x.

Example: $f(x) = |x|$ is continuous but not differentiable at 0.

In economics, continuity is a reasonable assumption (a small change in x gives a small change in y).

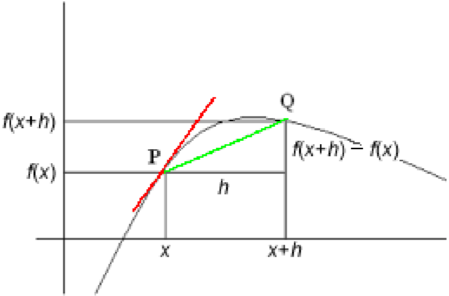
Differentiability tell us how smooth is that change

Geometric interpretation

If f is differentiable at x_0, the equation of the tangent line to the curve $y = f(x)$ at point $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

That is, the derivative of f at x_0 is the slope of this tangent line.



slope of the secant line through P and Q $= \frac{f(x+h)-f(x)}{h}$

slope of the tangent line through P $= f'(x)$

Economic example

In economics, the following crude approximation is used

$$f'(x) \approx f(x+1) - f(x).$$

Then the derivative is often called the marginal, since it is a good approximation of the marginal change of f at x. The less curved the graph of f, the better approximation.

Example: Consider a cost function $f(x)$ which measures the cost of manufacturing x units of an output. Then the marginal cost is the cost of making one more unit.

If the change in the amount is Δx instead of one unit, then one can use the approximation

$$f'(x)\Delta x \approx f(x+\Delta x) - f(x).$$

In differential notation

$$f'(x)dx \approx df.$$

Differentiation rules

Derivatives of elementary functions:

1. $f(x) = c, \quad f'(x) = 0$, where c is a constant.
2. $f(x) = x^n, \quad f'(x) = nx^{n-1}$, where n is a positive integer.
3. $f(x) = x^\alpha, \quad f'(x) = \alpha x^{\alpha-1}$, where $\alpha \in \mathbb{R}$ and $x > 0$.
4. $f(x) = \log(x), \quad f'(x) = \frac{1}{x}$, $x > 0$.
5. $f(x) = \sin(x), \quad f'(x) = \cos(x)$.
6. $f(x) = \cos(x), \quad f'(x) = -\sin(x)$.
7. $f(x) = e^x, \quad f'(x) = e^x$.
8. $f(x) = a^x, \quad f'(x) = a^x \log a$, $a > 0$.

Proof of 8: $(a^x)' = (e^{x\log(a)})' = a^x \log(a)$ (by the chain rule)

Example of 3: $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Differentiation rules

Theorem: If f and g are differentiable, then so are $f+g$, fg and f/g (the last one provided that $g(x) \neq 0$), and

- $(f+g)'(x) = f'(x) + g'(x)$,
- $(fg)'(x) = f'(x)g(x) + g'(x)f(x)$,   [Leibniz rule]
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$.

Differentiation rules

Chain rule:

Let f, g be two continuous functions. If f is differentiable at $x \in D_f$, and g is differentiable at $f(x) \in D_g$, then $h = g \circ f$ is differentiable at x and

$$h'(x) = g'(f(x))f'(x).$$

Example 1: The production cost of a firm is

$$C = f(x) = 4 + \log(x+1) + \sqrt{3x+1}.$$

By the chain rule

$$C' = f'(x) = \frac{1}{x+1} + \frac{3}{2\sqrt{3x+1}}.$$

If the production increases from 133 to 134, the production cost increases by $C'(133) \approx 0.08246$ units.

Example 2: $h(x) = u(x)^{v(x)}$ with $u(x) > 0$

We use **logarithmic differentiation**. Define

$$h(x) = u(x)^{v(x)}, \qquad u(x) > 0.$$

Take logs:

$$\ln h(x) = v(x) \ln u(x).$$

Differentiate both sides (product + chain rules):

$$\frac{h'(x)}{h(x)} = v'(x) \ln u(x) \; + \; v(x) \frac{u'(x)}{u(x)}.$$

Multiply by $h(x)$:

$$\boxed{h'(x) = u(x)^{v(x)} \left(v'(x) \ln u(x) + v(x) \frac{u'(x)}{u(x)} \right).}$$

Checks (special cases).

- If $v(x) \equiv c$ (constant): $h(x) = u(x)^c \Rightarrow h'(x) = c\, u(x)^{c-1} u'(x)$.
- If $u(x) \equiv a > 0$ (constant): $h(x) = a^{v(x)} \Rightarrow h'(x) = a^{v(x)} \ln(a)\, v'(x)$.

Some of the contributors

Differentiation rules

Theorem (derivative of the inverse): Let f be a strictly monotone continuous function differentiable at $x \in D_f$. Then the inverse f^{-1} is differentiable at $y = f(x)$ if and only if $f'(x) \neq 0$, and in this case

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

Proof : since $f^{-1}(f(x)) = x$, by the chain rule $(f^{-1})'(f(x))f'(x) = 1$

Example: $f(x) = e^x$, $f'(x) = e^x > 0$, therefore, $(\log)'(y) = \frac{1}{y}$ for all $y > 0$.

Higher-order derivatives: If f' is differentiable, its derivative is called the second derivative of f and denoted f''. Similarly, we can define higher-order derivatives, and $f^{(n)}$ is called the nth derivative of f. If $f^{(n)}$ is continuous, we say that f is n times continuously differentiable or C^n for short.

Limits

Theorem (L'Hôpital's rule): Let f, g be two C^1 functions in (a, b) with $g' \neq 0$ in (a, b). Assume that one of the next hypothesis is satisfied:

(a) $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$.

(b) $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \infty$ or $-\infty$.

Then,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

The proof uses the Mean value theorem (on slide 55)

Remark: x can be ∞ or $-\infty$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ can be ∞ or $-\infty$.

Example:

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} e^{\log(1+x)^{1/x}} = e^{\lim_{x\to 0} \frac{\log(1+x)}{x}} = e^{\lim_{x\to 0} \frac{1}{1+x}} = e$$

We have used l'Hôpital's rule in the third equality and the last property in page 28 in the second one.

Monotonicity

Theorem: Consider f differentiable in (a, b). Then

1. f is increasing on $[a, b] \iff f' \geq 0$ on (a, b).
2. f is decreasing on $[a, b] \iff f' \leq 0$ on (a, b).
3. f is constant on $[a, b] \iff f' = 0$ on (a, b).
4. If $f' > 0$ on (a, b), then f is strictly increasing on $[a, b]$.
5. If $f' < 0$ on (a, b), then f is strictly decreasing on $[a, b]$.

Observe that $f(x) = x^3$ is strictly increasing but $f'(0) = 0$, so the converse in 4. and 5. is not necessarily true.

Example: $f(x) = \frac{x^3}{3} + 2x^2 + 3x + 1$. We have

$$f'(x) = x^2 + 4x + 3 = (x + 1)(x + 3).$$

Therefore, f is strictly increasing on $(-\infty, -3]$ and $[-1, \infty)$, while f is strictly decreasing on $[-3, -1]$.

This implies that -3 is a local max and -1 is a local min (see next slide). However, there are no global optima since $\lim_{x \to \pm\infty} f(x) = \pm\infty$.

Optimal points: 1st order conditions

Let $f : D \to \mathbb{R}$. A point $x_0 \in D$ is called a local max (resp. local min) of f if there is an interval $(a, b) \subset D$ containing x_0 such that

$$f(x) \leq f(x_0), \qquad \text{for all } x \in (a, b)$$

(resp. $f(x) \geq f(x_0)$)

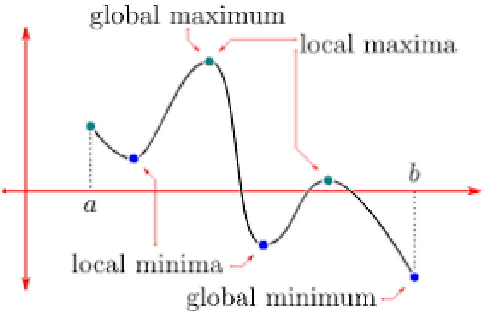
We say that x_0 is a global max (resp. global min) if

$$f(x) \leq f(x_0) \quad \text{for all } x \in D$$

(resp. $f(x) \geq f(x_0)$)

Boundary points are typically not considered as local max or min, we call a max or a min an optimum

Optimal points



Optimal points: 1st order conditions

Theorem: If f has a local optimum at $x \in (a, b)$ and f is differentiable at x, then $f'(x) = 0$.

Remark: The condition is only necessary, for example if $f(x) = x^3$, $f'(0) = 0$ but 0 is not a local max or min.

Points x such that $f'(x) = 0$ are called stationary points.

Theorem: Let f differentiable and $x \in (a, b)$ a stationary point. If there exists an interval (a^*, b^*) around x such that the function is strictly increasing (resp. decreasing) to the left of x and strictly decreasing (resp. increasing) to the right of x, then x is a local max (resp. min).

Economic example

Suppose you own a property whose market value will be $V(t)$ pesos t years from now. If the interest rate is R, this means that the present value is $P(t) = V(t)e^{-Rt}$. What is the optimal time t_0 to sell this property ? We maximize its present value with respect to t.

The first order condition is

$$P'(t) = (V(t)e^{-Rt})' = V'(t)e^{-Rt} - RV(t)e^{-Rt} = 0 \Rightarrow \frac{V'(t)}{V(t)} = R$$

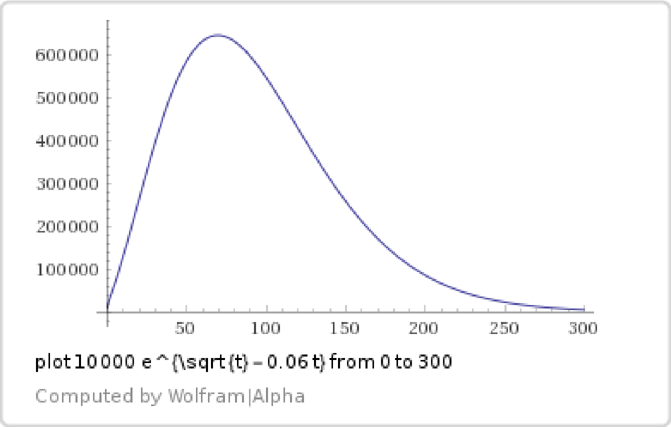
The optimal time to sell is when the rate of change of V equals the interest rate you can earn in the bank.

Example: If $V(t) = 10000e^{\sqrt{t}}$and $R = 6\%$ then the present value is

$$P(t) = 10000e^{\sqrt{t}-0.06t}.$$

The optimal time to sell is $t \approx 69.44$, which is the global max of P on $[0, \infty)$.

Plot of the present value $P(t)$



To check that a local optimum is global we need to look at the value of the function at the extreme points of the domain.
Here $P(0) = 10000$ and $\lim_{t\to\infty} P(t) = 0$.

Optimal points: 2nd order conditions

Theorem: Let $f : (a, b) \to \mathbb{R}$ a C^n function, and $x \in (a, b)$ a stationary point. If

$$f'(x) = f''(x) = \cdots = f^{(n-1)}(x) = 0, \quad \text{and} \quad f^{(n)}(x) \neq 0,$$

where n is even, (typically $n = 2$) then:

1. If $f^{(n)}(x) < 0$, the x is a local max.
2. If $f^{(n)}(x) > 0$, the x is a local min.

Example: Find the optimal points of $f(x) = \frac{\log^2 3x}{x}$, $x > 0$.

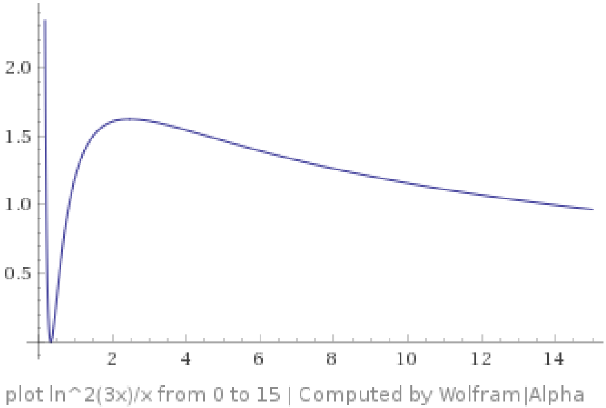
$$f'(x) = \frac{(2 - \log 3x)\log 3x}{x^2}$$

f' has two zeroes at $x_1 = 1/3$ and $x_2 = e^2/3$.

$$f''(x) = \frac{2(1 - 3\log 3x + \log^2 3x)}{x^3}$$

Since $f''(x_1) > 0$ and $f''(x_2) < 0$, x_1 is a local min and x_2 is a local max.

Plot of $f(x) = \frac{\log^2 3x}{x}, x > 0$



Since $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0} f(x) = \infty$ and $f(x_1) = 0$, $f(x_2) > 0$, x_1 is a global min but there a no global max.

Convexity and concavity

Theorem: Let f be a twice differentiable on (a, b). Then:

1. f is convex (concave) on $[a, b]$ if and only if $f''(x) \geq 0$ ($f''(x) \leq 0$) for all $x \in (a, b)$.
2. If $f''(x) > 0$ ($f''(x) < 0$) for all $x \in (a, b)$, then f is strictly convex (concave).

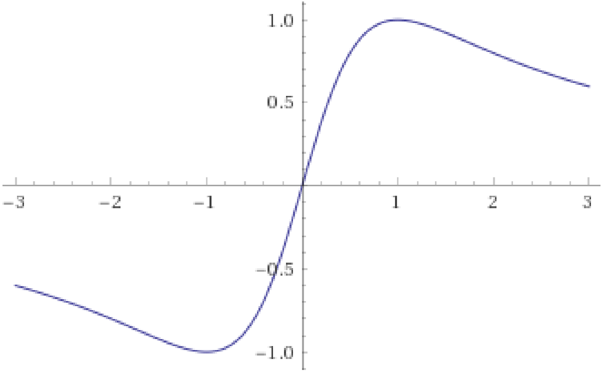
Example: $f(x) = \frac{2x}{x^2+1}$.

$$f'(x) = \frac{2(1-x^2)}{(x^2+1)^2} \qquad f''(x) = \frac{4(x^3-3x)}{(x^2+1)^3}.$$

Then, f is strictly convex on $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$ and strictly concave on $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$.

Theorem: If f and g are convex (or concave), then $f \circ g$ is convex (or concave).
Example: $f(x) = e^{x^2}$ is strictly convex on $[0, \infty)$.

Plot of $f(x) = \frac{2x}{x^2+1}$



plot 2x/(x^2+1) from -3 to 3 | Computed by Wolfram|Alpha

The points 0, $\sqrt{3}$ and $-\sqrt{3}$ are inflexion points. The points 1 and -1 are global max and min since $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

Convexity and concavity

Let f be a twice differentiable on (a,b). A point $x \in (a,b)$ is called an inflexion point if f changes at x from being concave to convex, or vice versa.

Theorem: Let $f : D_f \to \mathbb{R}$ C^n on (a,b). f has an inflexion point at $x \in (a,b)$ if and only if

$$f''(x) = \cdots = f^{(n-1)}(x) = 0, \quad \text{and} \quad f^{(n)}(x) \neq 0,$$

where n is odd.

(typically $n = 3$, that is, $f''(x) = 0$ and $f'''(x) \neq 0$)

Example: $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6 > 0$, so 0 is an inflexion point.

Mean-value theorem

Mean value theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad (\text{or } \; f(b) = f(a) + f'(c)(b - a)).$$

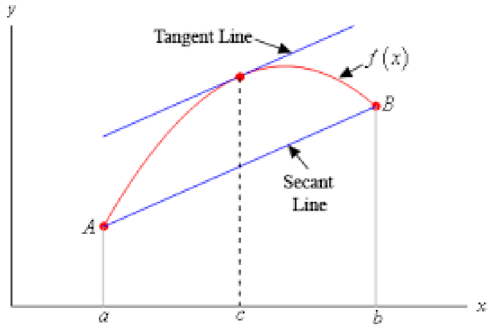
Proof: Set $h(x) = (f(b) - f(a))x - (b - a)f(x)$, and show that since $h(a) = h(b)$ it has a local max or min.

Geometric interpretation: $\frac{f(b)-f(a)}{b-a}$ is the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$. Thus, the mean value theorem says that there exists $c \in (a, b)$, such that the tangent line to f at c is parallel to the secant line.

Geometry of the mean-value theorem

Recall: ... there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad (\text{or } f(b) = f(a) + f'(c)(b - a)).$$



Taylor formula

Theorem (Taylor's formula of order n): Let f be $n+1$ times differentiable on (a,b), and let $x_0 \in (a,b)$ given. Then, for any $x \in (a,b)$, if $n=1$,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \text{error}.$$

If $n=2$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \text{error}.$$

If $n>2$,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \text{error}.$$

The error is small if x is close to x_0, therefore, this theorem says that differentiable functions may be locally approximated by a polynomial called the Taylor polynomial.

Taylor's error

The remainder term has the form

$$R_n^y(x) = \frac{f^{(n+1)}(y)}{(n+1)!}(x-x_0)^{n+1} \qquad \text{for some } y \text{ between } x_0 \text{ and } x.$$

If $f^{(n+1)}$ is bounded on (a,b) by a constant M, then

$$|R_n^y(x)| \le \frac{M}{(n+1)!}|x-x_0|^{n+1}.$$

Example: For $f(x) = \log(x)$ around $x_0 = 1$: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$. The degree-2 Taylor polynomial is $\log(x) \approx (x-1) - \frac{1}{2}(x-1)^2$. The error term is

$$R_2^y(x) = \frac{f^{(3)}(y)}{3!}(x-1)^3 = \frac{1}{3y^3}(x-1)^3,$$

which, for $x > 1$, is bounded by $\frac{1}{3}(x-1)^3$.