CSCC63 – Week 10 Tutorials

There are several questions here, you aren't expected to finish all of them today. But they'll be good practice for you.

1. Let's have some fun with the Subset-Sum reduction from class last week.

   To start, here are three 3CNF formulas:

$$\begin{aligned}
\phi_1 = \;& (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \\
& \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \\
\phi_2 = \;& (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \\
& \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \\
\phi_3 = \;& (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee x_3)
\end{aligned}$$

   (a) Run each of these formulas through the reduction from class to get a Subset-Sum instance.

   To get you started, here's the instance for ϕ_1:

   The table is

	x_1	x_2	x_3	c_1	c_2	c_3	c_4	c_5	c_6
x_1	1	0	0	0	0	0	0	1	1
$\overline{x_1}$	1	0	0	1	1	1	1	0	0
x_2	0	1	0	0	0	1	1	0	1
$\overline{x_2}$	0	1	0	1	1	0	0	1	0
x_3	0	0	1	0	1	0	1	1	0
$\overline{x_3}$	0	0	1	1	0	1	0	0	1
	0	0	0	1	0	0	0	0	0
	0	0	0	1	0	0	0	0	0
	0	0	0	0	1	0	0	0	0
	0	0	0	0	1	0	0	0	0
	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	1	0
	0	0	0	0	0	0	0	1	0
	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	1
t	1	1	1	3	3	3	3	3	3

   So $S_1 = \{100000011, 100111100, 10001101, 10110010, 1010110, 1101001, 100000, 100000, 10000, 10000, 1000, 1000, 100, 100, 10, 10, 1, 1\}$ and $t_1 = 111333333$.

(b) Now, for each of the three formulas ϕ_i and their associated tables S_i, look at each of the eight possible truth assignments.

For each truth assignment, does the truth assignment satisfy the formula?

To get you started, the truth assignment $\tau : x_1 = F, x_2 = F, x_3 = F$ does satisfy ϕ_1.

Overall, we get:

x_1	x_2	x_3	ϕ_1
F	F	F	T
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	F
T	T	T	F

How many associated subsets can you find?

> **Note: For the purposes of this exercise, we are considering two subsets to be different if they use different rows of our table, even if their numbers are otherwise the same.**

To get you started, $\tau : x_1 = F, x_2 = F, x_3 = F$ gives the subset
$\{100111100, 10110010, 1101001, 10000, 1000, 100, 100, 10, 10, 1, 1\}$.
Using this τ, we see:

- The numbers 100111100, 10110010, and 1101001 occur once in the table, and so can only be chosen in one way each.
- The numbers 10000 and 1000 each occur twice in the table, and we need to choose one of each, and so we have $2 \times 2 = 4$ ways of choosing them.
- The numbers 100, 10 and 1 each occur twice in the table, but we need both occurrences to get our subset, so there is one way of choosing these numbers.
- There are no other subsets associated with the truth assignment $x_1 = F, x_2 = F, x_3 = F$.

So this truth assignment gives four ways of choosing the numbers, giving four subsets.

If we look at all of the truth assignments, we get these numbers of subsets:

x_1	x_2	x_3	ϕ_1
F	F	F	4
F	F	T	0
F	T	F	0
F	T	T	4
T	F	F	0
T	F	T	0
T	T	F	0
T	T	T	0
			8

So ϕ_1 has two satisfying truth assignments, and S_1, t_1 have eight valid subsets.

(c) From part b, you can see that if a truth assignment does not satisfy ϕ_i, there is no associated subset of S_i. This is why, if none of the truth assignments is satisfying, there will be no valid subsets either.

However, if I were to tell you that a Subset-Sum instance had, say, eight solutions, is there any way you could tell how many solutions the corresponding ϕ had? Why not?

(d) Find a way to fix the Subset-Sum reduction so there is a one-to-one relation between the satisfying truth assignments for ϕ and the valid subsets of S. If you have this, you can use the S not only to tell whether its related ϕ is satisfiable, but to count the number of satisfying truth assignments.