The Dirichlet Function

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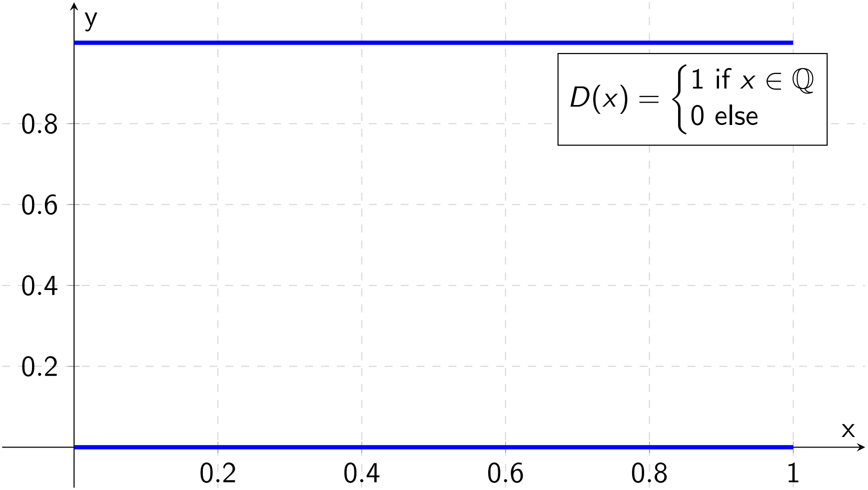
Definition

Named after 19th century German mathematician Johann Peter Gustav Lejeune Dirichlet.

$$D : [0,1] \to \mathbb{R}, \quad D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{else} \end{cases}$$

i.e. if $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ then $D(x) = 1$, else $D(x) = 0$.
e.g. $D(\frac{1}{2}) = 1$, $D(0.34) = 1$, $D(\frac{1}{\sqrt{5}}) = 0$, $D(\frac{\pi}{6}) = 0$.


y
x
0.2
0.4
0.6
0.8
1
$D(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ else} \end{cases}$

Riemann Integration

Definition

A *partition* P of $[a, b] = \{x : a \leq x \leq b\}$ is a finite sequence $(x_0, x_1, ..., x_{n-1}, x_n)$ where

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

This cuts up $[a, b]$ into n *blocks*.

e.g. One partition of $[0, 5]$ is $P = (0, 1, 3.5, 5)$ where $n = 3$ and $x_0 = 0$, $x_1 = 1$, $x_2 = 3.5$ and $x_3 = 5$.

Denote the i^{th} block by $B_i = [x_{i-1}, x_i]$ which has width $W_i = x_i - x_{i-1}$.

Definition

Consider a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$. The *upper sum* of f with respect to P is the number

$$U_{f,P} = \sum_{i=1}^{n} \overbrace{W_i}^{\text{width}} \overbrace{\sup_{x \in B_i} f(x)}^{\text{height}}$$

Similarly, the *lower sum* of f with respect to P is the number

$$L_{f,P} = \sum_{i=1}^{n} W_i \inf_{x \in B_i} f(x)$$

Riemann Integration

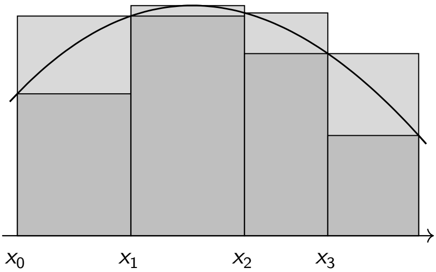


Figure: Upper and lower sums.

Definition

We say f is *Riemann integrable* if, for all $\varepsilon > 0$, there exists a partition P such that

$$U_{f,P} - L_{f,P} < \varepsilon$$

Is It Riemann Integrable?

For any partition P of $[0, 1]$, consider any block B_i of that partition.

We see that D takes values 0 and 1, no matter the block's width.

This is because the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{I}$ are *'dense'* in the real numbers $\mathbb{R}$.

And so, $U_{D,P} = 1$ and $L_{D,P} = 0$ for any partition P , so then

$$U_{D,P} - L_{D,P} = 1 \nless \varepsilon$$

and D is *not* Riemann integrable.

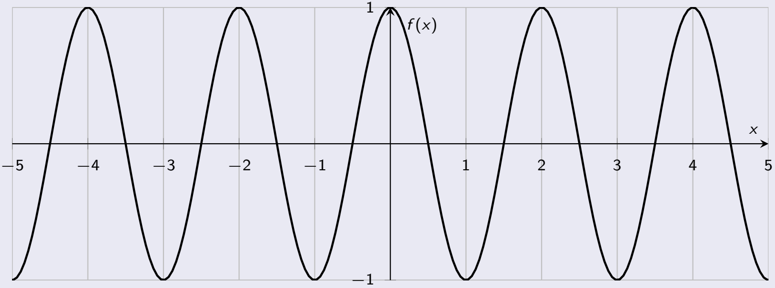
The Dirichlet Function

Closed Form

$$D(x) = \lim_{k\to\infty}\left(\lim_{j\to\infty}\cos^{2j}(k!\pi x)\right)$$

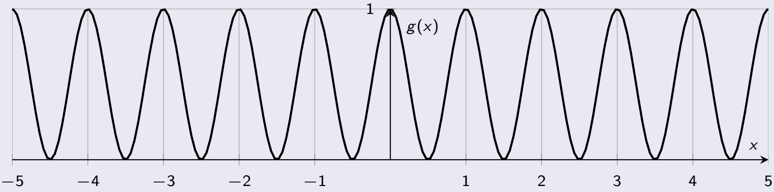
Proof

Consider the function $f(x) = \cos(\pi x)$. For integers k, $f(k) = \pm 1$.



Proof

Take $g(x) = f(x)^2 = \cos^2(\pi x)$ so that $g(k) = 1$ for integers k.



For non-integers q, $|f(q)| = |\cos(\pi q)| < 1$, so $0 \leq g(q) < 1$.

If we take the square many times, $g(q) \to 0$ for non-integers q, while $g(k) \to 1$ for integers k. [Desmos.]

Proof

Then define

$$Z(x) = \lim_{j\to\infty} g(x)^j = \lim_{j\to\infty} \cos^{2j}(\pi x) = \begin{cases} 1, & \text{if } x \in \mathbb{Z} \\ 0, & \text{else} \end{cases}$$

Remembering our proposed expression for D, we now have

$$D(x) = \lim_{k\to\infty}\left(\lim_{j\to\infty} \cos^{2j}(k!\pi x)\right) = \lim_{k\to\infty} Z(k!x)$$

Left To Show

$$D(x) = \lim_{k\to\infty} Z(k!x)$$

Proof

Take $x \in \mathbb{Q}$, then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Take any integer $k > q$. Now consider

$$\begin{aligned} Z(k!x) = Z\left(k! \cdot \frac{p}{q}\right) &= Z\left((1 \cdot 2 \cdot \ldots \cdot q \cdot \ldots \cdot k) \cdot \frac{p}{q}\right) \\ &= Z\Big((1 \cdot 2 \cdot \ldots \cdot 1 \cdot \ldots \cdot k) \cdot p\Big) \\ &= 1 \end{aligned}$$

This means that $Z(k!x) = 1$ for all k greater than q.
Then we know $D(x) = \lim_{k\to\infty} Z(k!x)$ for $x \in \mathbb{Q}$!

Proof

Take $x \notin \mathbb{Q}$. Then $k!x \notin \mathbb{Q}$ for all k.

So there must hold $Z(k!x) = 0$ for all k.

Then we know $\lim_{k \to \infty} Z(k!x) = 0$ for $x \notin \mathbb{Q}$!

So we've shown

$$D(x) = \lim_{j,k \to \infty} \cos^{2j}(k!\pi x)$$

as desired. $\square$

What Have We Done?

- D is not Riemann integrable.
- But $D(x) = \lim_{j,k\to\infty} \cos^{2j}(k!\pi x)$
- $\cos^{2j}(k!\pi x)$ is Riemann integrable for each $j, k = 0, 1, 2, \ldots$.
- Riemann integrability is not closed under limits!

The Monotone Convergence Theorem

Theorem [MCT for Sequences]

Let $(a_n)_{n=1}^{\infty}$ be a monotone and bounded sequence in $\mathbb{R}$. Then it converges to an element of $\mathbb{R}$.

Theorem [MCT for Functions]

Let $(f_n)_{n=1}^{\infty}$ be a monotone and bounded sequence in some function space S. Then it converges to an element of S.

The Dirichlet Function

Let $(a_n)_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} : x \geq 0\}$.

Then define the sequence of functions

$$f_n(x) = \begin{cases} 1, & \text{if } x = a_j \text{ for some } j \leq n \\ 0, & \text{else} \end{cases}$$

Each f_n is Riemann integrable since they each have only finitely many nonzero values.

The Monotone Convergence Theorem

The Dirichlet Function

$$f_n(x) = \begin{cases} 1, & \text{if } x = a_j \text{ for some } j \le n \\ 0, & \text{else} \end{cases}$$

Observe that

$$0 \le f_n(x) \le f_{n+1}(x) \le 1$$

for all n and for all x. So we have a monotone and bounded sequence. Also observe that

$$\lim_{n\to\infty} f_n(x) = D(x)$$