Lecture 2 : Calculus and Linear Algebra

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In many applications, a function f is given, and we are looking for F whose derivative is f.

Example: the marginal cost function C' is given (we know how cost changes according the production), and we want to find C.

The function F can be found by integration, which is the reverse process of differentiation.

Read Chapter 5 of Werner-Sotskov

Exercises: 5.1 (a)-(b), 5.2 (a)-(b), 5.3 (c) (Werner-Sotskov)

Indefinite integrals

A differentiable function F is called an antiderivative of a function f if $F'(x) = f(x)$, for all $x \in D_F = D_f$.

Theorem: Then, all antiderivatives of f are of the form $\tilde{F}(x) = F(x) + c$ where c is any constant.

The indefinite integral of f is defined as

$$\int f(x)dx = F(x) + c$$

Properties:

1. $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.
2. $\int cf(x)dx = c \int f(x)dx$.

Indefinite integrals

Some indefinite integrals:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \; n \neq -1$
2. $\int \frac{1}{x} dx = \log |x| + c$
3. $\int e^x dx = e^x + c.$
4. $\int \sin(x) dx = -\cos(x) + c$
5. $\int \cos(x) dx = \sin(x) + c$
6. $\int a^x dx = \frac{a^x}{\log a} + c, a > 0, a \neq 1$

Proof: differentiate the right hand side to obtain the expression inside the integral

Integration by substitution

Theorem: Assume that f has an antiderivative F, and let g continuously differentiable. Then setting $t = g(x)$, we obtain

$$\int f(g(x))g'(x)dx = \int f(t)dt = F(t) + c = F(g(x)) + c.$$

Examples:

1. $$\int (ax+b)^n dx = \frac{1}{a}\int t^n dt = \frac{t^{n+1}}{a(n+1)} + c = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

$(t = ax + b,\ dt = adx)$

2. $$\int \frac{e^x}{\sqrt[3]{1+e^x}}dx = \int t^{-1/3}dt = \frac{3}{2}t^{2/3} + c = \frac{3}{2}(1+e^x)^{2/3} + c$$

$(t = 1 + e^x,\ dt = e^x dx)$

Integration by parts

Theorem: Let u, v two differentiable functions. Then

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Proof: Apply the product rule to $(u(x)v(x))'$ and integrate.

Examples:

$$1. \int \log x dx = x \log x - \int dx = x(\log x - 1) + c$$

Apply the Theorem with $u = \log(x)$, $v' = 1$

$$2. \int \sin \sqrt{x} dx = 2 \int t \sin t \, dt = 2(-t \cos t + \int \cos t \, dt)$$
$$= 2(-t \cos t + \sin t) + c = 2(-\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x}) + c$$

First integrate by substitution $\sqrt{x} = t$, $2\sqrt{x}dt = dx$ then integrate by parts with $u = t$, $v' = \sin t$

The definite integral

The definite integral of $f : [a, b] \to \mathbb{R}_+$ continuous equals the area covered by the x-axis and the graph of f.

Notation: $\int_b^a f(x)dx$. Remark: If f is only bounded in $[a, b]$ with at most a finite number of discontinuities, the definite integral always exists.

Properties:

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.
3. If $c \in [a, b]$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
4. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.
5. If f, g are continuous in $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

The definite integral

Theorem: Let f continuous on $[a, b]$ with antiderivative F. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Moreover, the function $G(t) = \int_a^t f(x)dx$ is differentiable in $[a, b]$ and $G'(t) = f(t)$.

Example: The marginal cost of a firm producing one product is

$$C'(x) = 6 - \frac{60}{x+1}, \qquad x \in [0, 1000].$$

If the quantity produced changes form 300 to 400, the cost changes by

$$C(400) - C(300) = \int_{300}^{400} C'(x)dx = (6x - 60\log|x+1|)\Big|_{300}^{400}$$
$$\approx 582.79$$

Application of the integral: Proof of the number e

We can define the logarithm function as

$$\log x = \int_1^x \frac{1}{t} dt.$$

We then define the exponential function e^x as the unique inverse of $\log x$. In particular, $\log e = 1$, so $1 = \int_1^e \frac{1}{t} dt$.
We are going to show that

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

Let $t \in [1, 1 + \frac{1}{n}]$. Then

$$\int_1^{1+\frac{1}{n}} \frac{1}{1+\frac{1}{n}} dt \leq \int_1^{1+\frac{1}{n}} \frac{1}{t} dt \leq \int_1^{1+\frac{1}{n}} 1 dt.$$

Therefore

$$\frac{1}{n+1} \leq \log\left(1 + \frac{1}{n}\right) \leq \frac{1}{n}.$$

Application: Proof of the number e

Taking the exponential, we get

$$e^{\frac{1}{1+n}} \leq 1 + \frac{1}{n} \leq e^{\frac{1}{n}}.$$

Taking the $(n+1)th$ power of the left inequality and the nth power of the right, gives

$$\left(1 + \frac{1}{n}\right)^n \leq e \leq \left(1 + \frac{1}{n}\right)^{n+1}.$$

Divide the right inequality by $1 + \frac{1}{n}$ and combine with the left inequality to get

$$\frac{e}{1 + \frac{1}{n}} \leq \left(1 + \frac{1}{n}\right)^n \leq e.$$

Taking the limit as $n \to \infty$ we obtain the result.

Application of the integral: Taylor's formula $n = 2$

Applying 3 times the theorem in slide 8, we get

$$
\begin{aligned}
f(x) &= f(x_0) + \int_{x_0}^{x} f'(t_1)dt_1 \\
&= f(x_0) + \int_{x_0}^{x} f'(x_0)dt_1 + \int_{x_0}^{x}\int_{x_0}^{t_1} f''(t_2)dt_2dt_1 \\
&= f(x_0) + \int_{x_0}^{x} f'(x_0)dt_1 + \int_{x_0}^{x}\int_{x_0}^{t_1} f''(x_0)dt_2dt_1 + \int_{x_0}^{x}\int_{x_0}^{t_1}\int_{x_0}^{t_2} f'''(t_3)dt_3dt_2dt_1 \\
&= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \int_{x_0}^{x}\int_{x_0}^{t_1}\int_{x_0}^{t_2} f'''(t_3)dt_3dt_2dt_1
\end{aligned}
$$

Finally, using the intermediate value theorem, we get that there exists $y \in (x_0, x)$ such that

$$\int_{x_0}^{x}\int_{x_0}^{t_1}\int_{x_0}^{t_2} f'''(t_3)dt_3dt_2dt_1 = f'''(y)\frac{(x - x_0)^3}{3!}.$$

Chapter 5: Vectors

A vector is an ordered n-tuple

This is important in economics, since it can describe a bundle of commodities such that the ith value represents the quantity of the ith commodity.

Read Chapter 6 of Werner-Sotskov and Chapters 10 and 11 of Simon-Blume

Exercises: 6.2, 6.3, 6.4, 6.6, 6.7, 6.8 (Werner-Sotskov)

Definition

A vector $\mathbf{v} =$ is an ordered n-tuple of real numbers $v_1, \ldots, v_n$ called the coordinates, which points a direction between two n-dimensional points. Notation:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathbf{v}' = (v_1, \ldots, v_n).$$

The set of all real n-tuples is called the n-dimensional space $\mathbb{R}^n$:

$$\mathbb{R}^n = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| \, v_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\}$$

Special vectors: the zero vector $\mathbf{0}' = (0, 0, \ldots, 0)$, and the ith unit vector $\mathbf{e_i}' = (0, \ldots, 0, 1, 0, \ldots, 0)$, the 1 is at the ith component.

Operations on vectors

Sum of two vectors:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

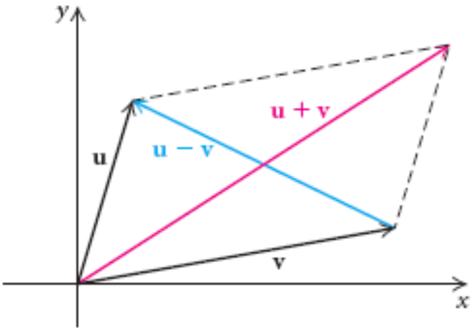
Multiplication of a vector by a scalar $\lambda \in \mathbb{R}$:

$$\lambda \mathbf{v} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}.$$

Difference of two vectors: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{1})\mathbf{v}$.

Theses properties satisfy the usual distributive, commutative and associative laws.

Sum and difference of two vectors



Scalar or inner product

The scalar or inner product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined as

$$\mathbf{u}'\mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

Properties:

1. Commutative: $\mathbf{u}'\mathbf{v} = \mathbf{v}'\mathbf{u}$.
2. Distributive: $\mathbf{u}'(\mathbf{v} + \mathbf{w}) = \mathbf{u}'\mathbf{v} + \mathbf{u}'\mathbf{w}$.
3. Not necessarily associative: $\mathbf{u}(\mathbf{v}'\mathbf{w}) \neq (\mathbf{u}'\mathbf{v})\mathbf{w}$.

Example: The total cost of production of a firm that produces the quantities $\mathbf{u}' = (30, 40, 10)$ of three products with cost of production $\mathbf{v}' = (20, 15, 40)$ is:

$$\mathbf{u}'\mathbf{v} = 20 \cdot 30 + 15 \cdot 40 + 40 \cdot 10 = 1600$$

Norm of a vector

The norm of a vector $\mathbf{v} \in \mathbb{R}^n$ is defined as

$$|\mathbf{v}| = \sqrt{\mathbf{v}'\mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

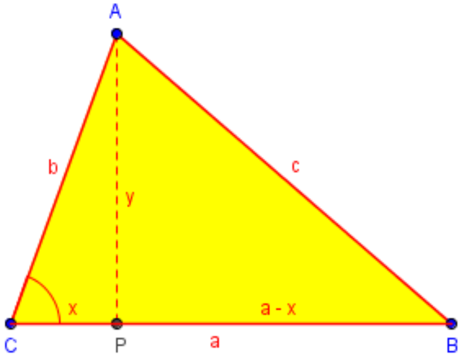
Geometric interpretation: The norm of a vector is its length (the distance to $\mathbf{0}$). The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $|\mathbf{u} - \mathbf{v}|$. This is the length of the vector connecting the terminal points of $\mathbf{u}$ and $\mathbf{v}$. If $|\mathbf{v}| = 1$, then $\mathbf{v}$ is called a unit vector.

Properties:

1. $|\mathbf{v}| \geq 0$ and $|\mathbf{v}| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $|\lambda \mathbf{v}| = |\lambda| \, |\mathbf{v}|$ for all $\lambda \in \mathbb{R}$.
3. $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}| \, |\mathbf{v}| \cos(\mathbf{u}, \mathbf{v})$. Proof: use the Law of Cosines
4. $\mathbf{u}'\mathbf{v} = |\mathbf{u}| \, |\mathbf{v}| \cos(\mathbf{u}, \mathbf{v})$ Proof: use 3. and $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u}'\mathbf{v}$.
5. Cauchy-Schwarz inequality: $\mathbf{u}'\mathbf{v} \leq |\mathbf{u}| \, |\mathbf{v}|$ Proof: use 4.
6. $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$. Proof: use 5. and $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u}'\mathbf{v}$

The Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$



Proof: by Pythagoras Theorem $x^2 + y^2 = b^2$ and $(a-x)^2 + y^2 = c^2$. Subtract both equations to eliminate y^2 to get $c^2 = a^2 + b^2 - 2ab(x/b)$.

Use the definition of the cosine to conclude.

Orthogonal vectors

Example: Let $\mathbf{u}' = (2, -1, 3)$, $\mathbf{v}' = (5, -4, -1)$. Then

$$\mathbf{u}'\mathbf{v} = 11 \leq |\mathbf{u}|\,|\mathbf{v}| = \sqrt{14}\sqrt{42} \approx 24.24.$$

What is the angle formed between $\mathbf{u}$ and $\mathbf{v}$?

$$\cos(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u}'\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{11}{\sqrt{14}\sqrt{42}} \approx 0.4537$$

Therefore the angle is approximately 63°.

Definition: Two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ are called orthogonal if the angle they form is 90°.

Since $\cos(90^\circ) = 0$, by Properties 4. and 5. we get that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u}'\mathbf{v} = 0$ and if and only if

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2.$$

Another proof of Cauchy-Schwarz inequality

We want to prove that $\mathbf{u}'\mathbf{v} \leq |\mathbf{u}|\,|\mathbf{v}|$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

If $|\mathbf{u}|$ and/or $|\mathbf{v}|$ are 0 it is trivial, so we assume they are both non-zero.

Consider the unit vectors $\mathbf{w_1} = \frac{\mathbf{u}}{|\mathbf{u}|}$ and $\mathbf{w_2} = \frac{\mathbf{v}}{|\mathbf{v}|}$. Then it suffices to show that

$$\mathbf{w_1}'\mathbf{w_2} \leq 1.$$

Since $|\mathbf{w_1} - \mathbf{w}_2|^2 \geq 0$, developing this inequality we conclude the desired proof.

Observe that if both vectors are non-zero, then we have equality if and only if $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \neq 0$, that is, the vectors are co-linear.

Two co-linear vectors are also called linearly dependent. This notion can be generalized to a family of m vectors in $\mathbb{R}^n$ as follows.

Linear dependence and independence

A linear combination of m vectors in $\mathbb{R}^n$ $\mathbf{v_1}, \ldots, \mathbf{v_m}$ is the vector in $\mathbb{R}^n$ defined as

$$\mathbf{u} = \lambda_1 \mathbf{v_1} + \cdots + \lambda_m \mathbf{v_m}.$$

where $\lambda_1, ..., \lambda_m$ are real numbers.

If $\lambda_i \geq 0$ for all $i = 1, \ldots, m$ and $\lambda_1 + \cdots + \lambda_m = 1$, then it is called convex combination.

m vectors in $\mathbb{R}^n$ $\mathbf{v_1}, \ldots, \mathbf{v_m}$ are said to be linearly dependent if there exist real numbers $\lambda_1, ..., \lambda_m$ not all zero such that

$$\lambda_1 \mathbf{v_1} + \cdots + \lambda_m \mathbf{v_m} = \mathbf{0}. \tag{1}$$

If this equation has the only solution $\lambda_1 = \cdots = \lambda_m = 0$, then $\mathbf{v_1}, \ldots, \mathbf{v_m}$ are said to be linearly independent.

Observe that (1) is a linear system of n equations and m unknowns (Chapter 8).

Linear dependence and independence

Remark: $\mathbf{v_1}, \ldots, \mathbf{v_m}$ are linearly dependent if and only if at least one of them equals a linear combination of all others.

Proof: if $\lambda_k \neq 0$, then $\mathbf{v_k}$ writes as a linear combination of all others.

Examples:

1. The vectors $\mathbf{v_1}' = (3, 1)$ and $\mathbf{v_2}' = (-9, -3)$ are linearly dependent since $\mathbf{v_2} = -3\mathbf{v_1}$. (the line through the vectors is $y = x/3$)
2. The vectors $\mathbf{v_1}' = (3, 1)$ and $\mathbf{v_2}' = (-1, 2)$ are linearly independent since the system $\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} = \mathbf{0}$ has the unique solution $\lambda_1 = \lambda_2 = 0$. (2 vectors are l.i. if and only if they don't lie in the same line and 3 vectors are l.i. if and only if they don't lie in the same plane)
3. The set of unit vectors $\mathbf{e_1}, \ldots, \mathbf{e_n}$ in $\mathbb{R}^n$ forms a set of linearly independent vectors.

Vector spaces

Any set of n linearly independent vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ forms a basis of $\mathbb{R}^n$ since any vector $\mathbf{u}$ in $\mathbb{R}^n$ writes as a linear combination of these vectors:

$$\mathbf{u} = \lambda_1 \mathbf{v_1} + \cdots + \lambda_n \mathbf{v_n}.$$

We say that they span $\mathbb{R}^n$.

The set of unit vectors $\mathbf{e_1}, \ldots, \mathbf{e_n}$ in $\mathbb{R}^n$ is called the standard basis of $\mathbb{R}^n$ since the scalars of the linear combination are the coordinates of $\mathbf{u}$.

Vector spaces

Theorem: If $\mathbf{v_1}, \ldots, \mathbf{v_n}$ forms a basis of $\mathbb{R}^n$, then every vector in $\mathbb{R}^n$ can be uniquely written as a linear combination of the basis.

Proof: Assume one vector has two different linear combination ans show that they are equal using linear independence.

Theorem: Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be a basis of $\mathbb{R}^n$, and let a vector $\mathbf{u_k}$ in $\mathbb{R}^n$ that writes as

$$\mathbf{u_k} = \lambda_1 \mathbf{v_1} + \cdots + \lambda_k \mathbf{v_k} + \cdots + \lambda_n \mathbf{v_n},$$

with $\lambda_k \neq 0$. The $\mathbf{v_1}, \ldots, \mathbf{v_{k-1}}, \mathbf{u_k}, \mathbf{v_{k+1}}, \ldots, \mathbf{v_n}$ is also a basis of $\mathbb{R}^n$. (since $\lambda_k \neq 0$ it will span the same space)

Therefore, $\mathbb{R}^n$ has infinitely many basis.

Vector spaces

If we consider the linear combinations of a set (that is, the span) of $k < n$ linearly independent vectors $\mathbf{v_1}, \ldots, \mathbf{v_k}$ of $\mathbb{R}^n$, we obtain a smaller space than $\mathbb{R}^n$ called a subspace of $\mathbb{R}^n$. We say they form a basis of this subspace of dimension k.

Therefore, a basis of a subspace is a set of linearly independent vectors that spans the subspace.

If the vectors $\mathbf{v_1}, \ldots, \mathbf{v_k}$ are linearly dependent, then they span a subspace of dimension $< k$. The dimension is the maximum number of linearly independent vectors in the family.

Vector spaces

Examples:

1. 2 linearly independent vectors in $\mathbb{R}^3$ span a plane (subspace of dimension 2).
2. 1 vector in $\mathbb{R}^2$ spans a line (subspace of dimension 1).
3. the set of solutions to an homogeneous system $A\mathbf{x} = \mathbf{0}$ where A is $m \times n$ matrix is a subspace of $\mathbb{R}^n$ (Chapter 7).

Examples

1. The vectors $\mathbf{v_1}' = (1, 0, 0)$, $\mathbf{v_2}' = (1, 2, 0)$ and $\mathbf{v_3}' = (1, 2, 3)$ form a basis of $\mathbb{R}^3$. The vector $\mathbf{u} = (3, 2, 1)$ writes as
$$(3, 2, 1) = 2\mathbf{v_1}' + \frac{2}{3}\mathbf{v_2}' + \frac{1}{3}\mathbf{v_3}'.$$
Therefore, $\{\mathbf{u}, \mathbf{v_2}, \mathbf{v_3}\}$, $\{\mathbf{v_1}, \mathbf{u}, \mathbf{v_3}\}$ and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{u}\}$ are also bases of $\mathbb{R}^3$.
2. The vectors $\mathbf{v_1}' = (2, 1, 0)$, $\mathbf{v_2}' = (1, -1, 2)$ and $\mathbf{v_3}' = (0, 3, -4)$ span a subspace of dimension 2 since they are linearly dependent and none is a multiple of the other. In fact, $\mathbf{v_1} - 2\mathbf{v_2} = \mathbf{v_3}$.
3. A basis of the *xy* plane in $\mathbb{R}^3$ is $\mathbf{v_1}' = (1, 2, 0)$, $\mathbf{v_2}' = (1, 3, 0)$.
4. The dimension of the subspace spanned by the vectors $\mathbf{v_1}' = (0, 1, 0, 1)$, $\mathbf{v_2}' = (1, 1, 0, 1)$, $\mathbf{v_3}' = (1, 0, 0, 0)$ and $\mathbf{v_4}' = (0, -1, 0, 1)$ is 3. A basis of this subspace is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}$ but not $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. In fact, $\mathbf{v_1} + \mathbf{v_3} = \mathbf{v_2}$, thus $\mathbf{v_4}$ cannot be replaced.

Chapter 6: Matrices and determinants

Matrices replace the use of tables that relates two different economic units.

The operation between matrices such as addition and multiplication simplifies the computations between the elements of different tables.

For example, to calculate how many units of raw material are required for the final production of a firm if some intermediate products are required.

Read Chapter 7 of Werner-Sotskov and Chapters 8 and 9 of Simon-Blume

Determinants are useful to compute the inverse of a matrix, to solve linear equations (Chapter 7) or to find eigenvalues (Chapter 9).

Exercises: 7.6, 7.9(b),(c),(d), 7.12, 7.14(a), 7.16, 7.18 (Werner-Sotskov)

Matrices

An $m \times n$ matrix is rectangular array of elements a_{ij} of the form

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

m is the number of rows, and n is the number of columns and we denote by $a_{i,j}$ the entry in row i and column j.

If $m = n$ it is called square matrix.

Matrices

If A is a $m \times n$ matrix with entries $a_{i,j}$ then its transpose A' is the $n \times m$ matrix with entries a_{ji}.

Example:

$$A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 7 & -1 & 0 & 4 \end{pmatrix} \qquad A' = \begin{pmatrix} 2 & 7 \\ 3 & -1 \\ 4 & 0 \\ 1 & 4 \end{pmatrix}$$

A square matrix is symmetric if $A' = A$ and antisymmetric if $A' = -A$.

Matrices

A $n \times n$ diagonal matrix and the Identity matrix are defined respectively as

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

A $n \times n$ is upper triangular matrix and lower triangular matrix are defined respectively as

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \qquad L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \dots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}.$$

The $m \times n$ zero matrix has all entries equal to zero.

Matrix operations

Addition of matrices: If $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices, then the matrix $A + B = (a_{ij} + b_{ij})$ is also $m \times n$.

Multiplication by a scalar: If $A = (a_{ij})$ is a $m \times n$ matrix and $\lambda \in \mathbb{R}$, then the matrix $\lambda A = (\lambda a_{ij})$ is also $m \times n$.

Definition: $-A = (-1)A$ and $A - B = A + (-B)$.

The addition and scalar multiplication satisfy the usual commutative, associative and distributive laws.

Matrix operations

Product: Let $A = (a_{ij})$ be a $m \times p$ matrix and $B = (b_{ij})$ a $p \times n$ matrix. Then the product $AB = (c_{ij})$ is the $m \times n$ matrix defined as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}.$$

Example:

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 7 & -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ 3 & -1 \\ 4 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 30 & 15 \\ 15 & 66 \end{pmatrix}$$

Matrix operations

Properties:

1. associative: $(AB)C = A(BC)$
2. distributive: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
3. Not commutative: $AB \neq BA$!
4. If A is an $n \times n$ matrix, $AI = IA = A$.
5. $(A')' = A$, $(A + B)' = A' + B'$, $(\lambda A)' = \lambda A'$, and
$$(AB)' = B'A'.$$

Remark: the matrix AA' is always square and symmetric.

Proof: $(AA')' = (A')'A' = AA'$.

Orthogonal matrices

A $n \times n$ matrix A is called orthogonal if $AA' = I$.

Remark: If A is orthogonal, the row (and column) vectors are pairwise orthogonal and unit vectors.

Proof: The diagonal terms of AA' are the square of the norm of the row and column vectors, and the non-diagonal terms are the scalar products.

Example:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix operations: elementary operations

The following columns (or row) operations in a matrix A are called elementary operations of type 1, 2 or 3 between columns (or rows):

1. Interchange two columns (or rows) of A.
2. Multiply all elements of a column (or row) of A by a scalar $\lambda \neq 0$.
3. Add to a column (or row) the multiple of another column (or row).

Determinants

If A is an $n \times n$ matrix, we denote by A_{ij} the submatrix obtained from A by deleting row i and column j. Then A_{ij} is a square matrix of order $(n-1) \times (n-1)$.

The determinant of an $n \times n$ matrix A is defined recursively as

$$\det(A) = |A| = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} |A_{1j}|.$$

For $n = 1$ we set $|A| = a_{11}$.

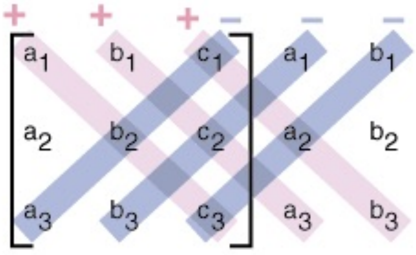
For $n = 2$, $|A| = a_{11}a_{22} - a_{12}a_{21}$.

For $n = 3$,

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{21} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} \\ &\quad - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned}$$

Determinants

A trick to remember the formula for $n = 3$ is:



$$\det A = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$

Determinants

Theorem (cofactor expansion of a determinant): If A is an $n \times n$ matrix the determinant can be computed expanding by any row i:

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|.$$

or any column j:

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|.$$

The determinants of the submatrices $|A_{ij}|$ are called minors, and the numbers $(-1)^{i+j}|A_{ij}|$ are called cofactors.

Example: Expanding by the second column:

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 0 & 2 \\ -1 & -4 & 2 \end{vmatrix} = (-1)^3 3 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + (-1)^5 (-4) \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = -16$$

Determinants

Properties of determinants:

1. det(A)=det(A').
2. If A is a lower or upper triangular matrix, then the determinant equals the product of the diagonal terms.
3. An elementary operation of type 1 changes the sign of the determinant.
4. An elementary operation of type 2 multiplies the determinant by λ.
5. An elementary operation of type 3 does not change the value of the determinant.
6. det(AB)= det(A)det(B).
7. The determinant of A is zero (singular matrix): if two rows or columns of A are equal, or if all elements of a row or columns are zero, or if a row or column is the sum of multiples of other rows or columns.

Using Properties 2. and 5., one can compute the determinant of any matrix by Gauss elimination.

Example:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -6 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & 3 \end{vmatrix} = -18$$

In the first step we have replaced r_2 by $r_2 - 3r_1$ and in the second we have replaced r_3 by $r_3 - r_1$.

Determinants

Let A be a $n \times n$ matrix, $\mathbf{x}$ a vector (variable) in $\mathbb{R}^n$ and $\mathbf{b}$ a vector (given) in $\mathbb{R}^n$. Then the equation

$$A\mathbf{x} = \mathbf{b}$$

is a linear system of n equations and n variables.

Assume that A is a non-singular matrix ($|A| \neq 0$), and let $A_j(\mathbf{b})$ the matrix obtained replacing column j of A by $\mathbf{b}$.

Cramer's rule: The unique solution to this system is given by

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|}, \dots, x_n = \frac{|A_n(\mathbf{b})|}{|A|},$$

Example:

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 0 & 2 \\ -1 & -4 & 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$$

Solution: $\mathbf{x}' = (\frac{75}{8}, -\frac{63}{16}, -\frac{19}{16})$.

Linear mappings

A mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is called linear if for all $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$A(\mathbf{x_1} + \mathbf{x_2}) = A(\mathbf{x_1}) + A(\mathbf{x_2}) \quad \text{and} \quad A(\lambda \mathbf{x}) = \lambda A(\mathbf{x}).$$

If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then there exists an $m \times n$ matrix A such that

$$A(\mathbf{x}) = A\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Examples of linear mappings

1. Consider the linear transformation of the plane corresponding to the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

   We have

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

   So the mapping corresponds to the reflexion across the y-axis.

2. The matrix of the linear transformation that rotates 45^o counterclockwise any planar vector is

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

   Observe that the first and second columns of A are, respectively,

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The inverse matrix

We say that a square matrix A is invertible if there exists a matrix A^{-1} called the inverse of A such that

$$AA^{-1} = A^{-1}A = I.$$

Properties:

1. $(A^{-1})^{-1} = A$.
2. $(AB)^{-1} = B^{-1}A^{-1}$.
3. $(A')^{-1} = (A^{-1})'$.
4. $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$ if $\lambda \neq 0$.
5. $|A^{-1}| = \frac{1}{|A|}$.

The inverse matrix

Theorem: A square matrix is invertible if and only if A is non-singular (that is, $|A| \neq 0$).

Theorem: If A is a square and non-singular matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

The inverse matrix

Theorem: If A is a square and non-singular matrix, then

$$A^{-1} = \frac{1}{|A|}((-1)^{i+j}|A_{ij}|)',$$

where $(-1)^{i+j}|A_{ij}|$ is the matrix of cofactors of A.

For $n = 2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

For $n > 2$ this formula is sometimes computationally inefficient.

In practice it is better to use elementary row operations Starting form the $n \times 2n$ matrix $(A \mid I)$, we proceed by elementary row operations until we arrive at $(I \mid A^{-1})$. Here only operations between ROWS are allowed (as when solving a system of equations).

Input-output model

n firms produce one good each.

a_{ij}=units of good i needed to produce 1 unit of good j

$A = (a_{ij})$ is called technology or input-output matrix.

$\mathbf{x} \in \mathbb{R}^n$ is the total amount of goods produced and $\mathbf{y} \in \mathbb{R}^n$ is the costumer demand vectors of the n goods.

$A\mathbf{x}$ represents the internal demand vector for each of the goods (units of each good to produce output of another good)

The supply and demand law tells us that the output $\mathbf{x}$ should be equal to the demand $A\mathbf{x} + \mathbf{y}$. We obtain the Input-output model:

$$\mathbf{x} = A\mathbf{x} + \mathbf{y} \quad \Leftrightarrow \quad (I - A)\mathbf{x} = \mathbf{y} \quad \Leftrightarrow \quad \mathbf{x} = (I - A)^{-1}\mathbf{y},$$

if $I - A$ is invertible!

Input-output model

Example:

Wood and Paper (US, 1998)

The following is the input-output chart for two sectors of the US economy. Create the technology matrix (the input-output matrix)

To	Wood	Paper
From Wood	36,000	7000
Paper	100	17,000
Total Output	120,000	120,000

Solution:

$$A = \begin{pmatrix} 0.3 & 0.0583 \\ 0.00083 & 0.14167 \end{pmatrix}$$

Input-output model

A family of examples of A is :

Theorem: If A is a square triangular matrix with all the elements of the diagonal equal to zero, then $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{n-1}.$$

Proof: check that $(I - A)(I - A)^{-1} = I$ and $(I - A)^{-1}(I - A) = I$. Use the fact that $A^n = 0$.

Example:

$$A = \begin{pmatrix} 0 & 3 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$$

Solution: $\mathbf{x}' = (66, 15, 4)$.