Characteristic Classes and Chern Classes

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What are Characteristic Classes and Chern Classes?

Characteristic classes are the natural analogue of a cohomology class for vector bundles.

Chern classes are a type of characteristic class for complex vector bundles.

Outline of Talk

(1) Vector Bundles.
  (a) Definitions.
  (b) Examples.
(2) Characteristic Classes.
  (a) The Cohomology Ring.
  (b) Characteristic Classes.
  (c) Chern Classes.
(3) Is $\mathbb{CP}^n$ Parallelisable?
  (a) Complex Projective Space, $\mathbb{CP}^n$.
  (b) Parallelisablity.
(4) Recovering Steifel-Whitney Classes.

What's a Vector Bundle?

Definition (Complex Vector Bundles)

A *complex vector bundle* over a topological space B is

(i) A topological space E called the *total space*, and

(ii) A continuous surjection $p : E \to B$ called the *projection map*

...together with the conditions...

(a) For each $b \in B$, the *fibre* $p^{-1}(b)$ is a complex vector space, and...

Definition (Complex Vector Bundles)

(b) (*Local Triviality*) Each point $b_0 \in B$ must possess a neighbourhood U with $p^{-1}(U)$ being homeomorphic to $U \times \mathbb{C}^r$. This homeomorphism $h : U \times \mathbb{C}^r \to p^{-1}(U)$ must map the fibres $p^{-1}(b)$ linearly onto $\{b\} \times \mathbb{C}^r$.

$$\begin{tikzcd} U \times \mathbb{C}^r \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where π_1 is the projection map onto the first coordinate.

What's a Vector Bundle?

Example (Tangent Bundle)

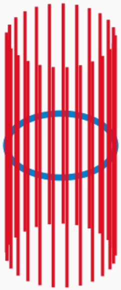
Fix a differentiable manifold B.

Each point $x \in B$ has a tangent space, $T_x B$. We take the space of pairs

$$TB = \{(x, y) : x \in B, y \in T_x B\}$$

and form the *tangent bundle*, with

1. Base space B,
2. Total space TB, and
3. Projection map $p : TB \to B$ given by $p(x, y) = x$.



Characteristic Classes

Fix an abelian group A. Recall the cohomology ring of B,

$$H^*(B;A) = \bigoplus_{i=0}^{\infty} H^i(B;A) = H^0(B;A) + H^1(B;A) + ...$$

A *characteristic class* c_i of vector bundles is an assignment of each bundle $E \to B$ to a cohomology class

$$c_0(E) + c_1(E) + ... \in \bigoplus_{i=0}^{\infty} H^i(B;A).$$

Characteristic Classes

Certain choices of B and A give rise to different classes:

(1) Stiefel-Whitney classes: $w_i(E) \in H^i(B; \mathbb{Z}_2)$,
$E \to B$ a real vector bundle.

(2) Chern classes: $c_i(E) \in H^{2i}(B; \mathbb{Z})$,
$E \to B$ a complex vector bundle.

(3) The Euler class: $e(E) \in H^n(B; \mathbb{Z})$,
$E \to B$ an oriented, n-dimensional real vector bundle.

What are Chern Classes?

Definition (Chern Classes)

Fix a complex vector bundle E over B. The Chern classes $c_k(E)$ are defined to be the unique classes satisfying:

(i) $c_0(E) = 1$.

(ii) (*Naturality*) If $f : B' \to B$ is a homeomorphism then recall f^*E is the pullback vector bundle over B'. Then

$$c_k(f^*E) = f^*c_k(E).$$

(iii) (*Whitney Sum Formula*) Let $F \to B$ be another complex vector bundle over B. Then

$$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E)c_{k-i}(F).$$

Definition (Chern Classes)

(iv) (*Normalisation*) The first Chern class of the tautological line bundle of $\mathbb{CP}^1$ is equal to -1 in $H^2(\mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}$.

The *total* Chern class of E is $c(E) = \sum_{i \geq 0} c_i(E)$. That is,

$$c(E) = c_0(E) + c_1(E) + ... \in H^*(B; \mathbb{Z}).$$

Complex Projective Space, $\mathbb{CP}^n$

Definition (Complex Projective Space)

Consider the non-zero vectors in $\mathbb{C}^{n+1} = \{(z_1, ..., z_{n+1}) : z_i \in \mathbb{C}\}$. We identify non-zero vectors in $\mathbb{C}^{n+1}$ with their scalar multiples.

$$(z_1, ..., z_{n+1}) \sim (\lambda z_1, ..., \lambda z_{n+1}),\ \lambda \in \mathbb{C}.$$

Then $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$.

$\mathbb{CP}^n$ has much more structure than just a topological space, it's a complex manifold.

$\mathbb{CP}^1$ is the *Riemann sphere*, $\mathbb{CP}^2$ is the *complex projective plane*.

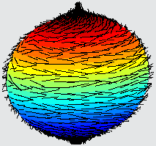
Parallelisability

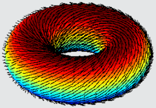
Definition (Parallelisability)

Fix a differentiable manifold M of dimension n.

We say M is *parallelisable* if there exist a family of smooth vector fields on the manifold, $V_1, \ldots, V_n$, such that every point p in M provides a basis $\{V_1(p), \ldots, V_n(p)\}$ of T_pM.

Example (Hairy Ball Theorem)





Parallelisability

Theorem (Milnor, Stasheff)

As before, M is a differentiable manifold. Then

$$M \text{ is parallelisable} \iff M \text{ has trivial tangent bundle.}$$

Is $\mathbb{CP}^n$ Parallelisable?

Recall the cohomology ring of $\mathbb{CP}^n$ over $\mathbb{Z}$,

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/\langle x^{n+1} \rangle.$$

Consider the tangent bundle, $T\mathbb{CP}^n$. If $T\mathbb{CP}^n$ were trivial, we would have $c(T\mathbb{CP}^n) = 1$.

This has total Chern class

$$c(T\mathbb{CP}^n) = (1+x)^{n+1} \in H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/\langle x^{n+1} \rangle.$$

Then $x \neq 0$ and $n \geq 0$, so $c(T\mathbb{CP}^n) \neq 1$.

Recovering Stiefel-Whitney Classes

Proposition

Regard an n-dimensional complex vector bundle $E \to B$ as a $2n$-dimensional real vector bundle.

Then

- $w_{2i+1}(E) = 0$, and
- $w_{2i}(E)$ is the image of $c_i(E)$ under the coefficient homomorphism $H^{2i}(B;\mathbb{Z}) \to H^{2i}(B;\mathbb{Z}_2)$.

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