

# **STA 437/2005**

Midterm Exam 2, Section LEC0201

**Date: March 14, 2025**

## **STA 437/2005, Midterm 2**

**Student Name:** \_\_\_\_\_  
(Please use capital letters)

**Student Number:** \_\_\_\_\_

**I am taking (please circle):**                      **STA437**                      **STA2005**

### **Instructions:**

- Fill out your name and student number on this page.
- Carefully edit your answer in the provided space. Use the last two pages for your own notes (this will not be graded).
- Show all your work to receive full credit.
- This exam has three problems. Each problem is on a separate page.
- No electronic devices, notes, or books are allowed during the exam.

**Good Luck!**

### Problem 1 (20 points)

Let  $X \sim N_3(\mathbf{0}, \Sigma)$ , where the covariance matrix is given by

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. (10 pts) What is the PCA transformation  $Z$  of  $X$  in this case?

Note that there is one eigenvector  $\mathbf{u}_3 := (0, 0, 1)$  with eigenvalue 1 (which can be seen immediately). All other eigenvectors are of the form  $(*, *, 0)$  so enough to check the eigenvectors of the upper  $2 \times 2$  submatrix. We have

$$\det\left(\begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix}\right) = 0 \text{ if and only if } \lambda = 1, 3.$$

The associated eigenvectors are  $\mathbf{u}_2 := \frac{1}{\sqrt{2}}(0, 1, -1)$ ,  $\mathbf{u}_1 := \frac{1}{\sqrt{2}}(0, 1, 1)$  respectively. So the transformation is  $Z = U^\top X$  where  $U$  has columns  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ .

2. (10 pts) Find the distribution of  $(Z_1, Z_2)$ .

Since  $\Sigma = U \Lambda U^\top$ , the distribution of  $Z$  is  $N_2(\mathbf{0}, \text{diag}(3, 1))$ .

## Problem 2 (20 points)

Consider the random vector  $X = (X_1, X_2)$ , where  $X$  is uniformly distributed on the unit square in  $\mathbb{R}^2$ , i.e., the probability density function of  $X$  is

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}, & \text{if } -1 \leq x_1, x_2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions.

1. (6 pts) Argue whether  $X$  follows spherical distribution.

No.  $Z$  is uniformly distributed on the unit square, its distribution is not rotationally invariant, which means it does not follow the spherical distribution. Alternatively, one can also argue from the PDF that, the PDF of  $Z$  depends on the individual coordinate of  $Z$ , rather than simply the norm of  $Z$ .

2. (6 pts) Determine which of the following vectors follow the same distribution as  $X$ :

$$Y_1 = \begin{pmatrix} X_2 \\ X_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -X_1 \\ X_2 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} \frac{\sqrt{3}}{2}X_1 - \frac{1}{2}X_2 \\ \frac{1}{2}X_1 + \frac{\sqrt{3}}{2}X_2 \end{pmatrix}.$$

Choose from the following options and briefly justify your answer:

- A. only  $Y_2$ .
- B. only  $Y_1$  and  $Y_2$ .
- C. only  $Y_2$  and  $Y_3$ .
- D. All of them.

B, only  $Y_1$  and  $Y_2$ . Since  $Z$  follows uniform distribution on the unit square, one can show that  $X_1$  and  $X_2$  are i.i.d. with marginal distribution  $U[-1, 1]$ . Therefore, it is invariant under permutation ( $Y_1$ ) and reflection ( $Y_2$ ), but not for a general rotation.

3. (8 pts) True or False (no need to provide justifications):

T / F:  $X_1, X_2$  have the same distribution.

T / F:  $X_1, X_2$  are independent.

T / F:  $X_1 \sim U[-1, 1]$ .

T / F: The copula of  $X_1$  and  $X_2$  is  $C(u, v) = uv$ .

Answer: T, T, T, T

### Problem 3 (20 points)

Consider a vector  $X = (X_1, X_2)$  whose distribution is a mixture of two Gaussian distributions with parameters:  $\pi_1 = 0.6$ ,  $\pi_2 = 0.4$  and

$$\mu_1 = (1, 2), \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mu_2 = (0, 0), \Sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(i) (10 pts) Denote by  $f_1(x)$ ,  $f_2(x)$  the densities of the two Gaussian components. Suppose  $f_1(3, 3) \approx 0.013$  and  $f_2(3, 3) \approx 0.005$ . Explain how you can use this information to compute the probability that the observation  $(3, 3)$  comes from the first Gaussian component.

We have

$$p(z = 1 | \mathbf{x} = (3, 3)) = \frac{\pi_1 p(\mathbf{x} = (3, 3) | z = 1)}{\pi_1 p(\mathbf{x} = (3, 3) | z = 1) + \pi_2 p(\mathbf{x} = (3, 3) | z = 2)}.$$

Since  $p(\mathbf{x} = (3, 3) | z = 1) = 0.013$  and  $p(\mathbf{x} = (3, 3) | z = 2) = 0.005$  we have everything we need to use this formula.

(ii) (10 pts) Compute  $\mathbb{E}(X_1 X_2)$ .

This follows by very standard arguments: First,  $\mathbb{E}(X_1 X_2) = \pi_1 \mathbb{E}(X_1 X_2 | z = 1) + \pi_2 \mathbb{E}(X_1 X_2 | z = 2)$ . Second,  $\mathbb{E}(X_1 X_2 | z = 1) = \text{cov}(X_1, X_2 | z = 1) + \mathbb{E}(X_1 | z = 1) \mathbb{E}(X_2 | z = 1)$ . Thus, we have

$$\mathbb{E}(X_1 X_2) = 0.6 \cdot (0 + 1 \cdot 2) + 0.4 \cdot (1 + 0) = 1.6.$$

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