MATH 366 Assignment 4 P1

1. a) This sequence is convergent because it is bounded by 0:

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

b) The sequence is convergent because $\frac{1}{n}$ is convergent.

$$\lim_{n \to \infty} \left(\frac{(-1)^n}{n}\right) \leq \lim_{n \to \infty} \frac{1}{n} = 0$$

Therefore, the sequence $\left(2 + \frac{(-1)^n}{n}\right)$ is also convergent:

$$\lim_{n \to \infty} \left(2 + \frac{(-1)^n}{n}\right) = 2 + \lim_{n \to \infty} \left(\frac{(-1)^n}{n}\right) = 2 + 0 = 2$$

2. a) $\log(z+1) = \int \frac{1}{z+1}\, dz$

Use geometric series

$$\frac{1}{z+1} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n (z)^n$$

$$\int \frac{1}{z+1}\, dz = \int \sum_{n=0}^{\infty} (-1)^n (z)^n\, dz = \sum_{n=0}^{\infty} \frac{(-1)^n (z)^{n+1}}{n+1} + C$$

$$\boxed{\log(z+1) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} + C}$$

At $z_0 = 0$ we have

$$\log(0+1) = \log(1) = 0 \;,\; \sum_{n=0}^{\infty} \frac{(-1)^n 0^{n+1}}{n+1} + C = C$$

Thus $C = 0$. So

$$\boxed{\log(z+1) = \sum_{n=0}^{\infty} \frac{(-1)(z^{n+1})}{n+1}}$$

b) At $z_0 = 0$ we have Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Then

$$e^z = e^{1+(z-1)} = e^1 e^{(z-1)} = e e^{z-1}$$

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

Thus at $z_0 = 1$,

$$\boxed{e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}}$$

MATH 366 Assignment 4 P3

c) At $z_0 = 0$, we have Maclaurin series

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \text{then}$$

$$\sin z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$$

Thus

$$z \sin z^2 = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+3}}$$

d) At $z_0 = 0$, we have Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n + (-z)^n}{n!}\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{n!} \left(1 + (-1)^n\right), \quad \left(1 + (-1)^n\right) = \begin{cases} 2, & \text{if } n = 2k \\ 0, & \text{if } n = 2k+1 \end{cases}$$

Thus $\boxed{\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}}$

a) Using ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1} z^{n+2}}{n+2}}{\frac{(-1)^n z^{n+1}}{n+1}}\right| = \lim_{n\to\infty}\left|-z\frac{n+1}{n+2}\right| = \boxed{|z|\lim_{n\to\infty}\frac{n+1}{n+2} = |z|}$$

which is less than 1 when $|z|<1$. Thus radius of convergence

$$\boxed{R=1}$$

b) Using ratio test

$$\lim_{n\to\infty}\left|\frac{e\frac{(z-1)^{n+1}}{(n+1)!}}{e\frac{(z-1)^n}{n!}}\right| = \lim\left|\frac{z-1}{n+1}\right| = \boxed{|z-1|\lim_{n\to\infty}\frac{1}{n+1} = 0}$$

which is less than 1 everywhere. Thus radius of convergence, $R=\infty$

c) $$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1} z^{4n+3+4}}{(2n+3)!}}{\frac{(-1)^n z^{4n+3}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|-z^4\frac{1}{(2n+3)(2n+2)}\right| = \boxed{|z^4|\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} \to 0}$$

which is less than 1 everywhere; $\boxed{R=\infty}$

MATH 366 Assignment 4 B5

3. d) $$\lim_{n\to\infty} \left| \frac{\frac{z^{2n+2}}{(2n+2)!}}{\frac{z^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| z^2 \frac{1}{(2n+1)(2n+2)} \right| = \boxed{|z|^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0}$$

which is less than 1 everywhere, $\boxed{R = \infty}$

4. $$(z^2-1)^{1/2} = (-(1-z^2))^{1/2} = i(1-z^2)^{1/2}$$

$$(1-z^2)^{1/2} = \underbrace{1}_{1} + \underbrace{\tfrac{1}{2}(-z^2)}_{2} + \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\frac{(-z^2)^2}{2!}}_{3} + \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)\frac{(-z^2)^3}{3!}}_{4}$$

$$(1-z^2)^{1/2} = 1 - \frac{z^2}{2} - \frac{z^4}{8} - \frac{z^6}{16} + \cdots$$

Therefore, $$\boxed{(z^2-1)^{1/2} = i\left(1 - \frac{z^2}{2} - \frac{z^4}{8} - \frac{z^6}{16} + \cdots\right)}$$

5. a) i. $$\frac{1}{z-i} + \frac{1}{z-1} = \frac{1}{z-1} + \frac{1}{(z-1)+1-i} = \frac{1}{z-1} + \frac{1}{1-i}\left(\frac{1}{1+\frac{z-1}{1-i}}\right)$$

We use geometric series

$$\frac{1}{1+\frac{(z-1)}{1-i}} = \sum_{n=0}^{\infty}\left(-\frac{(z-1)}{1-i}\right)^n$$

Thus

$$\frac{1}{z-i} + \frac{1}{z-1} = \boxed{\frac{1}{z-1} + \frac{1}{1-i} \sum_{n=0}^{\infty} \left(- \frac{(z-1)}{1-i} \right)^n}$$

Converges when

$$0 < \left| - \frac{(z-1)}{1-i} \right| < 1 \to 0 < \frac{|z-1|}{|1-i|} < 1 \to \boxed{0 < |z-1| < \sqrt{2}}$$

ii. $$\frac{1}{z-i} + \frac{1}{z-1} = \frac{1}{z-1} + \frac{1}{z-1} \cdot \frac{1}{1 + \frac{1-i}{z-1}}$$

Use geometric series

$$\frac{1}{1+\frac{1-i}{z-1}} = \sum_{n=0}^{\infty} \left(- \frac{1-i}{(z-1)} \right)^n$$

Thus $$\boxed{\frac{1}{z-i} + \frac{1}{z-1} = \frac{1}{z-1} + \frac{1}{z-1} \cdot \sum_{n=0}^{\infty} \left(- \frac{1-i}{(z-1)} \right)^n}$$

converges when

$$\boxed{\left| - \frac{1-i}{(z-1)} \right| < 1 \to |z-1| > \sqrt{2}}$$

b) $\frac{z+2}{z-1} = \frac{(z-1)+3}{z-1} = 1 + \frac{3}{z-1}$

i. $\frac{z+2}{z-1} = 1 - \frac{3}{1-z}$, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

Thus $\boxed{\frac{z+2}{z-1} = 1 - 3\sum_{n=0}^{\infty} z^n}$ converges when

~~or~~ $\boxed{0 < |z| < 1}$

ii $\frac{z+2}{z-1} = 1 + \frac{3}{z-1} = 1 + \frac{3}{z} \cdot \frac{1}{1-1/z}$

Use geometric series $\frac{1}{1-1/z} = \sum_{n=0}^{\infty} z^{-n}$

Thus $\boxed{\frac{z+2}{z-1} = 1 + \frac{3}{z}\sum_{n=0}^{\infty} z^{-n}}$ converges when

$\left|\frac{1}{z}\right| < 1 \rightarrow \boxed{|z| > 1}$

Pg

5. c. $\dfrac{z}{(z-1)(z+4)} = \dfrac{A}{z-1} + \dfrac{B}{z+4}, \quad A(z+4) + B(z-1) \equiv z$

$A + B = 1, \quad 4A - B = 0 \Rightarrow A = \frac{1}{5}, \quad B = \frac{4}{5}$

$$\frac{z}{(z-1)(z+4)} = \frac{1}{5}\left(\frac{1}{z-1} + \frac{4}{z+4}\right)$$

i. $\dfrac{1}{z-1} + \dfrac{4}{z+4} = \dfrac{1}{z-1} + \dfrac{4}{(z-1)+5} = \dfrac{1}{z-1} + \dfrac{4}{5} \cdot \dfrac{1}{1 + \frac{z-1}{5}}$

Use geometric series $\dfrac{1}{1+\frac{(z-1)}{5}} = \displaystyle\sum_{n=0}^{\infty} \left(-\frac{(z-1)}{5}\right)^n$

Thus $$\boxed{\frac{z}{(z-1)(z+4)} = \frac{1}{5}\left(\frac{1}{z-1} + \frac{4}{5}\sum_{n=0}^{\infty}\left(-\frac{(z-1)}{5}\right)^n\right)}$$

converges when

$\cancel{0 < \left|-\frac{(z-1)}{5}\right| < 1 \rightarrow 0 < |z-1| < 5}$

$$0 < \left|-\frac{(z-1)}{5}\right| < 1 \rightarrow \boxed{0 < |z-1| < 5}$$

MATH 316 Assignment 4 P9

5. c) ii

$$\frac{1}{z-1} + \frac{4}{(z-1)+5} = \frac{1}{z-1} + \frac{4}{z-1} \cdot \frac{1}{1+\frac{5}{z-1}}$$

$\hookrightarrow$ Use geometric series $\frac{1}{1+\frac{5}{z-1}} = \sum_{n=0}^{\infty} \left(-\frac{5}{(z-1)}\right)^n$

Thus $\boxed{\frac{z}{(z-1)(z+4)} = \frac{1}{5}\left(\frac{1}{z-1} + \frac{4}{z-1}\sum_{n=0}^{\infty}\left(-\frac{5}{(z-1)}\right)^n\right)}$

Converges when $\boxed{\left|-\frac{5}{(z-1)}\right| < 1 \to |z-1| > 5}$

6. a) The isolated singular point of $f(z) = \cancel{\frac{\sin z}{z^3}}\ \frac{\sin z}{z^3}$ is $\boxed{z=0}$

At this point we have Laurent series of $f(z)$

$$\frac{\sin z}{z^3} = \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \dots$$

We can see principle part is $\boxed{\frac{1}{z^2}}$

6. b) The isolated singular point of $f(z) = \frac{\cosh z}{z^2}$ is $\boxed{z=0}$

At this point we have Laurent series of $f(z)$

$$\frac{\cosh z}{z^2} = \frac{1}{z^2}\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right) = \left(\frac{1}{z^2} + \frac{1}{2!} + \frac{z^2}{4!} + \cdots\right)$$

We can see the principle part is $\boxed{\frac{1}{z^2}}$

c) $f(z)$ is a quotient of two functions and the denominator is 0 only if $z^2 - 3z = z(z-3) = 0$ so the singularities are

$z = 0$, $z = 3$

At $z = 0$, we have Laurent series of $f(z)$

$$\frac{z+2}{z^2-3z} = \frac{1}{z}\left(\frac{z+2}{z-3}\right) = \frac{1}{z}\left(\frac{z-3+5}{z-3}\right) = \frac{1}{z}\left(1 + \frac{5}{z-3}\right)$$

$$= \frac{1}{z}\left(1 - \frac{5/3}{1 - z/3}\right)$$

We use geometric series $\frac{1}{1-z/3} = 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 \cdots$

So $$\frac{z+2}{z^2-3z} = \frac{1}{z}\left(1 - \frac{5}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \left(\frac{z}{3}\right)^3 \cdots\right)\right)$$

$$= \left(1 - \frac{5}{3}\right)\left(\frac{1}{z}\right) - \frac{5}{3}\left(\frac{1}{3} + \frac{z}{3^2} + \cdots\right)$$

We can see the principle part is at $z=0$

$$\left(1 - \frac{5}{3}\right)\left(\frac{1}{z}\right) = \boxed{-\frac{2}{3z}}$$

At $z=3$ we have Laurent series

$$\frac{z+2}{z^2-3z} = \left(\frac{1}{z-3}\right)\left(\frac{z+2}{z}\right) = \left(\frac{1}{z-3}\right)\left(1 + \frac{2}{z}\right) = \left(\frac{1}{z-3}\right)\left(1 + \frac{2}{3+(z-3)}\right)$$

$$= \left(\frac{1}{z-3}\right)\left(1 + \frac{\frac{2}{3}}{1 + \frac{z-3}{3}}\right)$$

Use geometric series

$$\frac{1}{1 + \frac{z-3}{3}} = 1 - \left(\frac{z-3}{3}\right) + \left(\frac{z-3}{3}\right)^2 - \left(\frac{z-3}{3}\right)^3 + \cdots$$

So $$\frac{z+2}{z^2-3z} = \left(\frac{1}{z-3}\right) \cancel{\left(\frac{1}{z-3}\right)} \left(1 + \frac{2}{3}\left(1 - \frac{z-3}{3} + \left(\frac{z-3}{3}\right)^2 - \cdots\right)\right)$$

P12

$$= \left(1+\frac{2}{3}\right)\left(\frac{1}{z-3}\right) + \frac{2}{3}\left(-\frac{1}{3} + \frac{z-3}{3^2} - \cdots\right)$$

We can see principle part is

$$\left(1+\frac{2}{3}\right)\left(\frac{1}{z-3}\right) = \left(\frac{5}{3}\right)\left(\frac{1}{z-3}\right) = \boxed{\frac{5}{3(z-3)}} \quad \text{at } z=3$$

d) The isolated singular point of $f(z)$ is $ze^{1/z}$, $z=0$

At this point Laurent series of $f(z)$:

$$ze^{1/z} = z\left(1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots\right)$$

$$1 + z + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots$$

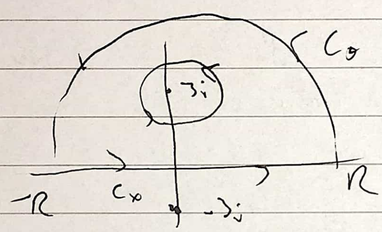
Principle part :

$$\frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots = \boxed{\sum_{n=1}^{\infty} \frac{1}{(n+1)!}\frac{1}{z^n}}$$

MATH 316 Assignment 4 P13

7. a) The integral is equal to integral of $\frac{1}{z^2+9}$ over real line $z = x$ with $x = -\infty \to \infty$. That complex function has two simple poles at $\pm 3i$ since $z^2+9 = (z+3i)(z-3i)$

Need to perform contour integration over path C $C = C_x + C_\theta$



Since $z = 3i$ is the only simple pole, Cauchy Formula gives us

$$\oint_C \frac{dz}{z^2+9} = \oint_C \frac{(z+3i)^{-1}}{z-3i} dz = 2\pi i \left(\frac{1}{\cancel{z} + 3i}\right) = 2\pi i \left(\frac{1}{3i+3i}\right)$$

$$= \boxed{\frac{\pi}{3}}$$

We need to figure out C_θ part. This part goes to zero when $R \to \infty$:

$$\left| \int_{C_\theta} \frac{dz}{z^2+9} \right| = \left| \int_0^\pi \frac{iRe^{i\theta}\, d\theta}{R_0^2 e^{2i\theta} + 9} \right| < \int_0^\pi \frac{R\, d\theta}{R^2-9} = \frac{\pi R}{R^2-9}$$

$\to 0$ as $R \to \infty$

Thus $$\int_{-\infty}^{\infty} \frac{dx}{x^2+9} = \frac{\pi}{3}$$

P14

7. b) $\dfrac{1}{(x^2+1)^2}$ is even function so

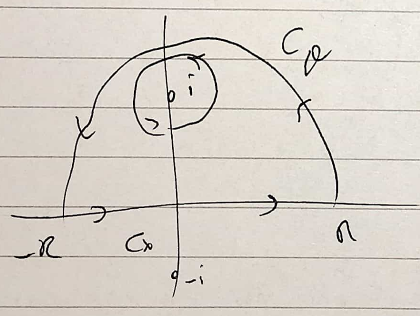
$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}.$$

Consider $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$.

The integral is equal to the integral of $\dfrac{1}{(z^2+1)^2}$ over real line $z>0$ with $x = -\infty \to \infty$. The complex function $\dfrac{1}{(z^2+1)^2}$ has two double poles at $z = \pm i$ since

$$(z^2+1)^2 = (z+i)^2\,(z-i)^2$$

Need to perform contour integration over the path C: $C = C_x + C_\rho$



Need to figure out contribution from the double pole by using generalized Cauchy formula

$$\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = 2\pi i \left(\frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\right)_{z=i}$$

$$= 2\pi i \left(\frac{-2}{(z+i)^3}\right)_{z=i} = 2\pi i\left(\frac{-2}{(i+i)^3}\right) = 2\pi i\left(\frac{-2}{-8i}\right) = \boxed{\frac{\pi}{2}}$$

MATH 361 Assignment 8 P 15

7. b) We need to figure out C_ρ part. This part goes to zero when $R \to \infty$:

$$\left|\int_{C_\rho} \frac{dz}{(z^2+1)^2}\right| = \left|\int_0^\pi \frac{iRe^{i\theta}\,d\theta}{(R^2e^{2i\theta}+1)^2}\right| < \int_0^\pi \frac{R\,d\theta}{(R^2-1)^2} = \frac{\pi R}{(R^2-1)^2}$$

$= 0$ when $R \to \infty$

Thus, $$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$$

c) $\frac{\cos 2x}{x^2+4}$ is even function so

$$\int_0^\infty \frac{\cos 2x}{x^2+4} = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos 2x}{x^2+4}\,dx$$

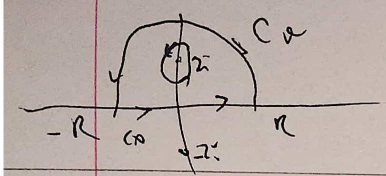
Consider $\int_{-\infty}^\infty \frac{\cos 2x}{x^2+4}\,dx$.

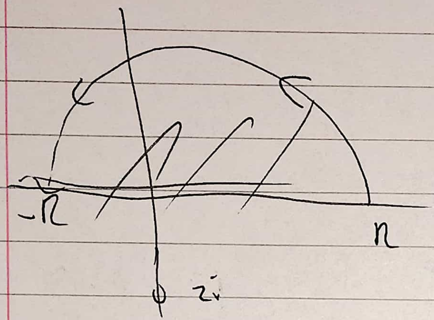
Integral is equal to the integral of $\frac{e^{2iz}}{z^2+4}$ over real line $z = x$ with $x = -\infty \to \infty$. That complex function has two simple poles at $z = \pm 2i$ since

$$z^2+4 = (z+2i)(z-2i)$$

Need to perform contour of integration over path C: $C = C_R + C_\rho$

P1b





Since $z=2i$, only simple pole inside our closed contour, Cauchy formula gives

$$\oint \frac{e^{i2z}}{z^2+4}\,dz = \int_C \frac{e^{i2z}(z+2i)^{-1}}{z-2i}\,dz = 2\pi i\left(\frac{e^{2iz}}{z+2i}\right)_{z=2i}$$

$$= 2\pi i\left(\left(\frac{e^{2i2i}}{2i+2i}\right)\right) = \boxed{2\pi i\left(\frac{e^{-4}}{4i}\right)}$$

Thus $\displaystyle\oint_C \frac{e^{i2z}}{z^2+4}\,dz = 2\pi i\left(\frac{1}{4\pi i}\right)e^{-4} = \boxed{\frac{\pi}{2}e^{-4}}$

Need to figure out $C\rho$ part; goes to zero as $R\to\infty$:

$$\left|\int_{C\rho}\frac{e^{2iz}}{z^2+4}\,dz\right| = \int_0^\pi \frac{e^{2iRe^{i\theta}}\, iRe^{i\theta}\,d\theta}{R^2e^{i2\theta}+4} \le \int_0^\pi \frac{R\,d\theta}{R^2-4} = \frac{\pi R}{R^2-4}$$

$=0$ as $R\to\infty$

Thus

$$\int_0^\infty \frac{\cos 2x}{x^2+4}\,dx = \frac{1}{2}\,\frac{\pi}{2}\,e^{-4} = \boxed{\frac{\pi}{4}e^{-4}}$$