SCIE3250 - Project Report

Bailey Whitbread

May 31, 2019

Contents

1 Abstract

The winding number of a closed curve, defined by a map $f : \mathbb{S}^1 \to \mathbb{C}$, around a point $p \in \mathbb{C}$ is an integer that counts the number of times the curve travels around p. Topological degree theory aims to generalize this notion of wrapping a suitable domain around a suitable codomain via a continuous mapping and assign a *'degree'* to these mappings.

Topological degree theory can be used to formulate existence theorems regarding solutions of nonlinear equations.

Our aim in this project is to work through Chapters 1 & 2 of Nirenberg's *Topics in Nonlinear Functional Analysis* [2]. This motivates us to define and study the degree of mappings on finite and infinite dimensional spaces. In finite dimensions, we define and study *Brouwer degree*, with *Leray-Schauder degree* being its analogue in infinite dimensional spaces.

Nirenberg begins [2] with the goal to solve nonlinear problems of the form

$$F(x) = 0$$

and goes onto illustrate the use of topology in solving these.

We use this illustration as a starting point, which requires us to understand a particular subfield of topology called homotopy theory. After this, we define the degree of a map and begin our study of degree theory. Once the theory of Brouwer and Leray-Schauder degree is presented, we present some fixed point theorems proved using our new tools.

2 Homotopy Theory

We present some definitions and results adapted from undergraduate course notes found in [1].

Definition 2.1. *Let X and Y be topological spaces and $f, g : X \to Y$ be continuous maps. A homotopy from f to g is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in X$.*

If a homotopy from f to g exists, we write $f \simeq g$ and say f and g are homotopic.

If $f \simeq k$ for some constant map k then we say f is homotopically trivial.

Example 2.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $f \simeq g$.*

To see this, define $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = (1-t)f(x) + tg(x)$.

F is continuous since it is the sum and product of continuous functions and we see that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ so that $f \simeq g$.

A corollary of this is that all continuous maps from $\mathbb{R}$ to $\mathbb{R}$ are homotopically trivial since the constant maps on $\mathbb{R}$ are continuous.

We can generalize this example but we require a definition before this.

Definition 2.2. *A subset $A \subset \mathbb{R}^n$ is convex if, for every $x, y \in A$, there holds*

$$(1-t)x + ty \in A, \qquad \textit{for all } t \in [0,1]$$

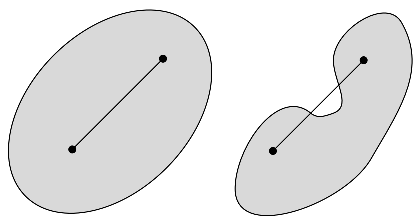


Figure 1: A convex set A requires that straight line segments between any two points in A are contained in A.

Proposition 2.1. *Let $A \subset \mathbb{R}^n$ be convex and endowed with the subspace topology inherited from $\mathbb{R}^n$ and let X be a topological space. If $f, g : X \to A$ are continuous then $f \simeq g$.*

Proof. Define $F : [0,1] \times X \to \mathbb{R}^n$ by $F(t,x) = (1-t)f(x) + tg(x)$. Convexivity of A tells us that $F(t,x) \in A$ for all $t \in [0,1]$ and $x \in X$. Observe $F(0,x) = f(x)$, $F(1,x) = g(x)$ and F is continuous because it is the sum and product of continuous functions so it is a homotopy from f to g. $\square$

For topological spaces X and Y, we denote the space of continuous functions $f : X \to Y$ by $C(X,Y)$.

Theorem 2.1. *As defined above, $\simeq$ is an equivalence relation on $C(X,Y)$.*

Proof. We must show reflexivity, symmetry and transitivity of $\simeq$.

(i) Reflexivity: $f \simeq f$ via the homotopy $F : [0,1] \times X \to Y$ defined by $F(t,x) = f(x)$. F is continuous since f is and there holds $F(0,x) = f(x) = F(1,x)$.

(ii) Symmetry: Let $f \simeq g$ via the homotopy $F : [0,1] \times X \to Y$. Then define the homotopy $G : [0,1] \times X \to Y$ by $G(t,x) = F(1-t,x)$ so that $g \simeq f$.

   G is continuous since $1-t$ is continuous in the subspace $[0,1]$ and there holds $G(0,x) = F(1,x) = g(x)$ and $G(1,x) = F(0,x) = f(x)$.

(iii) Transitivity: Let $f \simeq g$ and $g \simeq h$ via the homotopies $F : [0,1]\times X \to Y$ and $G : [0,1]\times X \to Y$. Then define the homotopy $H : [0,1] \times X \to Y$ by

$$H(t,x) = \begin{cases} F(2t,x), & \text{if } 0 \leq x \leq \frac{1}{2} \\ G(2t-1,x), & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

   so that $f \simeq h$.

   H is continuous since $2t$ and $2t-1$ are continuous in the subspace $[0,1]$ and F and G are homotopies and there holds $H(0,x) = F(0,x) = f(x)$ and $H(1,x) = G(1,x) = h(x)$.

as required. □

So we have an equivalence relation on $C(X,Y)$. Denote the space of equivalence classes by

$$[X,Y] := C(X,Y)/\simeq$$

with elements $[f] \in [X,Y]$ denoting the equivalence class of $f \in C(X,Y)$.

Example 2.2. *Denote the equivalence class of constant functions on X by $[\mathrm{const}_X]$. The previous example tells us that*

$$[\mathbb{R},\mathbb{R}] = \{[\mathrm{const}_{\mathbb{R}}]\}$$

and the previous proposition tells us that

$$[X,A] = \{[\mathrm{const}_X]\}$$

for topological spaces X and convex $A \subset \mathbb{R}^n$.

3 Brouwer Degree for $\mathbb{R}^n$

3.1 A Motivating Theorem

We begin with the aforementioned illustration of the use of homotopy theory to solve nonlinear problems. The statement of the following theorem is taken from [2].

Theorem 3.1. *Suppose $B := \bar{B}_1(0)$ is the closed unit ball in $\mathbb{R}^n$ and $F : B \to \mathbb{R}^k$ is continuous. Let $\phi : \partial B \to \mathbb{R}^k \setminus \{0\}$ denote the restriction of F to ∂B. That is, $\phi = F|_{\partial B}$. Then set*

$$\psi = \frac{\phi}{|\phi|} : \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$$

The following are equivalent:

(i) For every extension, F, of ϕ into B, there exists a solution to $F(x) = 0$.

(ii) The map ψ is homotopically nontrivial.

We present a proof given in [3].

Proof. We prove the forward implication by proving its logically equivalent contraposition. Let ψ be homotopically trivial. Then there exists a continuous $g : [0,1] \times \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$ with $g(0,x) = \psi(x)$ and $G(1,x) = p$ for some constant $p \in \mathbb{S}^{k-1}$. Now set

$$M := \sup_{x \in \mathbb{S}^{n-1}} |\phi(x)|$$

and also set

$$F(x) := \begin{cases} p, & \text{if } x = 0 \\ p\gamma_1(x), & \text{if } 0 < |x| \le \frac{1}{2} \\ g\Big(2 - 2|x|, \frac{x}{|x|}\Big)\gamma_2(x), & \text{if } |x| > \frac{1}{2} \end{cases}$$

where

$$\gamma_1(x) := \left|\phi\left(\frac{x}{|x|}\right)\right|\left(\frac{2|x|}{M+1}\right) + (1 - 2|x|)$$
$$\gamma_2(x) := \left|\phi\left(\frac{x}{|x|}\right)\right|\left[\frac{2 - 2|x|}{M+1} + 2|x| - 1\right]$$

Now F is continuous. To see this, note that F could only be discontinuous for $|x| = 0$ or $|x| = \frac{1}{2}$ since it is the composition, sum and product of continuous functions everywhere else. Then notice $\gamma_1(x) \to 1$ as $|x| \to 0$ so that

$$\lim_{x \to 0} F(x) = \lim_{|x| \to 0} F(x) = \lim_{|x| \to 0} p\gamma_1(x) = p = F(0)$$

and, for $|x| = \frac{1}{2}$, we have

$$\gamma_1(x) = \gamma_2(x) = \frac{1}{M+1}\left|\phi\left(\frac{x}{|x|}\right)\right|$$

Also $\gamma_1 > 0$ on $(0, \frac{1}{2}]$ and $\gamma_2 > 0$ on $(\frac{1}{2}, 1]$. For $x \in \mathbb{S}^{n-1}$, we have $F(x) = g(0,x)|\phi(x)|$ so that $\text{im } F \cap \{0\} = \varnothing$ (ie. F has no solutions).

To prove the reverse implication, a gain, we prove its contraposition. Let ϕ be such that it admits an extension $F : B \to \mathbb{R}^k \setminus \{0\}$. Then $\mu(x) = F(x)/|F(x)|$ defines a continuous map from B to $\mathbb{S}^{k-1}$.

Define $g : [0,1] \times \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$ via

$$g(t,x) = \mu((1-t)x)$$

Now g is well defined, continuous and for $x \in \mathbb{S}^{n-1}$, there holds

$$g(0,x) = \mu(x) = \frac{\phi(x)}{|\phi(x)|} = \psi(x)$$
$$g(1,x) = \mu(0) = \frac{F(0)}{|F(0)|}$$

for all x. Then we've just shown that ψ is homotopically trivial. □

We consider applications of this theorem in cases. If $n < k$ then every map $\psi : \mathbb{S}^{n-1} \to \mathbb{S}^{k-1}$ is homotopically trivial. To gain some intuition for this, think of the continuous image of $\mathbb{S}^1$ onto $\mathbb{S}^2$. This can always be deformed to a point. Then we cannot use the theorem to deduce the existence of any solutions to $F(x) = 0$.

If $n > k$ then our map ψ becomes unweildy and concluding if ψ is homotopically trivial or nontrivial becomes a difficult task.

We are specifically interested in the case $n = k$, where the homotopy class of ψ, $[\psi] \in [\mathbb{S}^{n-1}, \mathbb{S}^{k-1}]$, is determined by the 'degree' of ψ. It is a fact that ψ is homotopically trivial if and only if its degree is equal to 0.

3.2 Brouwer Degree

The following formulation of Brouwer degree is taken from [2]. The author considers smooth orientable manifolds. We focus our attention on simply $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n$ be open and bounded, $\phi : \bar{X} \to \mathbb{R}^n$ be continuously differentiable and $y_0 \in \mathbb{R}^n \setminus \phi(\partial X)$ be a regular value of ϕ.

We first define the degree of the map ϕ 'at' y_0.

Since y_0 is a regular value of ϕ, it follows from the implicit function theorem that the set $\phi^{-1}(y_0) = \{x \in \bar{X} : \phi(x) = y_0\}$ consists of isolated points in X. Also, since this set is compact, it is finite. That is,

$$\phi^{-1}(y_0) = \{x_1, ..., x_k\}$$

Definition 3.1. *If y_0 is a regular value of ϕ then*

$$d(y_0) := \sum_{j=1}^{k} sgn\ J_\phi(x_j)$$

The Jacobian of ϕ, J_ϕ, exists since ϕ lies in $C^1(\bar{X})$.

Definition 3.2. *A function $\mu \in C^\infty(\mathbb{R}^n)$ is admissible for y_0 and ϕ if its support lies in $\mathbb{R}^n \setminus \phi(\partial X)$ and*

$$\int_{\mathbb{R}^n} \mu(x)\ dx = 1$$

We are now ready to define Brouwer degree.

Definition 3.3. *Let μ be admissible for y_0 and ϕ. The Brouwer degree of ϕ with respect to X and y_0 is*

$$d(\phi, X, y_0) = \int_X (\mu \circ \phi)(x) J_\phi(x)\ dx$$

It is not obvious that $d(\phi, X, y_0)$ is well-defined. To see this, we check two claims which we present in the following proposition.

Proposition 3.1. *Let ϕ, X and y_0 be as above.*

(i) There indeed exists an admissible μ for y_0 and ϕ.

(ii) If μ and ν are admissible for y_0 and ϕ then

$$\int_X (\mu \circ \phi)(x) J_\phi(x)\ dx = \int_X (\nu \circ \phi)(x) J_\phi(x)\ dx.$$

Proof. Let ϕ, X and y_0 be as above.

(i) We explicity give such a μ. First define

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

where $C > 0$ is taken so that $\int_{\mathbb{R}^n} \mu(x)\ \mathrm{d}x = 1$ in the proceding definition of μ. Also, for $\varepsilon > 0$, define the family of functions

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\left(\frac{x}{\varepsilon}\right)$$

Now $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ for each $\varepsilon > 0$, $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\ \mathrm{d}x = 1$ and supp $\eta_\varepsilon = B_\varepsilon(0)$.

Then μ is given by $\mu(x) = \eta_\varepsilon(x - y_0)$ where $\varepsilon > 0$ is taken so that $B_\varepsilon(y_0)$ lies in $\mathbb{R}^n \setminus \phi(\partial X)$.

(ii) This is a consequence of Green's theorem and the proof is given in [2] in terms of manifolds and differential forms, which is beyond the scope of this project.

□

3.3 Properties of Brouwer Degree in $\mathbb{R}^n$

The properties and their respective proofs of Brouwer degree are found in [2].

Proposition 3.2. *For y_1 sufficiently close to y_0, there holds*

$$d(\phi, X, y_0) = d(\phi, X, y_1)$$

Proof. If μ is admissible for y_0 then it is also admissible for sufficiently close y_1 which lies in $B_\varepsilon(y_0)$.

This tells us that the degree of a map is constant as we vary y_0 over any connected component C of $\mathbb{R}^n \setminus \phi(\partial X)$. This constant value is sometimes written as $d(\phi, X, C)$. $\square$

Proposition 3.3. *If y_0 is a regular point of ϕ then*

$$d(y_0) = d(\phi, X, y_0)$$

As a consequence, we see that for $y_0 \notin \phi(\bar{X})$ (making y_0 a regular value of ϕ), there holds $d(\phi, X, y_0) = 0$.

Proof. Let $\phi^{-1}(y_0) = \{x_1, ..., x_k\}$ be the finite preimage of y_0 (discussed earlier). There exists disjoint neighbourhoods N_i of x_i such that ϕ is injective on each N_i. Set

$$N := \bigcap_{i=1}^{k} \phi(N_i)$$

which is a neighbourhood of y_0, since it is the finite intersection of neighbourhoods of x_i under a continuous map.

Let μ be admissible with support in N, so that

$$\begin{aligned} d(\phi, X, y_0) &= \int_X (\mu \circ \phi)(x) J_\phi(x) \, \mathrm{d}x \\ &= \sum_{j=1}^{k} \int_{N_j} (\mu \circ \phi)(x) J_\phi(x) \, \mathrm{d}x \\ &= \sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) \cdot \int_{\mathbb{R}^n} \mu(x) \, \mathrm{d}x \\ &= \sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) \\ &= d(y_0) \end{aligned}$$

$\square$

These previous two propositions tell us that $d(\phi, X, y_0)$ must be an integer equal to $d(y_1)$ for any regular value $y_1 \in C$ where C is any connected component lying in $\mathbb{R}^n \setminus \phi(\partial X)$.

Proposition 3.4. *Consider the continuous map $\phi : [0,1] \times \bar{X} \to \mathbb{R}^n$ and denote $\phi_t(x) := \phi(t,x)$. Then let $\phi_t \in C^1(X)$ for all $t \in [0,1]$. Also let $y_0 \notin \phi_t(\partial X)$ for all $t \in [0,1]$. Then $d(\phi_t, X, y_0)$ is independent of t.*

Proof. The set of points $Y = \{\phi_t(x) : x \in \partial X, t \in [0,1]\}$ is closed. Also note that, since $y_0 \notin \phi_t(\partial X)$ for all $t \in [0,1]$, we have $y_0 \notin Y$.

We choose an admissible μ with support in a small neighbourhood of y_0 disjoint from Y. This can be done since Y is closed. Then we have

$$d(\phi_t, X, y_0) = \int_X (\mu \circ \phi_t)(x) J_{\phi_t}(x) \, \mathrm{d}x$$

This is continuous as a function of t. Since Brouwer degree is an integer, for $d(\phi_t, X, y_0)$ to be continuous in t, it must be constant. □

Proposition 3.5. *Suppose X_i, with $i = 1, 2, ...$ is a sequence of disjoint open sets with $X_i \subset X$ for each i. Also let*

$$y_0 \notin \phi\left(\bar{X} \setminus \bigcup_{i=1}^{\infty} X_i\right)$$

Then $d(\phi, X_i, y_0)$ is zero, except for finitely many i, and

$$d(\phi, X, y_0) = \sum_{i=1}^{\infty} d(\phi, X_i, y_0)$$

Proof. Since $\phi(\bar{X} \setminus \bigcup_{i=1}^{\infty} X_i)$ is closed, there exists a neighbourhood N of y_0 disjoint from $\phi(\bar{X} \setminus \bigcup_{i=1}^{\infty} X_i)$ and let y be a regular value in N. Then there holds

$$d(\phi, X, y_0) = d(\phi, X, y), \quad d(\phi, X_i, y_0) = d(\phi, X_i, y)$$

Since y has a finite number of preimages, $\phi^{-1}(y)$ is contained in a finite number of the X_i's and the result follows immediately from proposition 3.3. □

In particular, we have

Proposition 3.6. *If $K \subset \bar{X}$ is closed and $y_0 \notin \phi(K) \cup \phi(\partial X)$, then*

$$d(\phi, X, y_0) = d(\phi, X \setminus K, y_0)$$

Proof. Let $X_i = X \setminus K$ for all $i = 1, 2, ...$ and apply the previous proposition. □

Proposition 3.7. *If ϕ is injective and preserves (reverses) the orientation of X, then at any point $y_0 \in \phi(X)$, $y_0 \notin \phi(\partial X)$, we have*

$$d(\phi, X, y_0) = 1 \ (or \ -1)$$

Proof. This follows directly from the definition of degree. In particular, if $\phi = \mathrm{Id}$ (or $-\mathrm{Id}$), then for $y_0 \in \phi(X) \cap (\mathbb{R}^n \setminus \phi(\partial X))$, we have

$$d(\phi, X, y_0) = 1 \text{ (or } (-1)^n)$$

$\square$

Proposition 3.8. *Suppose $\partial X = \varnothing$. Then $d(\phi, X, y)$ is defined for every $y \in \mathbb{R}^n$ and it is equal to 0.*

Proof. X is compact so that $\phi(X)$ is compact, so there is a point $y_0 \in \mathbb{R}^n$ that is not in $\phi(X)$. But then $d(\phi, X, y_0) = 0$ and $d(\phi, X, y)$ is independent of y which gives us the result. $\square$

We can extend degree to maps $\phi : \bar{X} \to \mathbb{R}^n$ that are simply continuous.

Definition 3.4. *Let $\phi : \bar{X} \to \mathbb{R}^n$ be continuous. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of $C^1(\bar{X})$ functions converging uniformly to ϕ. Then for every $p \notin \phi(\partial X)$, we consider the sequence*

$$\{d(\phi_n, X, p)\}_{n=1}^{\infty}$$

This sequence converges and its limit is not dependent on the choice of $\{\phi_n\}_{n=1}^{\infty}$. Then we define

$$d(\phi, X, p) = \lim_{n \to \infty} d(\phi_n, X, p)$$

to be the degree of ϕ at p.

We need to justify the claims made in this definition. The following justification is outlined in [4]. Let $\mathrm{dist}(x_0, X) = \inf\{|x_0 - x| : x \in X\}$ denote the distance between a point and set. Let $d = \mathrm{dist}(p, \phi(\partial X))$ which is nonzero since $p \notin \phi(\partial X)$.

Since $\phi_n \to \phi$ uniformly, choose N sufficiently large so that if $n > N$ then $\|\phi_n - \phi\|_\infty \leq \frac{1}{2}d$, where $\|\cdot\|_\infty$ denotes the uniform norm $\|f\|_\infty = \sup\{|f(x)| : x \in \mathrm{domain}(f)\}$.

p is not contained in any ball centered at $\phi(x)$, $x \in \partial X$, with radius $\frac{1}{2}d$ so p cannot be expressed as

$$p = t\phi_n(x) + (1-t)\phi_m(x)$$

for any $t \in [0,1]$, $x \in \partial X$, $n, m > N$, since $\phi_n(x)$ and $\phi_m(x)$ belong to some such balls.

Then fix $n, m > N$ and apply Proposition 3.4 to see that eventually

$$d(\phi_n, X, p) = d(\phi_m, X, p)$$

In other words, the desired limit exists and is independent of choice of $\{\phi_n\}_{n=1}^{\infty}$ since we took an arbitrary such sequence above.

For continuous ϕ, all properties of degree in this chapter, whose formulation makes sense, hold for ϕ.

4 Leray-Schauder Degree for Banach Spaces

We wish to extend our notion of degree to infinite dimensional spaces. The approach taken here is that of [2]. However, details are light and so we refer to [5] which handles Leray-Schauder degree in an identical setting, but with more detail.

We consider a real Banach space X, an open and bounded $\Omega \subset X$, continuous operators $\phi : \bar{\Omega} \to X$ of the form

$$\phi = I - K$$

where I is the identity operator and K is some compact linear operator (to be defined) and $y_0 \notin \phi(\partial\Omega)$.

Definition 4.1. *Let A and B be real normed spaces. A continuous linear operator $K : A \to B$ is a compact linear operator if $\overline{K(C)}$ is compact in B for every bounded $C \subset A$.*

4.1 Leray-Schauder Degree

We seek to define $d(\phi, \Omega, y_0)$. We construct a family of mappings, K_ε, that approximate K and which have finite dimensional range. Then, we can define $d(\phi, \Omega, y_0)$ using the degree of $I - K_\varepsilon$ relative to an appropriate finite dimensional subset of X.

Proposition 4.1. *Let A and B be real normed spaces, $M \subset A$ be bounded and $K : M \to B$ compact. For all $\varepsilon > 0$, there exists a continuous map $K_\varepsilon : M \to B$ whose range, $K(M)$, is finite dimensional (ie. has a finite basis), with*

$$\|Ku - K_\varepsilon u\| < \varepsilon$$

for all $u \in M$.

Proof. $M \subset A$ is bounded so that $\overline{K(M)}$ is compact since K is compact. Then it is covered by a finite collection of balls in B with radius ε. Denote this collection by $B_\varepsilon(v_i)$, $i = 1, ..., N$, with each $v_i \in \overline{K(M)}$.

For each i, we define $m_i : M \to [0, \infty)$ by

$$m_i(x) = \max\{0, \varepsilon - \|Kx - v_i\|\}.$$

We remember K is a continuous linear operator so each m_i is continuous in x. For a given $x \in M$, there is some i such that $m_i(x)$ is nonzero. In particular, choose i such that $Kx \in B_\varepsilon(v_i)$ so that $\|Kx - v_i\| < \varepsilon$.

This prevents division by 0 when we define

$$\theta_i(x) = \frac{m_i(x)}{\sum_{k=1}^{N} m_k(x)}$$

for each $i = 1, ..., N$. Now θ_i must be continuous too, as it's the sum and quotient of continuous m_i. If we fix x and sum $\theta_i(x)$ over i, we obtain

$$\sum_{i=1}^{N} \theta_i(x) = \sum_{i=1}^{N} \frac{m_i(x)}{\sum_{k=1}^{N} m_k(x)} = \frac{1}{\sum_{k=1}^{N} m_k(x)} \sum_{i=1}^{N} m_i(x) = 1$$

Also, $\theta_i(x) = 0$ if and only if $m_i(x) = 0$. Then $m_i(x) = 0$ precisely when x is such that $\|Kx - v_i\| \geq \varepsilon$ which happens only when $Kx \notin B_\varepsilon(v_i)$ so we see that

$$\text{supp } \theta_i = \overline{K^{-1}(B_\varepsilon(v_i))}$$

Finally, define $K_\varepsilon : M \to B$ by

$$K_\varepsilon(x) = \sum_{i=1}^{N} \theta_i(x) v_i$$

Again, K is continuous, and writing K_ε in the form above allows us to readily see that $T_\varepsilon(M)$ is contained in a finite dimensional linear space spanned by $\{v_1, ..., v_N\}$. Further, there holds

$$\|Kx - K_\varepsilon x\| = \left\| Kx - \sum_{i=1}^{N} \theta_i(x) v_i \right\| = \left\| \sum_{i=1}^{N} \theta_i(x)(Kx - v_i) \right\|$$

Then $\theta_i(x) = 0$ except for x such that $\|Kx - v_i\| < \varepsilon$. This, and remembering $\sum_{i=1}^{N} \theta_i(x)$, tells us that

$$\|Kx - K_\varepsilon x\| = \left\| \sum_{i=1}^{N} \theta_i(x)(Kx - v_i) \right\| = \sum_{i=1}^{N} \|\theta_i(x)(Kx - v_i)\| \ < \varepsilon$$

as required. □

We denote the finite dimensional linear space spanned by $\{v_1, ..., v_N\}$ by N_ε. Additionally, we write ϕ_ε to mean $I - K_\varepsilon$. We're now ready to define $d(\phi, \Omega, y_0)$.

Definition 4.2. *Let X be a real Banach space, $\Omega \subset X$ be open and bounded, $\phi : \bar{\Omega} \to X$ be a continuous operator with $\phi = I - K$ for some compact K, and $y_0 \notin \phi(\partial\Omega)$. Then set $\delta = dist(y_0, \partial\Omega)$ and $\varepsilon < \frac{\delta}{2}$.*

The Leray-Schauder degree of ϕ is given by

$$d(\phi, \Omega, y_0) = d_B(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0)$$

where d_B denotes the Brouwer degree of ϕ_ε, which is defined since $N_\varepsilon \cap \Omega$ is a finite dimensional linear subspace of Ω.

Now it is unclear that $d(\phi, \Omega, y_0)$ is independent on choice of K_ε. We show that this is the case by employing a proposition of Nirenberg (Proposition 1.8.1 in [2]), stated below without proof.

Proposition 4.2. *Take $\Omega \subset \mathbb{R}^n$ to be open and bounded. Consider $\mathbb{R}^n$ as the direct sum of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ where $n_1 + n_2 = n$. That is, for each $x \in \mathbb{R}^n$, there uniquely corresponds $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ such that $x = x_1 + x_2$. Consider the map $F : \bar{\Omega} \to \mathbb{R}^n$ given by*

$$F(x) = x + \phi(x)$$

where $\phi : \bar{\Omega} \to \mathbb{R}^{n_1}$ is some map. If $y \in \mathbb{R}^{n_1}$ and $y \notin F(\partial\Omega)$ then

$$d(F, \Omega, y) = d(F|_{\Omega \cap \mathbb{R}^{n_1}}, \Omega \cap \mathbb{R}^{n_1}, y)$$

To see that degree is independent of K_ε, take K_η to be another approximation of K. Now let $\hat{N}$ be a finite dimensional space containing N_ε and N_η. Then, by our proposition,

$$d(\phi_\varepsilon, N_\varepsilon \cap \Omega, y_0) = d(\phi_\varepsilon, \hat{N} \cap \Omega, y_0)$$
$$d(\phi_\eta, N_\eta \cap \Omega, y_0) = d(\phi_\eta, \hat{N} \cap \Omega, y_0)$$

Then we take the homotopy $(t, x) \mapsto t\phi_\varepsilon(x) + (1 - t)\phi_\eta(x)$ and homotopy invariance of Brouwer degree tells us that $d(\phi_\varepsilon, \hat{N} \cap \Omega, y_0) = d(\phi_\eta, \hat{N} \cap \Omega, y_0)$.

Now all results presented for finite dimensional degree theory can be extended to our maps $\phi = I - K$ by taking approximations ϕ_ε.

4.2 A Class of Compact Operators

We first define a particular type of operator and demonstrate that it yields some nice examples of compact operators.

Definition 4.3. *Let K be some function. Then the integral operator T with kernal K is given by*

$$Tu(s) = \int K(s, t)u(t)\ dt$$

Note that this is a formal definition and the integral need not make sense for all K and u. We give some a concrete example of a family of compact operators.

Example 4.1. *Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Consider the operator $T : C[0, 1] \to C[0, 1]$ given by*

$$Tu(s) = \int_0^1 K(s, t)u(t)\ dt$$

We have a product of continuous functions so this integral is defined. The fundamental theorem of calculus tells us that Tu is continuous in s so T is indeed a map from $C[0, 1]$ to $C[0, 1]$. Linearity of the integral tells us that T must be linear.

To see that T is compact, we refer to the Arzela-Ascoli theorem in [3] which is a result sufficient to show compactness.

Roughly, we take $\{u_n\}_{n=1}^\infty$ to be a sequence in $C[0, 1]$ bounded by some $M > 0$ (under the uniform norm). Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(s, t) - K(s', t')| < \varepsilon$ whenever

$|(s,t)-(s',t')|<\delta$. *So* $\{Tu_n\}_{n=1}^{\infty}$ *is a sequence of continuous functions such that*

$$|Tu_n(s)-Tu_n(s')| \leq \int_0^1 |K(s,t)-K(s',t)||u_n(t)| \; dt \leq \varepsilon \int_0^1 |u_n(y)| \; dy \leq \varepsilon \|u_n\| \leq M\varepsilon$$

whenever $|s-s'|<\delta$.

We have shown $\{Tu_n\}_{n=1}^{\infty}$ *to be a family in* $C[0,1]$ *satisfying 'equicontinuity'. Then Arzela-Ascoli tells us that there exists a subsequence* $\{Tu_{n_k}\}_{k=1}^{\infty}$ *that converges uniformly to a continuous function.*

Uniform convergence implies convergence in $C[0,1]$ *so* $\{Tu_{n_k}\}_{k=1}^{\infty}$ *converges. Then* T *is compact because the image of any bounded sequence contains a convergent subsequence (see 'sequential compactness' in [8]).*

5 Applications

5.1 Brouwer Fixed Point Theorem

We now present a primary application of degree theory - Brouwer's fixed point theorem. We first need some lemmas before we can prove this result.

We remind ourselves of a property of Brouwer degree.

Proposition 5.1. *If $X \subset \mathbb{R}^n$ is open and bounded, $\phi : \bar{X} \to \mathbb{R}^n$ is continuous, $K \subset \bar{X}$ is closed and $y_0 \notin \phi(K) \cup \phi(\partial X)$, then*

$$d(\phi, X, y_0) = d(\phi, X \setminus K, y_0)$$

We use this to prove the following proposition.

Proposition 5.2. *Let X, ϕ and K be as above, $p \notin f(\bar{X})$. Then*

$$d(\phi, X, p) = 0$$

Proof. Take $K = \bar{X}$ in the previous proposition to see that

$$d(\phi, X, p) = d(\phi, X \setminus \bar{X}, p) = d(\phi, \varnothing, p) = 0$$

□

An important consequence is that if $d(\phi, X, p)$ is defined and nonzero then $\phi(x) = p$ for at least one $x \in X$. We now use this fact in the following lemma.

Lemma 5.1. *If $g : \overline{B_r(0)} \to \overline{B_r(0)}$ is continuous then there exists $x \in \overline{B_r(0)}$ such that $g(x) = x$ (ie. g has a fixed point).*

Proof. Set $f(x) = x - g(x)$. Now fixed points of g correspond to solutions of f. If f has a solution on $\partial B_r(0)$ then we're done. So assume $f(x) \neq 0$ on $\partial B_r(0)$.

Note that $d(\mathrm{Id}_{B_r(0)}, B_r(0), 0) = 1$ and set

$$H(t, x) = tf(x) + (1-t)\mathrm{Id}_{B_r(0)}(x) = t(x - g(x)) + (1-t)x = x - tg(x)$$

H is continuous in t and x since it is the sum and product of continuous functions and $H(0, x) = \mathrm{Id}_{B_r(0)}(x)$ and $H(1, x) = f(x)$ so H is a homotopy from $\mathrm{Id}_{B_r(0)}$ to f.

Then homotopy invariance of degree tells us that

$$d(f, B_r(0), 0) = d(\mathrm{Id}_{B_r(0)}, B_r(0), 0) = 1$$

and we see that $f(x) = 0$ must have at least one solution on $B_r(0)$ as a consequence of the previous proposition. This means g always has a fixed point as desired. □

As we approach the statement and proof of Brouwer's fixed point theorem, we require one last definition.

Definition 5.1. *Let $S \subset \mathbb{R}^n$. A continuous map $R : \mathbb{R}^n \to S$ is a retraction of $\mathbb{R}^n$ onto S if $R|_S = Id_S$. That is, if $R(x) = x$ for all $x \in S$.*

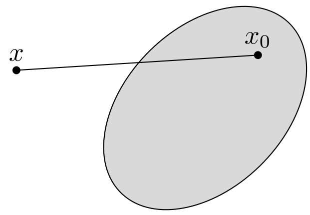
Lemma 5.2. *If $C \subset \mathbb{R}^n$ is closed and convex then there exists a retraction of $\mathbb{R}^n$ onto C.*

Proof. We give such a retraction. Fix $x_0 \in C$. Then we have a retraction $R : \mathbb{R}^n \to C$ given by

$$R(x) = \begin{cases} x, & \text{if } x \in C \\ y, & \text{if } x \notin C \end{cases}$$

where we give a description of y now.

Consider the unique line segment connecting our fixed x_0 and input x.



We consider the intersection between the boundary of C and this line segment which we denote Γ. That is,

$$\Gamma = \{x(1-t) + tx_0 : t \in [0,1]\} \cap \partial C$$

Then C being closed ensures that ∂C is nonempty and convexivity of C ensures that this line segment uniquely intersects the boundary. Then we define y to be the unique element of Γ.

It remains to be shown that R is continuous but we can easily see this since small variations of x will result in small variations of the intersection between x and x_0. $\square$

Now we're ready for Brouwer's fixed point theorem.

Theorem 5.1. *Let $C \subset \mathbb{R}^n$ be closed, bounded and convex. If $g : C \to C$ is continuous then there exists a fixed point of g.*

Proof. Boundedness of C tells us that there exists $r > 0$ such that $C \subset B_r(0)$ (ie. C is contained in some ball centered at the origin). The previous lemma ensures existence of a retraction of $\mathbb{R}^n$ onto C, R.

Now set $G = g \circ R : \overline{B_r(0)} \to C \subset \overline{B_r(0)}$. This is a well defined map and continuous since its the composition of continuous g and R.

Then lemma 5.1 tells us that there exists a fixed point of G in $\overline{B_r(0)}$. That is, $G(x) = g(R(x)) = x$ for some $x \in \overline{B_r(0)}$. Since im G lies in C, there exists $x' \in C$ with $G(x') = x'$. However $R = \mathrm{Id}$ on C so that $x' = G(x') = g(R(x')) = g(x')$ and we have our desired fixed point of g. □

We recall that topological spaces X and Y are homeomorphic if there exists a homeomorphism $f : X \to Y$ (continuous bijection with continuous inverse). We use the properties of homeomorphisms to extend the theorem of Brouwer.

Theorem 5.2. *Let $C \subset \mathbb{R}^n$ be closed, bounded and convex. Let $\tilde{C}$ be homeomorphic to C. Then $\tilde{C}$ also enjoys the fixed point property. That is, if $f : \tilde{C} \to \tilde{C}$ is continuous then f has a fixed point.*

Proof. To see this, let $h : C \to \tilde{C}$ be a homeomorphism and $f : \tilde{C} \to \tilde{C}$ be continuous. The map $F : C \to C$ given by $F(x) = h^{-1}(f(h(x)))$ is continuous since it's the composition of continuous h^{-1}, f and h.

Brouwer's fixed point theorem tells us that F has some fixed point, say, $x_0 \in C$. Then

$$F(x_0) = x_0 \Leftrightarrow h^{-1}(f(h(x_0))) = x_0 \Leftrightarrow f(h(x_0)) = h(x_0)$$

Then set $y_0 = h(x_0)$ so that $f(y_0) = y_0$ as required. □

5.2 Schauder Fixed Point Theorem

Brouwer's fixed point theorem asserts that any continuous map on a closed, bounded and convex subset of $\mathbb{R}^n$ possesses a fixed point. Unfortunately such a result does not hold true for infinite dimensional spaces. We consider the following example illustrating this failure.

Example 5.1. *Consider l^2, whose elements are complex sequences $\{z_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |z_n|^2$ converges. We write $\{z_n\}_{n=1}^{\infty} = z = (z_1, z_2, ...)$. We can impose the norm $\|z\| = \sum_{n=1}^{\infty} |z_n|^2$.*

Let B be the closed unit ball in l^2 and $f : B \to B$ be given by

$$f(z) = (\sqrt{1 - \|z\|^2}, z_1, z_2, ...)$$

We claim this is continuous without proof. Suppose $z = (z_1, ...)$ is a fixed point of f. Then $\|z\| = 1$ since $\|f(z)\| = 1$ for all $z \in B$. On the other hand, if $z = (\sqrt{1 - \|z\|^2}, z_1, z_2, ...)$ then $z_1 = 0$, $z_2 = z_1$, $z_3 = z_2$ and so on so that $z = (0, 0, 0, ...)$ and $\|z\| = 0$, a contradiction.

We present a theorem of Nirenberg.

Theorem 5.3. *Let Ω be a closed and bounded subset of a Banach space X. Then $f : \Omega \to X$ is compact if and only if f is a uniform limit of finite dimensional mappings (ie. mappings with ranges contained in a finite dimensional subspace).*

Before presenting this proof, we require a definition.

Definition 5.2. *The convex hull of a set A is the smallest convex set that contains A. Here, smallest means that there does not exist a convex subset B of the convex hull such that $A \subset B$.*

Proof. Let f be compact. Then $\overline{f(\Omega)}$ is compact in X. For $\varepsilon > 0$, we cover this compact set with a finite collection of open balls, written $B_\varepsilon(x_i)$, $i = 1, ..., J$ with each $x_i \in \overline{f(\Omega)}$.

We consider a 'partition of unity' on $\overline{f(\Omega)}$. That is, a collection $\psi_i : \overline{f(\Omega)} \to [0, \infty)$ such that

$$\sum_{i=1}^{J} \psi_i(x) = 1, \psi_i = 0 \text{ off of } B_\varepsilon(x_i)$$

Now set

$$f_\varepsilon(x) = \sum_{i=1}^{J} \psi_i(f(x)) x_i$$

Then $f_\varepsilon(x)$ is in the convex hull of $\{x_1, ..., x_J\}$. Also, much like in our construction of K_ε for Leray-Schauder degree,

$$\|f(x) - f_\varepsilon(x)\| = \left\| \sum_{i=1}^{J} \psi_i(f(x))(x_i - f(x)) \right\|$$

If $\psi_i(f(x)) > 0$ then $f(x) \in B$ and $\|x_i - f(x)\| < \varepsilon$ so $\|f - f_\varepsilon\| < \varepsilon$ uniformly in x. $\square$

We can now prove the Schauder fixed point theorem.

Theorem 5.4. *Let Ω be a closed, convex and bounded subset of a Banach space X. If $f : \Omega \to \Omega$ is compact then f has a fixed point.*

Proof. Let f_ε be an approximation of f as above and $N_\varepsilon = \text{span}\{x_1, ..., x_J\}$.

Ω is convex and $f_\varepsilon(\Omega)$ is contained in the convex hull of $f(\Omega)$, so we have $f_\varepsilon : \Omega \to \Omega \cap N_\varepsilon$.

So f_ε maps a closed and bounded set, $N_\varepsilon \cap \Omega$, into itself. Brouwer's fixed point theorem tells us that we have a fixed point, x_ε.

We send $\varepsilon \to 0$. Compactness tells us that $f_\varepsilon(x_\varepsilon)$ has a convergent subsequence which we denote $f_\varepsilon(x_\varepsilon)$.

Then $x_\varepsilon = f_\varepsilon(x_\varepsilon)$ limits to some x_0. But

$$\|x_\varepsilon - f(x_\varepsilon)\| = \|f_\varepsilon(x_\varepsilon) - f(x_\varepsilon)\| \leq \varepsilon$$

so $f(x_\varepsilon) \to x_0$ and $f(x_0) = x_0$ as desired. $\square$

5.3 Fixed Points of Integral Operators

Recall our brief study of integral operators. We extend this to a nonlinear example.

Example 5.2. *Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous and bounded by $M > 0$. Then consider the map $T : B_r \to C[0,1]$ where*

$$B_r = \{u \in C[0,1] : \|u\|_\infty \leq r\}$$

given by

$$Tu(s) = \int_0^1 K(s,t) f(t, u(t)) \; dt$$

Then T is a compact linear operator for any $r > 0$.

Proposition 5.3. *T has a fixed point. That is, there exists some $u \in C[0,1]$ such that $Tu(s) = u(s)$.*

Proof. Take $u \in C[0,1]$ and we have

$$\begin{aligned}
|Tu(s)| &= \left| \int_0^1 K(s,t) f(t, u(t)) \; \mathrm{d}t \right| \\
&\leq \int_0^1 |K(s,t) f(t, u(t))| \; \mathrm{d}t \\
&= \int_0^1 |K(s,t)||f(t, u(t))| \; \mathrm{d}t \\
&\leq M \cdot \sup_{s,t} |K(s,t)|
\end{aligned}$$

So T is a compact map of the ball $\{u \in C[0,1] : \|u\| \leq M \cdot \sup_{s,t} |K(s,t)|\}$ into itself and the Leray-Schauder fixed point theorem gives us the result. □

6 Appendix 1 - Calculus

The following definitions are found in [6, 7, 8].

Let $X \subset \mathbb{R}^n$ be open and bounded.

Definition 6.1. *The class of functions $f : X \to \mathbb{R}^n$ that are continuous on X is denoted $C^0(X;\mathbb{R}^n)$. We omit mention of the codomain and write $C^0(X)$ when there's no possibility of confusion.*

Definition 6.2. *The class of functions $f : X \to \mathbb{R}^n$ that have continuous k^{th} derivative on X is denoted $C^k(X)$. The class of functions $f : \bar{X} \to \mathbb{R}^n$ with uniformly continuous k^{th} derivative on X is denoted $C^k(\bar{X})$.*

Definition 6.3. *Let $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^k$ be continuously differentiable (i.e., $f \in C^1(\mathbb{R}^n)$) with $f = (f_1, f_2, ..., f_k)$ for $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, ..., k$. Then the Jacobian matrix of f at x_0 is the matrix*

$$J_f(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \dots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(x_0) & \dots & \frac{\partial f_k}{\partial x_n}(x_0) \end{pmatrix}$$

If $n = k$ then we can consider the (Jacobian) determinant of $J_f(x_0)$ which we write as simply $f'(x_0)$.

Definition 6.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be continuously differentiable.*

(i) A point $x_0 \in \mathbb{R}^n$ is a regular point if the Jacobian matrix of f at x_0, $J_f(x_0)$, has maximal rank. That is, when $J_f(x_0)$ is expressed in row echelon form, the number of nonzero rows is equal to $min(n, k)$.

(ii) If x_0 is not a regular point then we call it a critical point.

(iii) A point $y_0 \in \mathbb{R}^k$ is called a critical value if its preimage $f^{-1}(y_0)$ contains a critical point. That is, if there exists a critical point x_0 such that $f(x_0) = y_0$.

(iv) If y_0 is not a critical value then we call it a regular value.

(v) If $n = m$ then regular points correspond to x_0 such that $J_f(x_0)$ is invertible and critical points correspond to x_0 such that $J_f(x_0)$ is not invertible.

7 Appendix 2 - Topology

We present relevant material taken from [9].

7.1 Topological Spaces

Definition 7.1. *A topology on a set X is a collection of subsets τ such that*

(i) $\varnothing, X \in \tau$.

(ii) $\{U_i\}_{i \in I} \subset \tau$ *for some arbitrary index set I implies* $\bigcup_{i \in I} U_i \in \tau$.

(iii) $\{U_i\}_{i=1}^n$ *for some $n \in \mathbb{N}$ implies* $\bigcap_{i=1}^n U_i \in \tau$.

If $U \in \tau$ then we say U is open in X and $U^c = X \setminus U$ is closed in X.

Definition 7.2. *Let $\{(X_i, \tau_i)\}_{i=1}^n$ be a finite family of topological spaces, $X = \prod_{i=1}^n X_i$ (interpreted as a Cartesian product) and define the i^{th} projection π_i by*

$$\pi_i : X \to X_i, \quad \pi_i(x_1, x_2, ..., x_i, ..., x_n) = x_i.$$

Then the product topology on X is the topology generated by

$$\bigcup_{i=1}^{n} \{\pi_i^{-1}(U) : U \text{ is open in } X_i\}.$$

Definition 7.3. *Let X be a topological space and $A \subset X$. Then the subspace topology on A is given by*

$$\{A \cap U : U \text{ is open in } X\}.$$

Definition 7.4. *Suppose X is a topological space and $A \subset X$. The interior of A is the union of all open subsets of A. The closure of A is the intersection of all closed supersets of A.*

Definition 7.5. *A topological space X is Hausdorff if, for each pair of distinct points $x_1, x_2 \in X$, there exist disjoint open sets U_1 and U_2 containing x_1 and x_2 respectively.*

Remark 7.1. *Sequence limits, if they exist, are unique in a Hausdorff space.*

7.2 Continuous Functions

Definition 7.6. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \to Y$. Then*

(i) f is continuous if $f^{-1}(V) \in \tau_X$ for all $V \in \tau_Y$.

(ii) f is a homeomorphism if f is bijective and both f and its inverse, $f^{-1} : Y \to X$, are continuous. Then we say that X and Y are homeomorphic.

Remark 7.2. *Constant functions are continuous, inclusion maps ($f : A \subset X \to X$, $f(a) = a$) are continuous and continuity is preserved under composition of functions.*

Remark 7.3. *Let $f : X \to \prod_{i=1}^n X_i$ be given by $f(x) = (f_1(x), ..., f_n(x))$. Then f is continuous $\Leftrightarrow$ all the f_i's, given by $f_i : X \to X_i$, are continuous.*

7.3 Connectedness and Compactness

Definition 7.7. *A seperation of a topological space X is a pair of nonempty and disjoint open sets U and V such that $U \cup V = X$. We say X is connected if there does not exist a seperation of X.*

Remark 7.4. *If $f : X \to Y$ is continuous and X is connected then its image is connected.*

Definition 7.8. *Given x and y in a topological space X, paths in X from x to y are continuous maps $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. We say X is path connected if there exists a path between any two points in X.*

Definition 7.9. *Let X be a topological space. A collection of sets $\{U_i\}$ covers X if*

$$X \subset \bigcup_i U_i$$

We call $\{U_i\}$ an open cover of X if all sets in $\{U_i\}$ are open.

Definition 7.10. *A topological space X is compact if, for all open covers $\{U_i\}$, there exists a finite subcollection that also covers X.*

Remark 7.5. *Images of compact spaces under continuous maps are compact.*

Remark 7.6. *If $f : X \to Y$ is bijective and continuous, X is compact and Y is Hausdorff, then f is in fact a homeomorphism.*

Remark 7.7 (Heine-Borel). *For $S \subset \mathbb{R}^n$, S is compact $\Leftrightarrow$ S is closed and bounded.*

8 Appendix 3 - Functional Analysis

We present relevant material taken from [10].

8.1 Metric, Normed and Banach Spaces

Definition 8.1. *A metric space is a pair (X, d) such that $d : X \times X \to [0, \infty)$ that satisfies:*

(i) $d(x, y) = 0 \Leftrightarrow x = y$.

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Definition 8.2. *A mapping $f : X \to Y$ between metric spaces (X, d_x) and (Y, d_Y) is continuous at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \varepsilon$. f is continuous on X if its continuous at every $x_0 \in X$.*

Definition 8.3. *A norm on a vector space V over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a map $\| \cdot \| : V \to [0, \infty)$ that satisfies:*

(i) $\|x\| = 0 \Leftrightarrow x = 0$.

(ii) $\|\alpha x\| = |a|\|x\|$ *for all* $x \in V$ *and* $\alpha \in \mathbb{K}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$.

The pair $(V, \| \cdot \|)$ is called a normed space.

Remark 8.1. *A normed space $(V, \| \cdot \|)$ induces a metric d via $d(x, y) = \|x - y\|$. This means every normed space induces a metric space.*

Definition 8.4. *A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is Cauchy if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ dependent only on ε such that $d(x_m, x_n) < \varepsilon$ for every $m, n > N$. A metric space is complete if every Cauchy sequence converges to some $x_0 \in X$.*

Remark 8.2. *Convergence of a sequence in a metric space always implies the sequence is Cauchy since we can take N such that $d(x, x_n) < \frac{\varepsilon}{2}$ for all $n > N$ so that*

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Definition 8.5. *A normed space $(V, \| \cdot \|)$ is a Banach space if the metric space induced by its norm is complete.*

8.2 Linear Operators

Definition 8.6. *A linear operator between two vector spaces X and Y over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a map $T : X \to Y$ such that $T(\alpha x + y) = \alpha T(x) + T(y)$ for all $x, y \in X$ and $\alpha \in \mathbb{K}$. We sometimes write Tx for $T(x)$.*

Remark 8.3. *If T, X and Y are as above, then $T(0_X) = 0_Y$.*

Definition 8.7. *A linear operator $T : X \to Y$ between normed spaces X and Y is bounded if there exists $c \in [0, \infty)$ such that $\|Tx\| \leq c\|x\|$ for all $x \in X$ (where the norms are taken in the appropiate spaces).*

Remark 8.4. *If $x = 0$ then we trivially have $\|Tx\| \leq c\|x\|$ so we can take $x \neq 0$ and, equivalently, consider c such that*

$$\frac{\|Tx\|}{\|x\|} \leq c$$

This motivates the following definition.

Definition 8.8. *If $T : X \to Y$ is a linear operator between normed spaces then the operator norm of T is*

$$\|T\|_* = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

If X is the zero normed space (ie. only contains 0) then we define $\|T\|_* = 0$.

Remark 8.5. *Equivalently, we can express the operator norm as*

$$\|T\|_* = \sup_{\|x\|=1} \|Tx\|$$

Remark 8.6. *If $T : X \to Y$ is a linear operator between normed spaces, then T is continuous $\Leftrightarrow$ T is bounded, and if T is continuous at any $x_0 \in X$ then it is continuous on all of X.*

References

[1] Suciu, Alexandru I. *MATH4565 Topology Notes*, Northeastern University, 2010.

[2] Nirenberg, Louis. *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New edition, 1974.

[3] Grotowski, Joseph R. *MATH4401 Advanced Analysis Notes*, University of Queensland, 2005.

[4] Schwartz, J. T. *Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, 1st edition, 1969.

[5] Lloyd, Noel Glynne. *Degree Theory*, Cambridge University Press, 1st Edition, 1978.

[6] Adams, Robert A. Essex, Christopher. *Calculus: A Complete Course*, Pearson Prentice Hall, 2009.

[7] Rudin, Walter. *Principles of Mathematical Analysis*, McGraw-Hill, 3rd edition, 1976.

[8] Pugh, Charles Chapman. *Real Mathematical Analysis*, Springer International Publishing, 2nd edition, 2015.

[9] Munkres, James R. *Topology*, Prentice Hall, 2nd Edition, 2000.

[10] Kreyszig, Erwin. *Introductory Functional Analysis with Applications*, John Wiley & Sons, 1st Edition, 1989.