STA 437/2005: Methods for Multivariate Data

Week 4: Non-Gaussian Distributions

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Elliptical distributions

Why Study Elliptical Distributions?

- Generalize the multivariate normal distribution.
- Model data with heavy tails or outliers.
- Maintain symmetry and linear correlation structures.
- Applications in finance, insurance, and environmental studies.

Spherical Distributions

Orthogonal Matrices: $O(m) = \{U \in \mathbb{R}^{m \times m} : U^\top U = I_m\}$.

Spherical distribution

A random vector $X \in \mathbb{R}^m$ has a *spherical distribution* if for any $U \in O(m)$:

$$X \stackrel{d}{=} UX.$$

Characteristic function satisfies: $\psi_X(\boldsymbol{t}) = \psi_{UX}(\boldsymbol{t}) = \psi_X(U^\top \boldsymbol{t})$ and so **equivalently** $\psi_X(\boldsymbol{t})$ depends only on $\|\boldsymbol{t}\|$. Thus, the same applies to the density:

$$f_X(\mathbf{x}) \; = \; h(\|\boldsymbol{x}\|) \qquad \text{for some } h \text{ (generator)}.$$

Examples of Spherical Distributions

Standard normal distribution $Z \sim N_m(0, I_m)$ is a simple example.

Spherical scale mixture of normals

If $Z \sim N_m(0, I_m)$ and a random variable $\tau > 0$ is independent of Z, then:

$$X = \frac{1}{\sqrt{\tau}} Z$$

has a spherical distribution.

Indeed: Let $U \in O(m)$, then

$$UX \; = \; \frac{1}{\sqrt{\tau}} UZ \; \stackrel{d}{=} \; \frac{1}{\sqrt{\tau}} Z \; = \; X.$$

Moment Structure of Spherical Distributions

Spherical symmetry implies:

$$\mathbb{E}[X] = 0,$$
$$\mathrm{var}(X) = cI_m, \quad \text{for some } c \geq 0.$$

Indeed: $\mathrm{var}(X) = \mathrm{var}(UX) = U\mathrm{var}(X)U^\top$ for any $U \in O(m)$

For $X = \frac{1}{\sqrt{\tau}}Z$ with $Z \sim N(0, I_m)$, $\tau > 0$, $\tau \perp\!\!\!\perp Z$:

$$\mathrm{var}(X) = \mathbb{E}[\tau^{-1}]I_m.$$

Indeed: $\mathbb{E}[X] = \mathbb{E}[\frac{1}{\sqrt{\tau}}Z] = \mathbb{E}[\frac{1}{\sqrt{\tau}}]\mathbb{E}[Z] = \mathbf{0}_m$ and so

$$\mathrm{var}(X) \;=\; \mathbb{E}XX^\top - \mathbb{E}[X]\mathbb{E}[X]^\top \;=\; \mathbb{E}[\tfrac{1}{\tau}ZZ^\top] \;=\; \mathbb{E}[\tfrac{1}{\tau}]\mathbb{E}[ZZ^\top] \;=\; \mathbb{E}[\tfrac{1}{\tau}]I_m$$

Independence of $\|X\|$ and $\frac{X}{\|X\|}$

Key Property: For spherical distributions, the norm $\|X\| = \sqrt{X^\top X}$ is independent of the direction $\frac{X}{\|X\|}$. **Proof Sketch:**

- Let $U \in O(m)$. Then:

$$\frac{X}{\|X\|} \stackrel{d}{=} \frac{UX}{\|UX\|} = U\frac{X}{\|X\|}.$$

- $\frac{X}{\|X\|}$ is rotationally invariant, implying uniform distribution on the unit sphere.

Polar Coordinates

Definition: In $\mathbb{R}^m$, polar coordinates represent $\mathbf{x}$ as:

$$\mathbf{x} = r\mathbf{u}(\boldsymbol{\theta}),$$

where $r = \|\mathbf{x}\|$ is the radial coordinate, and $\boldsymbol{\theta}$ are angular coordinates. **Jacobian**

Determinant:

$$J(r, \boldsymbol{\theta}) = r^{m-1} \prod_{i=2}^{m-1} \sin^{m-i}(\theta_{i-1}).$$

Implication: If $f(\mathbf{x}) = g(\|\mathbf{x}\|^2)$, then:

$$f(\mathbf{x})d\mathbf{x} = g(r^2)r^{m-1}drd\boldsymbol{\theta}.$$

Elliptical Distributions

Definition: A random vector $X \in \mathbb{R}^m$ has an elliptical distribution if:

$$X = \mu + \Sigma^{1/2} Z,$$

where Z is a spherical random vector, and $\Sigma^{1/2}$ is the square root of a positive semi-definite matrix Σ. **Examples:**

- Multivariate normal: $Z \sim N_m(0, I_m)$.
- Scale mixtures of normals.

Why Elliptical Distributions?

- Flexibility: Models data with heavy tails.
- Retains properties like symmetry and linear correlation structure.
- Applications:
  - Finance: Captures extreme losses/gains.
  - Environmental studies: Models outliers in natural events.

Scale Mixtures of Normals

Representation:

$$X = \mu + \frac{1}{\sqrt{\tau}}\Sigma^{1/2}Z,$$

where $Z \sim N_m(0, I_m)$ and $\tau > 0$ is independent of Z. **Special Cases:**

- $\tau \equiv 1$: Multivariate normal.
- $\tau \sim \frac{1}{k}\chi^2_k$: Multivariate t-distribution.
- $\tau \sim \mathrm{Exp}(1)$: Multivariate Laplace.

Covariance and Correlation in Elliptical Distributions

- Σ is the **scale matrix** but not the covariance matrix.
- Actual covariance:
$$\mathrm{Var}(X) = c\Sigma, \quad c > 0.$$
- Correlation structure is still governed by Σ.

Copula models