PHIL 379
Assignment 3

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1. We can let the model be *M* = *<D,I>* such that *D* is the domain *{1,2,3}* and *I* is the interpretation where *1, 2, 3* $\in$ *D.*

(i) In order to make the sentence true, we can define the property F of x as *F(x): 'x is 1'*. Then we read: "There exists y in {1,2,3} such that for all x in {1,2,3}, x is 1 if and only if x = y. This can be intuitively seen as true.

In order to make the the sentence false, we can define the property *F* of x as *F(x): 'x is positive number in domain'*. This can also be intuitively seen as false as there are multiple numbers that are positive, there doesn't exist one that is equal to all of them.

(ii) In order to make the sentence true, we can define the property *R* of two arguments as *R: 'g is the same as h'*. We can see that when x is the same as y and x is the same as z, then it implies y equals z. And so choosing R as this property makes the sentence true.

In order to make the sentence false, we can define the property *R* of two arguments as *R: 'g is not the same as h'*. We can see that when x is not the same as y and x is not the same as z, then the implication y equals z is not necessarily true so it does not imply so. And so choosing R as that property makes the sentence false.

(iii) In order to make the sentence true, we can define the property *R* of two arguments as *R: 'g is the same as h'*. In addition, we need to define the function *f(y)* as *f(1) = f(2) = f(3) = 1*. Defining *R* and *f* in that way leads to the sentence being always true.

In order to make the sentence false, we let the property *R* remain the same. However, we change the function *f(y)* as *f(1) = f(2) = f(3) = any positive number in domain*. As such, there are cases where the sentence is not true and so is false.

2. We can let the model be M for our language L, where $L = \{ y \in M \mid M \models R(x,y) \}$. Some arbitrary point $x \in M$ would then be matched with set $L(x)$ of all those which are satisfying R with x.

Sentence (ii) states that there exists y such that for all x, $y \in L(x)$. This is equivalent to saying that the intersection of all $L(x)$ is not empty.

Sentence (i) states that for all $x \in M$, there exists y such that $y \in L(x)$. So then $L(x)$ cannot be empty.

As such, the following conditional can be constructed using the conclusions of the above two sentences: "If intersection of all $L(x)$'s are not empty then no $L(x)$ can be empty." This shows that the second statement implies the first.

Note: I also did an informal symbolic argument from what I learned in *Logic I* though I'm not sure if it is correct to do the same in *Logic II*. Maybe you can just provide me with some feedback on if it is okay to do this or not:

1.	$\neg[\exists_y \forall_x (R(x,y)) \longrightarrow \forall_x \exists_y (R(x,y))]$	(assumption, indirect proof)
2.	$\neg[\neg(\exists_y \forall_x (R(x,y))) \vee \forall_x \exists_y (R(x,y))]$	(conditional line 1)
3.	$\exists_y \forall_x (R(x,y))\ \&\ \neg \forall_x \exists_y (R(x,y)$	(negation line 2)
4.	$\exists_y \forall_x (R(x,y))$	(disjunction line 3)
5.	$\neg\forall_x \exists_y (R(x,y)$	(disjunction line 3)
6.	$\forall_x (R(x,a))$	(existential instantiation flag a, line 4)
7.	$\neg\exists_y (R(b,y)$	(universal instantiation, line 5)
8.	$R(b,a)$	(universal instantiation, line 6)
9.	$\neg R(b,a)$	(existential instantiation flag a, line 7)
10.	$\neg R(b,a)\ \&\ R(b,a)$	(addition, contradiction)
11.	$\exists_y \forall_x (R(x,y)) \longrightarrow \forall_x \exists_y (R(x,y))$	(by indirect proof)

3. We can define the open formula F as $p(x) \vee \neg p(y)$. Applying the universal closure to it produces $\forall_x \forall_y (p(x) \vee \neg p(y))$ which is invalid. However, applying the existential closure to it $\exists_x \exists_y (p(x) \vee \neg p(y))$ makes the formula valid.

4. For models M and $M+$ of a language L, we have for every sentence S of L

$$M \vDash L \quad <\!\!-\!\!> \quad M+ \vDash L \ (1)$$

That is, S is true in M if and only if S is true in M+. To show this, we proceed by induction on complexity.

To start, we introduce the notion of a nonlogical predicate R and some closed terms $t_1, t_2, \ldots, t_n$ that will be useful for later. The atomic sentence has atomic clause in the definition of truth as

$$M \vDash R(t_1, t_2, \ldots, t_n) <\!\!-\!\!> R^M (t_1{}^M, t_2{}^M, \ldots, t_n{}^M)$$
$$M+ \vDash R(t_1, t_2, \ldots, t_n) <\!\!-\!\!> R^{M+} (t_1{}^{M+}, t_2{}^{M+}, \ldots, t_n{}^{M+})$$

Since $M+$ is an extension, we also have j as a correspondence which is useful to introduce now but will be more important later. This gives us the following:

$$R^M(t_1{}^M, t_2{}^M, \ldots, t_n{}^M) <\!\!-\!\!> R^{M+} (j(t_1{}^{M+}), j(t_2{}^{M+}), \ldots, j(t_n{}^{M+}))$$
$$R^{M+} (j(t_1{}^M), j(t_2{}^M), \ldots, j(t_n{}^M)) <\!\!-\!\!> R^{M+} (t_1{}^{M+}, t_2{}^{M+}, \ldots, t_n{}^{M+})$$

The above equivalences show (1) if we didn't have identity or functions symbols.

However, in our case since $M+$ is defined strictly as an extension, the identity and function symbols are present and so the above is not sufficient. Since the identity is present, we have to use correspondence j to show

$$e_1 = e_2 <\!\!-\!\!> j(e_1) = j(e_2)$$

Where e is some element in |M| = |M+|. This simply is the condition we are starting with given that it is an extension, that j is one to one and it must be so as j is defined as a correspondence.

For function symbols, for any closed term t there is

$$j(t_1{}^M) = t_1{}^{M+}$$

Expanding on this, using the definition of extension and supposing the closed terms $t_1, t_2, \ldots, t_n$ hold for some function f, we have

$$j(f((t_1), (t_2), \ldots, (t_n))^M) = j(f^M((t_1{}^M), (t_2{}^M), \ldots, (t_n{}^M))) = f^{M+}(j((t_1{}^M), (t_2{}^M), \ldots, (t_n{}^M)))$$
$$= f^{M+}((t_1{}^{M+}), (t_2{}^{M+}), \ldots, (t_n{}^{M+})) = f((t_1), (t_2), \ldots, (t_n))^{M+}$$

The proof is now complete and shows that any sentence S of L is true in M if and only if S is also true in M+[1].

[1] Boolos, G. S., Burgess, J. P., & Jeffrey, R. C. (2010). *Computability and logic*. pg 142. Cambridge: Cambridge Univ. Press.