CSCB63 Tutorial 6 — BFS produces shortest paths

An important result about Breadth-First Search is that it finds the shortest paths from the start vertex. In this tutorial, we will prove that BFS correctly computes distances from the start vertex to each other vertex.

```
BFS(s):
  for all v:
    v.seen := false
    v.d = inf
  queue := new Queue()
  queue.enqueue(s)
  s.seen := true
  s.d := 0                            // distance from s to s

  while not queue.is_empty():
    u := queue.dequeue()
      for each v in u's adjacency list:
       if not v.seen:
         v.seen := true
         v.d = u.d + 1                // distance from s to v
         queue.enqueue(v)
```

Definition 1. Let $G = (V, E)$ be a directed or undirected graph, with $v, u \in V$. Define the *shortest-path distance* $\delta(u, v)$ from u to v as the minimum number of edges on any path from u to v in G. If there is no path from u to v, then define $\delta(u, v) = \infty$.

Lemma 1. Let $G = (V, E)$ be a directed or undirected graph, let $s \in V$. Then for any edge $(u, v) \in E$,

$$\delta(s, v) \leq \delta(s, u) + 1$$

Proof. □

Lemma 2. Let $G = (V, E)$ be a directed or undirected graph, let $s \in V$, and suppose we run `BFS(s)`. Then for any vertex v, at each point during the execution of the algorithm (including at termination),

$$\delta(s, v) \leq v.d$$

Proof. By induction on the number of `enqueue` calls.

□

Next we prove that at every point during BFS, the values $v.d$ of all nodes v in the queue are either all the same or look like this: $..., k, k, k + 1, k + 1, ...$ Formally:

Lemma 3. If during execution of BFS the queue contains vertices $(v_1, v_2, \ldots, v_n)$ where v_1 is at the head of the queue, then

$$v_n.d \leq v_1.d + 1 \text{ and } v_i.d \leq v_{i+1}.d \text{ for all } 1 \leq i < n$$

Proof. By induction on the number of queue operations (both enqueue and dequeue).

□

Lemma 4. Suppose vertex u is enqueued before vertex v during BFS. Then $u.d \leq v.d$ at the time v is enqueued.

Proof. Follows from the previous Lemma. $\square$

Theorem 1. Upon termination of BFS on a graph $G = (V, E)$ from a start vertex $s \in V$, for every node $v \in V$, $v.d = \delta(s, v)$.

Proof. $\square$