MATH 366 Assignment 3

p1

1. a) $$\int_1^2 (3t-2i)\,dt = \frac{1}{3}\frac{(3t-2i)^3}{3}\Big|_1^2 = \frac{1}{3}\left(\frac{(3\cdot 2-2i)^3}{3} - \frac{(3\cdot 1-2i)^3}{3}\right)$$

$$= \frac{1}{9}\left[(6-2i)^3 - (3-2i)^3\right] = \frac{1}{9}\left(6^3 - 3\cdot 6^2\cdot 2i + 3\cdot 6\cdot(2i)^2 - (2i)^3 - 3^3 + 3\cdot 3^2\cdot 2i - 3\cdot 3\cdot(2i)^2 + (2i)^3\right)$$

$$= \frac{1}{9}(153 - 162i) = \boxed{17-18i}$$

b) $$\int_0^\pi e^{it}\,dt = \int_0^\pi e^{it}\,dt = \frac{1}{i}e^{it}\Big|_0^\pi = \frac{1}{i}\left(e^{i\pi} - e^{i\cdot 0}\right) = \frac{1}{i}\left(e^{i\pi} - e^0\right) = \frac{1}{i}(e^{i\pi}-1)$$

$$\frac{1}{i}(\cos\pi + i\sin\pi - 1) = \frac{1}{i}(-1 + i\cdot 0 - 1) = \frac{1}{i}(-2) = \boxed{2i}$$

2. i) $$\int_C f = \int_C (x^2+iy^2)\,dz = \int_C (x^2\,dx - y^2\,dy) + i\int_C (x^2\,dy + y^2\,dx)$$

$$= \int_{-1}^1 \left(x^2\,dx - (x^3)^2\,d(x^3)\right) + i\int_{-1}^1 \left(x^2\,d(x^3) + (x^3)^2\,dx\right) = \int_{-1}^1 (x^2 - x^6 3x^2)\,dx + i\int_{-1}^1 (x^2 3x^2 + x^6)\,dx$$

$$= \int_{-1}^1 (x^2 - 3x^8)\,dx + i\int_C (3x^4 + x^6)\,dx = \left(\frac{x^3}{3} - \frac{3x^9}{9}\right)\Big|_{-1}^1 + i\left(\frac{3x^5}{5} + \frac{x^7}{7}\right)\Big|_{-1}^1$$

$$= 2\left(\frac{x^3}{3} - \frac{x^9}{3}\right)\Big|_0^1 + 2i\left(\frac{3x^5}{5} + \frac{x^7}{7}\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{1}{3}\right) + 2i\left(\frac{3}{5} + \frac{1}{7}\right)$$

$$= \boxed{\frac{52}{35}i}$$

P2

ii) $\int_C f = \int_C (3x^2 - 2iy)\,dz = \int_C (3x^2 dx + 2y\,dy) + i\int_C (-2y\,dx + 3x^2 dy)$

$= \int_0^1 (3x^2 + 2x^3\, 3x^2)\,dx + i\int_0^1 (-2x^3 + 3x^2\, 3x^2)\,dx = \int_0^1 (3x^2 + 6x^5)\,dx + i\int_0^1 (-2x^3 + 9x^4)\,dx$

$= (x^3 + x^6)\Big|_0^1 + i\left(-\frac{x^4}{2} + 9\frac{x^5}{5}\right)\Big|_0^1 = (1+1) + i\left(-\frac{1}{2} + \frac{9}{5}\right)$

$= \boxed{2 + \frac{13}{10} i}$

iii) Put $C_1: y = 0,\ 0 \le x \le 1$ and $C_2: x = 1,\ 0 \le y \le 1$. Then

$\int_C \bar{z}\,dz = \int_{C_1} \bar{z}\,dz + \int_{C_2} \bar{z}\,dz = \int_{C_1} (x - iy)(dx + i\,dy) + \int_{C_2} (x-iy)(dx + i\,dy)$

$= \int_{C_1} (x\,dx + y\,dy) + i\int_{C_1} (-y\,dx + x\,dy) + \int_{C_2} (x\,dx + y\,dy) + i\int_{C_2} (-y\,dx + x\,dy)$

$= \int_0^1 (x\,dx + 0) + i\int_{C_1} (-0\,dx + 0 \cdot x) + \int_0^1 (1 \cdot 0 + y\,dy) + i\int_0^1 (-y \cdot 0 + 1\,dy)$

$= \int_0^1 x\,dx + \int_0^1 y\,dy + i\int_0^1 dy = \frac{x^2}{2}\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 + i = \frac{1}{2} + \frac{1}{2} + i = \boxed{1+i}$

iv) C: ~~$y = 0$~~ $y = x,\ 0 \le x \le 1$

$\int_C \bar{z}\,dz = \int_C (x - iy)(dx + i\,dy) = \int_C (x\,dx + y\,dy) + i\int_C (-y\,dx + x\,dy) =$

$= \int_0^1 (x\,dx + x\,dx) + i\int_0^1 (-x\,dx + x\,dx) = 2\int_0^1 x\,dx + i\int_0^1 0 =$

$= x^2\Big|_0^1 = \boxed{1}$

MATH 366 Assignment 3 P3

2 b) If the function is analytic, then the integral of this function does not depend on the path of integration. Since $f(z) = \bar{z}$, then $u = x$ and $v = -y$. Then $\frac{du}{dx} = 1 \neq -1 = \frac{dv}{dy}$. So $f(z) = \bar{z}$ is not analytic in each point. This explains that the results from iii and iv are different.

3. Consider function $f(z) = \frac{\cancel{z}\, 2z^2 - 5}{(z^2+1)(z^2+4)}$, then for $|z| = R$, we have

$$|f(z)| = \left|\frac{2z^2 - 5}{(z^2+1)(z^2+4)}\right| = \frac{|2z^2 - 5|}{|z^2+1|\,|z^2+4|}.$$ Since $|z+w| \leq |z| + |w|$ and $|z-w| \geq |z| - |w|$ for any $z, w \in \mathbb{C}$,

$$|f(z)| = \frac{|2z^2 - 5|}{|z^2+1|\,|z^2+4|} = \frac{|2z^2 + (-5)|}{|z^2 - (-1)|\,|z^2 - (-4)|} \leq \frac{|2z^2| + |-5|}{|z^2 - |-1||\,||z^2| - |-4||}$$

$$= \frac{2|z|^2 + 5}{||z|^2 - 1|\,||z|^2 - 4|}.$$ Since $|z| = R > 2$, then $|z|^2 - 1 = R^2 - 1 > 0$ and

$|z|^2 - 4 = R^2 - 4 > 0$. So,

$$|f(z)| \leq \frac{2|z|^2 + 5}{||z|^2 - 1|\,||z|^2 - 4|} = \frac{2R^2 + 5}{(R^2-1)(R^2-4)} = M$$

Note that if C upper half of circle $|z| = R$, then length of l of C is equal πR. Hence, $\boxed{\left|\int_C f\right| \leq M \cdot l = \frac{2R^2+5}{(R^2-1)(R^2-4)}\pi R = \frac{\pi R(2R^2+5)}{(R^2-1)(R^2-4)}}$

p4

4. Use Cauchy Integral Theorem

a) $z^2+3=0$, $z_1 = i\sqrt{3}$, $z_2 = i\sqrt{3}$. Since $|z_1| = \sqrt{3} > 1$, $|z_2| = \sqrt{3} > 1$

this means that f is analytic inside contour $|z|=1$. Then by CIT

we have $\boxed{\int_c f = 0}$

b) $\cos z = 0$,

$\frac{e^{iz}+e^{-iz}}{2} = 0$, $e^{iz}+e^{-iz} = 0$, $e^{iz} + \frac{1}{e^{iz}} = 0$, $e^{2iz}+1=0$, $e^{2iz} = -1$,

$2iz = \ln|-1| + i(\pi + 2\pi k)$, $k \in \mathbb{Z}$, $2iz = \ln 1 + i(\pi + 2\pi k)$, $k \in \mathbb{Z}$

$2iz = i(\pi + 2\pi k)$, $k \in \mathbb{Z}$, $\boxed{z = \frac{\pi}{2} + \pi k, \ k \in \mathbb{Z}}$

Since $|z| \geq \frac{\pi}{2} > 1$, this means f is analytic inside contour $|z|=1$.

Then by CIT, we have ~~$\oint_c f$~~ $\boxed{\int_c f = 0}$

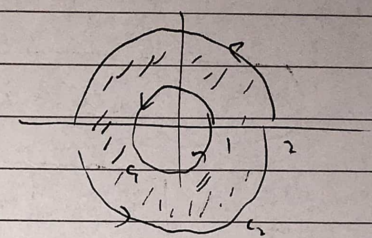
c) $z^2+4z+4=0$, $(z+2)^2=0$, $z=-2$. Since $|z| = 2 > 1$, this means

that f is analytic inside contour $|z|=1$. Then by CIT, we have

~~$\oint$~~ $\boxed{\int_c f = 0}$

MATH 316 Assignment 2 P5

5. Consider contour $C_2 \cup C_1^-$



If the function f is analytic inside contours $C_2 \cup C_1^-$, then by CIT we have $\int_C f = 0$. Then

$$0 = \int_{C_2 \cup C_1^-} f = \int_{C_2} f + \int_{C_1^-} f = \int_{C_2} f - \int_{C_1} f \quad \text{and} \quad \int_{C_1} f = \int_{C_2} f.$$

Therefore in all cases, it is sufficient to prove that f is analytic on $1 \le |z| \le 2$.

a) $5z^2 + 3 = 0$, $5z^2 = -3$, $z^2 = -3/5$, $z = \pm i\sqrt{3/5}$.

Since $|z| = |\pm i\sqrt{3/5}| = \sqrt{3/5} < 1$, this means f is analytic on $1 \le |z| \le 2$.

Hence $\cancel{\int_{C_1} f = \int f}$ $\int_C f = \int_{C_2} f$.

b) $\sin z = 0$, $\frac{e^{iz} - e^{-iz}}{2i} = 0$, $e^{iz} - e^{-iz} = 0$, $e^{2iz} - 1 = 0$, $e^{2iz} = 1$,

$2iz = \ln|1| + i(0 + 2\pi k)$, $k \in \mathbb{Z}$, $2iz = i2\pi k$, $k \in \mathbb{Z}$, $z = \pi k, k \in \mathbb{Z}$

for $k = 0$, we have $z = 0$ and $|z| = 0 < 1$.

for $k \in \mathbb{Z} \setminus \{0\}$, we have $|z| \ge \pi > 1$.

Hence f is analytic on $1 \le |z| \le 2$ and so $\int_{C_1} f = \int_{C_2} f$

P6

c) $e^z + 1 = 0$, $e^z = -1$, $z = \ln|-1| + i(\pi + 2\pi k)$, $k \in \mathbb{Z}$

$z = i(\pi + 2\pi k)$, $k \in \mathbb{Z}$.

So $|z| = |\pi + 2\pi k| \geq \pi > 2$ for each $k \in \mathbb{Z}$. This means that f is analytic on $1 \leq |z| \leq 2$. Hence $\int_{C} f = \int_{C_2} f$