Lecture 1 : Calculus

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Mathematics Brush-up

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Goal of the course

Review core tools of calculus that power modern economics, finance, and data-driven policy.

These tools are crucial in courses on micro/macro, econometrics, finance, and machine learning.

Warm up before the Master starts: sharpen intuition, not just technique.

Main references:

1. *Mathematics for Economists* by C. P. Simon and L. Blume
2. *Mathematics of Economics and Business* by F. Werner and Y. N. Sotskov

Cultivate a Critical Mindset

Technical knowledge requires effort.

- **If a step feels “magical”, stop.** Ask *why* it works, not just *how*.
- **Challenge every answer — especially given by LLMs.**
  Large-language models can sound confident yet be wrong; verify with first principles or a textbook.
- **Use multiple lenses.**
  Check special cases, sketch a graph, test numerically, or search for a counterexample.

Index

Sequences

A sequence is an ordered list of real numbers $(a_1, a_2, \ldots)$ indexed by $n \in \mathbb{N}$.

Notation: $\{a_n\}_{n=1}^{\infty}$ where each $a_n \in \mathbb{R}$.

Define by a formula or a recursion.

Examples:

1. $a_n = \dfrac{n-1}{2n+1}$ $(a_1 = 0, a_2 = \frac{1}{5}, a_3 = \frac{2}{7}, \ldots)$
2. $a_1 = 2,\ a_{n+1} = 2a_n - 1$ $(2, 3, 5, 9, \ldots)$
3. $a_1 = 1, a_2 = 1,\ a_{n+1} = a_n + a_{n-1}$ (Fibonacci)
4. Arithmetic: $a_n = a_1 + (n-1)d$
5. Geometric: $a_n = a_1 r^{n-1}$

Why sequences show up in modern econ/data

Where they appear:

- **User metrics:** daily active users (DAU) over days n.
- **Inflation tracking:** monthly CPI/CPIH index a_n.
- **Training loss:** $a_n =$ loss after n gradient steps (decreasing sequence).
- **Carbon budgets:** remaining allowance after n years.

Read Ch. 2 Werner-Sotskov. Exercises: 2.4, 2.5, 2.7 (Werner-Sotskov)

Small motivating examples

1. **EAR vs APR in FinTech.** Let P be the initial investment. If a platform advertises an *annual percentage rate* (APR) R but compounds m times/year, then after n years:

$$a_n = P\left(1 + \frac{R}{m}\right)^{mn}.$$

The one-year *effective annual rate* (EAR) is:

$$\left(1 + \frac{R}{m}\right)^{m} - 1.$$

(APR is the quoted yearly rate without compounding; APY/EAR accounts for compounding.)

2. **Creator economy revenue.** If the number of views per video is v_n and the cost-per-thousand views (CPM) is c, total revenue over n videos is:

$$s_n = \sum_{k=1}^{n} c\, v_k.$$

Properties of a sequence

A sequence is increasing if $a_{n+1} \geq a_n$ (strictly if $>$).

Decreasing if $a_{n+1} \leq a_n$ (strictly if $<$).

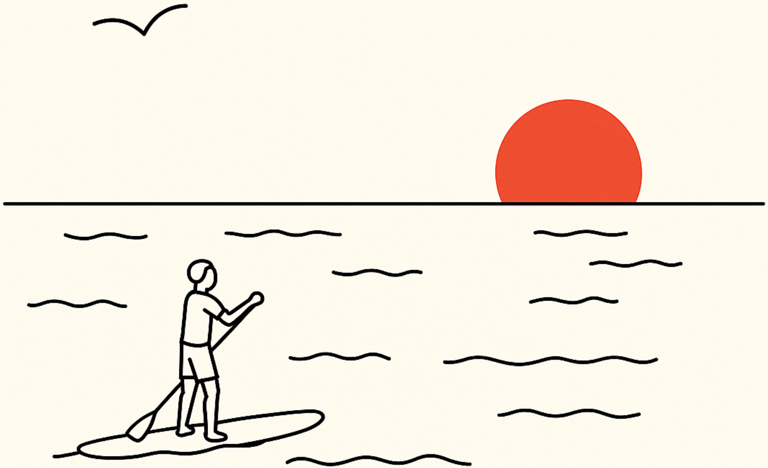
Monotone if increasing or decreasing.

Bounded if $\exists C > 0$ s.t. $|a_n| \leq C$ for all n.

Examples:

1. $a_n = 2(n-1)^2 - n$ is strictly increasing (since $a_{n+1} - a_n = 4n - 3 > 0$)
2. $a_n = 1/n^2$ is bounded ($0 < a_n \leq 1$)
3. **Training loss:** if a learning rate is sensible, $a_n =$loss often decreases and is bounded below by 0 $\Rightarrow$ convergent in many practical cases.

Can a strictly increasing sequence be bounded?



Limit of a sequence

A real number a is the limit of $\{a_n\}$ if as $n \to \infty$, a_n gets arbitrarily close to a. Notation: $\lim_{n\to\infty} a_n = a$.

A sequence is convergent if it has a limit; otherwise it diverges.

Examples

1. $a_n = \frac{1}{n^2} \to 0$
2. $a_n = (-1)^n$ has no limit
3. $a_n = n^2 \to \infty$
4. $a_n = r^n \to 0$ if $|r| < 1$; diverges if $r > 1$; no limit if $r \leq -1$.

Epsilon-definition

Definition: $a = \lim_{n\to\infty} a_n$

$$\forall \epsilon > 0 \quad \exists N_\epsilon \quad \text{s.t.} \;\; \forall n > N_\epsilon : \qquad |a_n - a| < \epsilon.$$

Example: $a_n = \dfrac{1}{n} \to 0$. For any $\epsilon > 0$, take $N_\epsilon > \frac{1}{\epsilon}$.

Checks:

- $\lim a_n = a$ if and only if $\lim |a_n - a| = 0$.
- If $a_n = a$ for all n, then $\lim a_n = a$.

If formal definition hard to work with, try to squeezing with easier bounds

If $0 < \frac{1}{2n^2-1} \leq \frac{1}{n}$ for $n \geq 1$, then $\lim \frac{1}{2n^2-1} = 0$.

Limits that matter in practice

Classics

1. $\left(1+\frac{1}{n}\right)^n \to e$
2. $\left(1+\frac{x}{n}\right)^n \to e^x$ for any real x

Continuous compounding & APY: if APR is R, then

$$\lim_{m\to\infty}\left(1+\frac{R}{m}\right)^m = e^R, \quad \text{EAR} = e^R - 1.$$

Example: APR $= 8\% \Rightarrow$ EAR $\approx 8.329\%$.

Monotone convergence and algebra of limits

Theorem

Every bounded monotone sequence is convergent.

(Just bounded or just monotone is not enough in general)

Example: the sequence $a_1 = 1,\ a_{n+1} = \sqrt{3a_n}$ is bounded by 3 (proof by induction) and strictly increasing (since $\frac{a_{n+1}}{a_n} > 1$), thus convergent.

Algebra of limits

If $\lim a_n = a$ and $\lim b_n = b$, then

$$\lim(a_n + b_n) = a + b, \quad \lim(a_n b_n) = ab, \quad \lim \frac{a_n}{b_n} = \frac{a}{b} \text{ if } b_n, b \neq 0.$$

Partial sums and series

The nth partial sum s_n of $\{a_n\}$ is

$$s_n = \sum_{k=1}^{n} a_k.$$

Examples

1. Arithmetic sequence $a_n = a_1 + (n-1)d$: $\quad s_n = na_1 + \frac{n(n-1)}{2}d$
2. Geometric sequence $a_n = a_1 r^{n-1}$, $r \neq 1$: $\quad s_n = a_1 \frac{1-r^n}{1-r}$

Use-cases:

- Let *margin* be the profit earned per customer per period and p be the customer retention rate. The total profit from a single customer over n periods is: $s_n = \sum_{k=1}^{n} \text{margin} \cdot p^k$.
- Let E_k be the emissions produced in period k (e.g., in tonnes of CO_2). Then the total emissions over n periods are: $s_n = \sum_{k=1}^{n} E_k$, which can be compared to a fixed carbon budget.

Series and convergence

A series is the limit (if it exists) of partial sums:

$$\sum_{k=1}^{\infty} a_k \ := \ \lim_{n\to\infty} s_n .$$

Geometric series: converges if and only if $|r| < 1$ with

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}.$$

Customer Lifetime Values: Consider the example on the previous slide. With margin m and monthly retention $p < 1$, discounted at monthly rate d,

$$\text{CLV} \ = \sum_{k=1}^{\infty} \frac{m\, p^k}{(1+d)^k} \ = \ \frac{m\,\frac{p}{1+d}}{1-\frac{p}{1+d}} \ = \ \frac{mp}{1+d-p}.$$

Chapter 2: Functions of one variable

A one-variable function assigns a unique $y \in \mathbb{R}$ to each $x \in \mathbb{R}$: $y = f(x)$.

x: independent (exogenous) variable; y: dependent (endogenous) variable.

Examples in econ/data: cost, revenue, demand, utility, probability of purchase vs. price, click-through rate vs. ad spend, etc.

Read Ch. 3 Werner-Sotskov; Simon-Blume 2.1-2.2, 5.1-5.3.

Graph and domain

Definition

The graph of f is $\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$.

The domain D is the set of x where $f(x)$ is defined.

Notation: $f : D \to \mathbb{R}$.

Example: $f(x) = \frac{1}{x}$ has $D = \mathbb{R} \setminus \{0\}$.

Sometimes we restrict the domain (e.g., prices $x \geq 0$).

Properties of functions

A function $f : D \to \mathbb{R}$ is increasing if $f(x_1) \leq f(x_2)$ for any $x_1 < x_2$ in D (strict if $<$).

Decreasing: reverse inequality.

Bounded: $\exists C > 0$ s.t. $|f(x)| \leq C$ for all $x \in D$.

Examples:

- **Price-demand curve** often decreasing on realistic intervals.
- **Learning curve** (error vs. epochs) typically decreasing and bounded below by 0.

Convex/concave functions

For $x_1 < x_2$ and $t \in [0, 1]$, the convex combination is $tx_1 + (1 - t)x_2$.

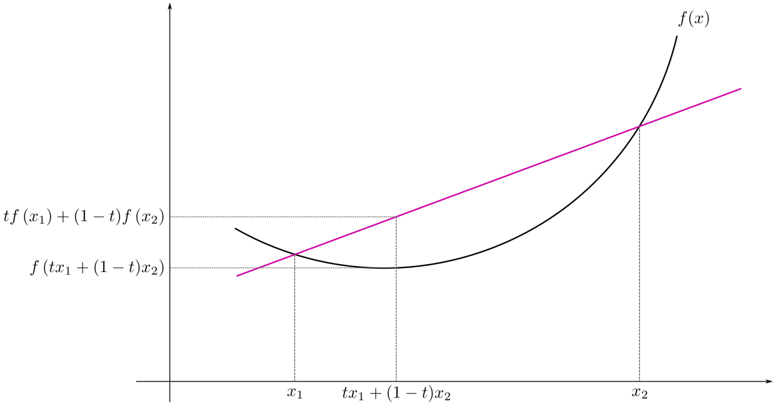
A set $D \subset \mathbb{R}$ is convex if it contains convex combinations.

A function $f : D \to \mathbb{R}$ on convex D is convex if

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$$

(and concave if $\geq$). Strict versions replace $\leq$ by $<$ for $t \in (0, 1)$ and $x_1 \neq x_2$.

Convex function



Quasi-convex/concave functions

f is quasi-convex if

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Quasi-concave if $\geq \min\{f(x_1), f(x_2)\}$.

Facts: convex $\Rightarrow$ quasi-convex; concave $\Rightarrow$ quasi-concave.

Any strictly increasing function is both quasi-convex and quasi-concave.

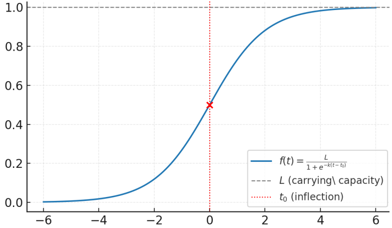
Economic examples: Indifference curves are typically quasi-convex; profit in price can be quasi-concave on feasible ranges.

Economic example: logistic growth (adoption curves)

The logistic function

$$f(t) = \frac{L}{1 + e^{-k(t-t_0)}}$$

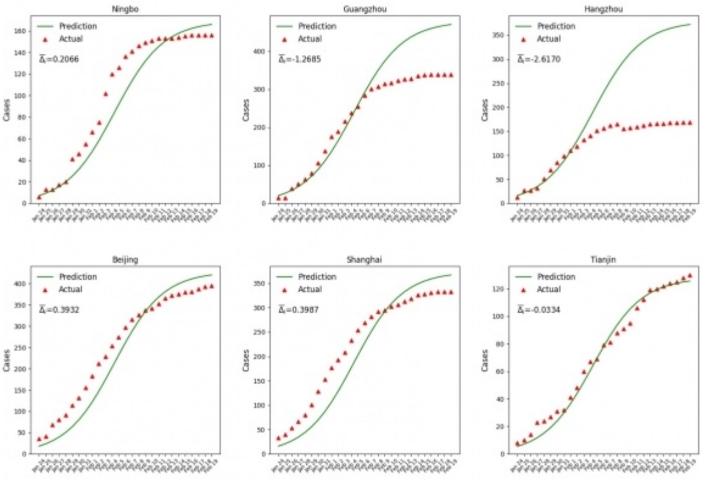
models technology/product adoption or viral spread. Early exponential growth, mid-phase inflection at t_0, saturation near L (market size).



Interpretation: growth accelerates, then slows as saturation/frictions (capacity, regulation, budgets) bind. Same shape often appears in learning curves (accuracy vs. data).

Epidemics and beyond

Logistic curves fit epidemic growth, but also market penetration, platform adoption, content diffusion. Check the following epidemic curves over time of the COVID-19 in China



100
80
Model 2033
Model 2025
≈70%

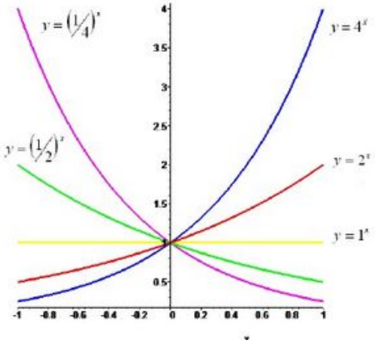

100
80
Model 2033
Model 2025
≈70%

Examples of functions

1. Monomials: $f(x) = ax^k$
2. Polynomials: sums of monomials
3. Rational: ratios of polynomials
4. Exponential: $f(x) = a^x$
5. Trig: $\sin x$, $\cos x$, ...
6. Linear: $f(x) = mx + b$ (slope m, intercept b; also $y - f(x_0) = m(x - x_0)$)
7. Natural exponential: $f(x) = e^x$ ($e^x = \lim_n (1 + \frac{x}{n})^n$)
8. Natural log: $f(x) = \log x$ ($e^{\log x} = x$, $\log(e^x) = x$)

Econ view: demand curves (often decreasing), cost curves (often convex), utility (often concave), isoelastic forms, log-sum-exp in discrete choice.

$f(x) = 1^x$	Horizontal line with y-intercept 1
$f(x) = a^x, \quad 0 < a < 1$	Exponential decay
$f(x) = a^x, \quad a > 1$	Exponential growth



Differentiation measures sensitivity: how $f(x)$ changes with small changes in x.

Examples: price $\to$ demand sensitivity, ad spend $\to$ clicks, carbon tax $\to$ emissions, risk $\to$ portfolio value.

Calculus also underpins extrema, monotonicity, convexity — and the local behavior behind optimization and ML.

Read Ch. 4 Werner-Sotskov; Simon-Blume Chs. 2-5.

Exercises: 4.6(c), 4.10, 4.14, 4.15(a)-(c), 4.16(d), 4.17(b) (Werner-Sotskov), 5.5 (b)-(c)-(e) (Simon-Blume)

Limit of a function

Let $f : D \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. $L = \lim_{x \to x_0} f(x)$ if for any sequence $x_n \in D$ with $x_n \to x_0$, we have $f(x_n) \to L$.

Remark: x_0 need not be in D (removable/essential discontinuities matter in econ too).

Example: $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ since $0 \leq |x \sin \frac{1}{x}| \leq |x|$.

Algebra of limits

If $\lim f = y_1$ and $\lim g = y_2$ at x_0, then

$$\lim(f+g) = y_1 + y_2, \quad \lim(fg) = y_1 y_2, \quad \lim \frac{f}{g} = \frac{y_1}{y_2} \text{ if } y_2 \neq 0,\ g \neq 0 \text{ near } x_0.$$

Continuity

Definition

f is continuous at $x_0 \in D$ if $\lim_{x \to x_0} f(x) = f(x_0)$. If f is continuous at every point of $B \subset D$, we say f is continuous on B.

Theorem: If f, g are continuous at x_0 then so are $f + g$ and fg. If moreover, $g(x) \neq 0$ around x_0, then $\frac{f}{g}$ is continuous at x_0.

Heine-Borel fact (1D): continuous $f : [a, b] \to \mathbb{R}$ is bounded.

Composition and inverse

Two important results

(composition) If f, g are continuous, so is $h = g \circ f$; moreover

$$\lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right).$$

(inverse) If $f : [a, b] \to \mathbb{R}$ is strictly monotone and continuous with $f(a) = c, f(b) = d$, then the inverse $f^{-1} : [c, d] \to [a, b]$ is also strictly monotone and continuous.

Econ: Let p denote the price of a good and q the quantity sold. The *demand function* $q(p)$ gives the quantity q consumers will buy at price p. Equivalently, the *inverse demand function* $p(q)$ gives the price at which q units can be sold. Revenue as a function of quantity is:

$$R(q) = p(q) \cdot q,$$

which involves compositions.

Derivative

Let $f : (a, b) \to \mathbb{R}$. f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. Denote this limit by $f'(x)$.

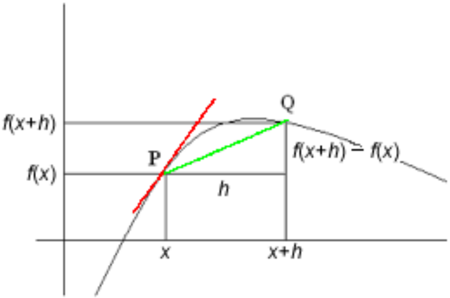
Fact: differentiable $\Rightarrow$ continuous (not conversely: $|x|$ is continuous but not differentiable at 0).

Interpretation: local linear approximation — crucial in marginal analysis, price elasticity, and gradient-based optimization.

Geometric interpretation

If f is differentiable at x_0, the tangent line at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$



slope of secant through P,Q: $\frac{f(x+h)-f(x)}{h}$; slope of tangent at P: $f'(x)$

A “marginal” is a derivative

In economics, a **marginal** quantity measures how much something changes when we increase an input by a small amount. Mathematically, we often approximate

$$f(x+1) - f(x) \;\approx\; f'(x),$$

when the change is 1 unit.

Examples:

- **Marginal revenue:** $\frac{d}{dq}[p(q)\,q]$ at the current sales level q. The derivative tells you how much extra revenue you gain (or lose) by selling one more unit.
- **Click-through sensitivity:** $\frac{d\,\mathrm{CTR}}{d(\text{ad spend})}$ at the budget currently in use. Here CTR (click-through rate) is a function of ad spending; the derivative measures how much CTR improves with an extra euro of spending.

If the change is Δx rather than 1: $\quad f(x+\Delta x) - f(x) \approx f'(x)\,\Delta x.$

In differential notation: $\quad df \approx f'(x)\,dx$

Derivatives of elementary functions

1. $c' = 0$
2. $(x^n)' = nx^{n-1}$ for $n \in \mathbb{N}$
3. $(x^\alpha)' = \alpha x^{\alpha-1}$ for $\alpha \in \mathbb{R}$, $x > 0$
4. $(\log x)' = \dfrac{1}{x}$, $x > 0$
5. $(\sin x)' = \cos x$
6. $(\cos x)' = -\sin x$
7. $(e^x)' = e^x$
8. $(a^x)' = a^x \log a$, $a > 0$

Proof of 8: $(a^x)' = (e^{x \log a})' = a^x \log a$ (chain rule)

Product/quotient rules

If f, g are differentiable:

- $(f+g)' = f' + g'$
- $(fg)' = f'g + g'f$ [Leibniz rule]
- $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$ where $g \neq 0$ near the point.

Example (revenue): $R(q) = p(q)\,q$, then $R'(q) = p'(q)\,q + p(q)$.

Chain rule

If $h = g \circ f$ and f, g are differentiable at the relevant points, then

$$h'(x) = g'(f(x))\, f'(x).$$

Example (cost of production):

$$C(x) = 4 + \log(x+1) + \sqrt{3x+1} \quad \Rightarrow \quad C'(x) = \frac{1}{x+1} + \frac{3}{2\sqrt{3x+1}}.$$

Example (choice models): In discrete choice models, the logistic function $\sigma(x) = \frac{1}{1+e^{-x}}$ maps a score x (reflecting how attractive an option is) to the probability of choosing it. Its derivative $\sigma'(x) = \sigma(x)[1 - \sigma(x)]$ measures how sensitive the choice probability is to changes in x.

Differentiating $u(x)^{v(x)}$

Let $u(x) > 0$. For $h(x) = u(x)^{v(x)}$, taking logs gives:

$$\log h = v \log u \quad \Rightarrow \quad \frac{h'}{h} = v' \log u + v\,\frac{u'}{u}.$$

Hence

$$\boxed{h'(x) = u(x)^{v(x)}\Big(v'(x)\log u(x) + v(x)\,\frac{u'(x)}{u(x)}\Big)}$$

Econ example: A startup's valuation is $V(t) = V_0[1 + g(t)]^{T(t)}$, where $g(t)$ is the projected annual growth rate and $T(t)$ the number of years that growth lasts. Both $g(t)$ and $T(t)$ vary with t.

Some of the contributors (in form of *caganers*)

Derivative of the inverse

Theorem: If f is strictly monotone, continuous, and differentiable at x with $f'(x) \neq 0$, then f^{-1} is differentiable at $y = f(x)$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

Since $f^{-1}(f(x)) = x$, chain rule gives $(f^{-1})'(f(x))f'(x) = 1$.

Example: $f(x) = e^x \Rightarrow (\log)'(y) = 1/y$.

Higher-order derivatives: If f' is differentiable, its derivative is called the second derivative of f and denoted f''. Similarly, we can define higher-order derivatives, and $f^{(n)}$ is called the nth derivative of f. If $f^{(n)}$ is continuous, we say that f is n times continuously differentiable or C^n for short.

L'Hôpital's rule

Theorem: Let f, g be C^1 on (a, b) with $g' \neq 0$ on (a, b). If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad \text{or} \quad \pm\infty,$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{(if the RHS limit exists).}$$

Proof idea uses the Mean Value Theorem (slide 53).

Example:

$$\lim_{x \to 0} (1+x)^{1/x} = \exp\left(\lim_{x \to 0} \frac{\log(1+x)}{x}\right) = \exp\left(\lim_{x \to 0} \frac{1}{1+x}\right) = e.$$

Monotonicity via derivatives

If f is differentiable on (a, b):

1. f increasing on $[a, b] \iff f'(x) \geq 0$ on (a, b).
2. f decreasing on $[a, b] \iff f'(x) \leq 0$ on (a, b).
3. f constant on $[a, b] \iff f'(x) = 0$ on (a, b).
4. $f'(x) > 0$ on $(a, b) \Rightarrow f$ strictly increasing on $[a, b]$.
5. $f'(x) < 0$ on $(a, b) \Rightarrow f$ strictly decreasing on $[a, b]$.

Converse of 4-5 can fail at isolated points (e.g., x^3).

Example: $f(x) = \frac{x^3}{3} + 2x^2 + 3x + 1$; $f'(x) = (x+1)(x+3)$ determines monotonicity intervals and local extrema.

Optimal points: first-order conditions

Local optima

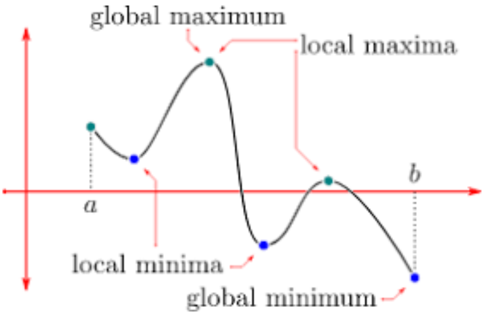
Let $f : D \to \mathbb{R}$. A point $x_0 \in D$ is a local max if $\exists (a, b) \subset D$ with $x_0 \in (a, b)$ and $f(x) \leq f(x_0)$ for all $x \in (a, b)$. Similarly for local min with $\geq$.

If f has a local optimum at interior x and is differentiable, then $f'(x) = 0$ (necessary, not sufficient).

Economic flavor: In profit maximization, the optimal quantity q^* often satisfies the first-order condition $f'(q^*) = 0$, which, for profit $\pi(q) = R(q) - C(q)$, means *marginal revenue* $R'(q)$ equals *marginal cost* $C'(q)$.

If x is stationary ($f'(x) = 0$) and f is strictly increasing to the left and strictly decreasing to the right, then x is a local max (reverse for min).

Optimal points



Example: Timing a sale (present value)

Suppose an assets market value is $V(t)$ at time t. At a constant annual interest rate R, the **present value** of selling at time t is

$$P(t) = V(t)\, e^{-Rt},$$

where e^{-Rt} discounts future cash flows.

Goal: Choose t to maximize $P(t)$.

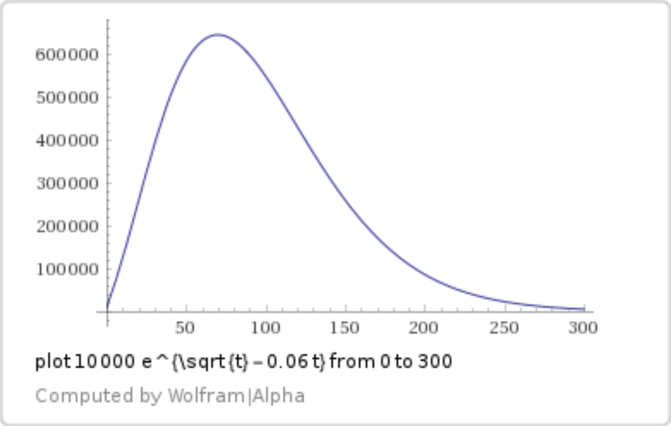
FOC (set derivative to zero):

$$P'(t) = V'(t)e^{-Rt} - R\, V(t)e^{-Rt} = 0 \quad \Rightarrow \quad \frac{V'(t)}{V(t)} = R.$$

Interpretation: Sell when the assets *instantaneous percentage growth rate* (matches) the *interest rate you could earn in the bank*.

Example: If $V(t) = 10000\, e^{\sqrt{t}}$ and $R = 6\%$, then $P(t) = 10000\, e^{\sqrt{t} - 0.06\, t}$, which is maximized near $t \approx 69.44$.

Plot of the present value $P(t)$



To certify a global optimum on an interval, compare values at boundaries and critical points. Here $P(0) = 10000$, $\lim_{t\to\infty} P(t) = 0$.

Second-order conditions

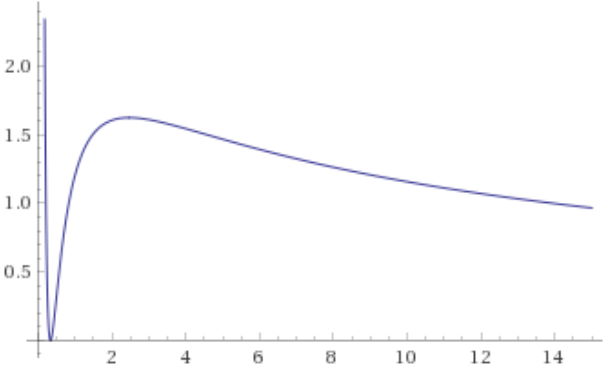
Test: If $f'(x_*) = 0$ and $f''(x_*) < 0$ (> 0), then x_* is a strict local max (min). Higher-order tests extend this when $f''(x_*) = 0$.

Example: For $f(x) = \dfrac{\log^2(3x)}{x}$ ($x > 0$):

$$f'(x) = \frac{(2 - \log 3x)\log 3x}{x^2}, \quad f''(x) = \frac{2(1 - 3\log 3x + \log^2 3x)}{x^3}.$$

Critical points at $x_1 = \frac{1}{3}$ (min), $x_2 = \frac{e^2}{3}$ (max).

Plot of $f(x) = \dfrac{\log^2 3x}{x}$, $x > 0$



plot ln^2(3x)/x from 0 to 15 | Computed by Wolfram|Alpha

$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $f(x_1) = 0$, $f(x_2) > 0 \Rightarrow x_1$ global min, no global max.

Convexity and concavity

If f is twice differentiable on (a, b):

1. f convex on $[a, b] \Leftrightarrow f''(x) \geq 0$ for all $x \in (a, b)$.
2. f concave on $[a, b] \Leftrightarrow f''(x) \leq 0$ for all $x \in (a, b)$.
3. $f'' > 0$ $(< 0) \Rightarrow f$ strictly convex (concave).

Theorem: If f and g are convex (or concave), then $f \circ g$ is convex (or concave).
Example: $f(x) = e^{x^2}$ is strictly convex on $[0, \infty)$.

Economics: costs often convex; utility often concave; regularized losses in ML are designed to be convex for tractable optimization.

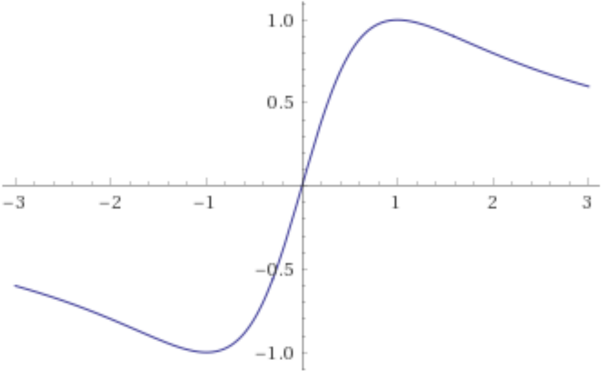
Example of changing curvature

Let $f(x) = \dfrac{2x}{x^2+1}$. Then

$$f'(x) = \frac{2(1-x^2)}{(x^2+1)^2}, \qquad f''(x) = \frac{4(x^3-3x)}{(x^2+1)^3}.$$

So f is strictly convex on $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$ and strictly concave on $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$.

Plot of $f(x) = \dfrac{2x}{x^2+1}$



plot 2x/(x^2+1) from −3 to 3 | Computed by Wolfram|Alpha

Inflexion points at $x \in \{-\sqrt{3}, 0, \sqrt{3}\}$. Global max at $x = 1$, min at $x = -1$, and $\lim_{|x|\to\infty} f(x) = 0$.

Inflexion points

If f is C^n on (a, b), an inflexion point at x_* occurs when curvature changes sign. A sufficient test:

$$f''(x_*) = \cdots = f^{(n-1)}(x_*) = 0, \quad f^{(n)}(x_*) \neq 0, \quad n \text{ odd}.$$

Example: $f(x) = x^3$ has $f''(0) = 0$, $f^{(3)}(0) = 6 \neq 0 \Rightarrow$ inflexion at 0.

Mean-value theorem

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{(equivalently } f(b) = f(a) + f'(c)(b - a)\text{).}$$

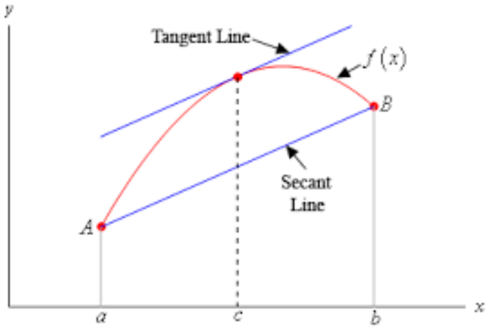
Proof idea: consider $h(x) = (f(b) - f(a))x - (b - a)f(x)$; since $h(a) = h(b)$, h has an interior extremum.

Interpretation: the instantaneous slope matches the secant slope at some point.

Geometry of the mean-value theorem

Recall: $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Taylor formula

Theorem (Taylor's formula of order n): Let f be $n+1$ times differentiable on (a, b), and let $x_0 \in (a, b)$ given. Then, for any $x \in (a, b)$, if $n = 1$,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \text{error}.$$

If $n \geq 2$,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \text{error}.$$

The error is small if x is close to x_0, therefore, this theorem says that differentiable functions may be locally approximated by a polynomial called the Taylor polynomial.

Economics/data: local linear/quadratic approximations underlie elasticity (log-linearization), Delta method, and second-order price responses.

Taylor remainder and a handy expansion

Remainder (Lagrange form)

$$R_n^y(x) = \frac{f^{(n+1)}(y)}{(n+1)!}(x - x_0)^{n+1} \quad \text{for some } y \text{ between } x_0 \text{ and } x.$$

If $|f^{(n+1)}| \leq M$ near x_0, then $|R_n^y(x)| \leq \dfrac{M}{(n+1)!}|x - x_0|^{n+1}$.

Example (log around 1): With $x_0 = 1$,

$$\log x \approx (x-1) - \tfrac{1}{2}(x-1)^2, \quad R_2^y(x) = \frac{1}{3y^3}(x-1)^3.$$

Finance quick check: For small r, $\log(1+r) \approx r - \frac{r^2}{2} \Rightarrow \text{EAR} \approx \text{APR} - \frac{1}{2}\text{APR}^2$ (small correction).

Short comment about small-o notation

Writing $|R_n^y(x)| \leq \dfrac{M}{(n+1)!}|x - x_0|^{n+1}$ we conclude $\frac{R_n^y(x)}{|x-x_0|^n} \leq \dfrac{M}{(n+1)!}|x - x_0|$ and so

$$\lim_{x \to x_0} \frac{R_n^y(x)}{|x - x_0|^n} \; = \; 0.$$

This we write as $R_n^y(x) = o(|x - x_0|^n)$.

In particular: If f is twice differentiable:

$f(x+h) \; = \; f(x) + f'(x)h + o(|h|)$.

The generalization of this last expression to multivariate functions will provide powerful insights into their optimization.