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1. $z = a + ib = r(\cos\theta + i\sin(\theta))$, $r = \sqrt{a^2+b^2}$, $\theta = \arctan\left(\frac{b}{a}\right)$

a) $\left(\frac{-1+5i}{3+3i}\right)\left(\frac{3-3i}{3-3i}\right) = \frac{-3+3i+15i-15i^2}{9-9i^2} = \frac{12+18i}{18}$

$= \boxed{\frac{2}{3} + i, \quad a = \frac{2}{3}, \ b = 1}$

$r = \sqrt{\left(\frac{2}{3}\right)^2 + 1^2} = \sqrt{\frac{4}{9}+1} = \sqrt{\frac{13}{9}} = \boxed{\frac{\sqrt{13}}{3} = r}$

$\theta = \arctan \frac{1}{2/3} = 0.983$

$\therefore \boxed{\frac{-1+5i}{3+3i} = \frac{\sqrt{13}}{3}\left(\cos 0.983 + i\sin 0.983\right) = \frac{2}{3} + i}$

b. $\frac{3}{i} + \frac{i}{4} = \frac{3}{i}\frac{(-i)}{(-i)} + \frac{i}{4} = \frac{-3i}{1} + \frac{i}{4} = \boxed{\frac{-11i}{4}, \quad a = 0, \ b = -\frac{11}{4}}$

$r = \sqrt{0^2 + \left(\frac{-11}{4}\right)^2} = \sqrt{\frac{121}{16}} = \boxed{\frac{11}{4} = r}$

$\theta = \lim_{a \to 0} \arctan \frac{-11/4}{a} = -\arctan\left(\lim_{a\to 0} \frac{11}{4a}\right) = -\arctan(\infty) = -\frac{\pi}{2}$

$\therefore \boxed{\frac{3}{i} + \frac{i}{4} = 0 - \frac{11}{4}i = \frac{11}{4}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)}$

c) $(2+i)(-1-3i)(3+2i) = (-2 + -6i - i - 3i^2)(3+2i)$

$= (-2-7i+3)(3+2i) = (1-7i)(3+2i) = 3 + 2i - 21i - 14i^2$

$= 17 - 19i$, $a = 17$, $b = -19$

$r = \sqrt{17^2 + (-19)^2} = \sqrt{650} = \boxed{5\sqrt{26} = r}$

$\theta = \arctan \frac{-19}{17} \simeq \boxed{-0.840 = \theta}$

$\boxed{(2+i)(-1-3i)(3+2i) = 17 - 19i = \cancel{5\sqrt{26}(\cos\theta + i\sin} \; 5\sqrt{26}(\cos -0.840 + i \sin -0.840)}$

2. First way

Binomial formula: $(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n} b^n$

In our case: $(a+b)^5 = \binom{5}{0} a^5 + \binom{5}{1} a^4 b + \binom{5}{3} a^2 b^3 + \binom{5}{4} a b^4 + \binom{5}{5} b^5$

$= a^5 + 5a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5 a b^4 + b^5$ (used calculator for ( ) coefficients)

Then $(2-i)^5 = (2 + (-i))^5$

$= 2^5 + 5 \cdot 2^4 \cdot (-i) + 10 \cdot 2^3 \cdot (-i)^2 + 10 \cdot 2^2 \cdot (-i)^3 + 5 \cdot 2 \cdot -i^4 + (-i)^5$

$32 - 5i \cdot 16 + 10 \cdot 8 \cdot -1 + 10 \cdot 4 \cdot i + 10 \cdot 1 + -i$ [illegible]

$\boxed{(2-i)^5 = -38 - 41i}$

2. Second way

First: $(2-i)$; $r = \sqrt{2^2+(-1)^2} = \sqrt{5}$ $\theta = \arctan\left(\frac{-1}{2}\right) = -0.463596101$

$$(2-i) = \sqrt{5}\left(\cos(-0.463596101) + i\sin(-0.463596101)\right)$$

$$(2-i)^5 = (\sqrt{5})^5\left(\cos(5\cdot -0.463596101) + i\sin 5\cdot(-0.463596101)\right)$$

$$25\sqrt{5}\,(\cancel{\cos}\ -0.678856 + -0.734271 i)$$

$$= \boxed{-37.9992 + -i41.0470}$$

$$\approx -38 - 41i$$

As we can see, by applying Binomial Formula and Integer Power Formula (1.4-2), we achieve same result

$$\boxed{(2-i)^5 = -38-41i}$$

3. Recall definition of conjugate: ~~$z = a+ib = re^{i\theta}$~~

$z = a+ib = re^{i\theta}$, $\bar{z} = a-ib = \rho e^{-i\theta}$

$= r(\cos\theta + i\sin\theta)$, $= r(\cos\theta - i\sin\theta)$

Then
$$\frac{\bar{z}^k}{z^k} = \frac{(\rho e^{-i\theta})^k}{(\rho e^{i\theta})^k} = \frac{\rho^k e^{-ik\theta}}{\rho^k e^{ik\theta}} = \rho^k e^{-ik\theta}$$
(since $r\angle\theta$ and $r\angle\theta$ are conjugates)

$$\therefore (\bar{z})^k = \rho^k e^{-ik\theta} = \overline{z^k} \Leftrightarrow (\bar{z})^k = \overline{z^k}$$

4. Recall following properties of complex numbers:

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$$

$$\overline{(z^k)} = (\bar{z})^k$$

$$0 = 0 + i0 \Leftrightarrow \bar{0} = \overline{0 + i0} = 0 - i0 = 0 \Leftrightarrow 0 = \bar{0} = 0$$

Then when z_0 is a root of

$$a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 = 0$$

it means

$$a_n z_0^n + a_{n-1} z_0^{n-1} + a_1 z_0 + a_0 = 0$$

Apply conjugation

$$\overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots a_1 z_0 + a_0} = \bar{0}$$

Then $\overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots a_1 z_0 + a_0} = \overline{a_n z_0^n} + \overline{a_{n-1} z_0^{n-1}} + \cdots \overline{a_1 z_0} + a_0$

$$= a_n \overline{z_0^n} + a_{n-1} \overline{z_0^{n-1}} + \cdots a_1 \bar{z}_0 + a_0 = a_n (\bar{z}_0)^n + a_{n-1} (\bar{z}_0)^{n-1} + \cdots a_1 (\bar{z}_0) + a_0$$

Conclusion

$$a_n (\bar{z}_0)^n + a_{n-1} (\bar{z}_0)^{n-1} + \cdots a_1 (\bar{z}_0) + a_0 = \bar{0} = 0$$

Then $a_n (\bar{z}_0)^n + a_{n-1} (\bar{z}_0)^{n-1} + \cdots a_1 (\bar{z}_0) + a_0 = 0$ this means

number $\bar{z}_0$ is a root of polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 = 0$$

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5. Let $z_1 = r_1 e^{i\theta_1}$ where $\theta_1 = \arg z_1$

$z_2 = r_2 e^{i\theta_2}$ where $\theta_2 = \arg z_2$

Then a) $\arg z_1 z_2 = \arg z_1 + \arg z_2$ if $z_1, z_2 \neq 0$

$z_3 = z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \Leftrightarrow$

$\arg(z_3) = \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

$\boxed{\text{Conclusion: } \arg z_1 z_2 = \arg z_1 + \arg z_2 \text{ if } z_1, z_2 \neq 0 \text{ is } \underline{\text{true}}}$

b) $\arg \bar{z} = -\arg z$ if z is not real

$z = re^{i\theta} \Leftrightarrow \arg z = \theta$

$\bar{z} = re^{-i\theta} \Leftrightarrow \arg \bar{z} = -\theta = -\arg z$

$\boxed{\text{Conclusion: } \arg \bar{z} = -\arg z \text{ if } z \text{ is not real is } \underline{\text{true}}}$

c) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ if $z_1, z_2 \neq 0$

$z_3 = \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \Leftrightarrow$

$\arg(z_3) = \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

Conclusion: $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ $z_1, z_2 \neq 0$ is true

6. a) $1 + z + z^2 + \dots z^n = \frac{z^{n+1} - 1}{z-1}, \quad n \geq 1$

① Basestep: $n=1$

$$1 + z = \frac{z^{1+1} - 1}{z-1}$$

$$= \frac{z^2 - 1}{z-1}$$

$$= \frac{(z-1)(z+1)}{(z-1)}$$

$1+z = z+1$ ✓ $n=1$

② Inductive step for $n=k$, assume

$$1+z+z^2+\dots+z^k = \frac{z^{k+1}-1}{z-1} \quad \text{(IH)}$$

To show $n=k+1$

$$1+z+z^2+\dots+z^{k+1} = \frac{z^{k+2}-1}{z-1}$$

③ Proof: $1+z+z^2+\dots z^{k+1} = \underbrace{(1+z+z^2+\dots z^{k})}_{\checkmark} + z^{k+1}$

$$= \frac{z^{k+1}-1}{z-1} + z^{k+1} \quad \text{(IH)}$$

$$= \frac{z^{k+1}-1}{z-1} + \frac{z^{k+1}}{z-1} = \frac{\cancel{z^{k+1}} - 1 + z^{k+1+1} - \cancel{z^{k+1}}}{z-1}$$

$$= \boxed{\frac{z^{k+2}-1}{z-1}}$$

④ Conclusion:

This it is shown

$$1+z+z^2+\dots z^n = \frac{z^{n+1}-1}{z-1}, \forall n \geq 1$$

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b) By definition, $\sin(n\theta) = \dfrac{e^{in\theta} - e^{-in\theta}}{2i} \Leftrightarrow 2i \cdot \cancel{\sin\theta}\sin(n\theta) = (e^{in\theta} - e^{-in\theta})$

Denote that $\cos(n\theta) = Re(\cos(n\theta) + i\sin(n\theta)) = Re(e^{in\theta}) = Re(z)$

Then

$$1 + \cos\theta + \cos 2\theta + \dots + \cos(n\theta) = Re(1 + z + z^2 + \dots z^n) = Re\left(\frac{z^{n+1} - 1}{z - 1}\right)$$

$$= Re\left(\frac{(e^{i\theta})^{n+1} - 1}{e^{i\theta} - 1}\right) = Re\left(\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}\right)$$

$$= Re\left(\frac{e^{\frac{i(n+1)\theta}{2}}\left(e^{\frac{i(n+1)\theta}{2}} - e^{\frac{-i(n+1)\theta}{2}}\right)}{e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)}\right)$$

$$= \left[\frac{e^x}{e^y \cdot a} = \frac{e^{x-y}}{a}, \text{ then } \frac{e^{\frac{i(n+1)\theta}{2}}}{e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}\right)} = \frac{e^{\frac{i(n+1-1)\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}\right]$$

$$= Re\left(\frac{e^{\frac{i(n+1)\theta}{2} - \frac{i\theta}{2}}\left(2i \cdot \sin\left(\frac{n+1}{2}\theta\right)\right)}{2i\sin(\theta/2)}\right) = Re\left(\frac{e^{\frac{in\theta}{2}}\sin\left(\frac{n+1}{2}\theta\right)}{\sin(\theta/2)}\right)$$

(2xy)

$$= Re\left(\frac{\left(\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta}{2}\right)\right)\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)$$

$$\operatorname{Re}\left(\frac{\cos\left(\frac{n\theta}{2}\right)\cdot\sin\left(\frac{n+1}{2}\theta\right)+i\sin\left(\frac{n\theta}{2}\right)\sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)$$

$$=\frac{\cos\frac{n\theta}{2}\cdot\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}} \qquad \cos x\sin y=\frac{\sin(y-x)+\sin(y+x)}{2}$$

$$=\frac{\frac{1}{2}\cdot\left(\sin\left(\left[\frac{n+1}{2}-\frac{n}{2}\right]\theta\right)+\sin\left(\left[\frac{n+1}{2}+\frac{n}{2}\right]\theta\right)\right)}{\sin\frac{\theta}{2}}$$

$$\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}+\frac{\sin\left(\left[n+\frac{1}{2}\right]\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}=\boxed{\frac{1}{2}+\frac{\sin\left(\left[n+\frac{1}{2}\right]\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}}$$

$$\boxed{\text{Conclusion: } 1+\cos\theta+\cos(2\theta)+\cdots+\cos(n\theta)=\frac{1}{2}+\frac{\sin\left(\left[n+\frac{1}{2}\right]\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}}$$

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7. By definition, function $f(z) = u(x,y) + iv(x,y)$ is analytic if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In our case

$$f(z) = \cancel{e^{z^2}} = e^{x^2-y^2}\left[\cos(2xy) + i\sin(2xy)\right] =$$

$$\begin{cases} u(x,y) = e^{x^2-y^2}\cdot\cos(2xy) \\ v(x,y) = e^{x^2-y^2}\cdot\sin(2xy) \end{cases}$$

1) Then $\frac{\partial u}{\partial x} = \frac{d}{dx}\left[e^{x^2-y^2}\cdot\cos(2xy)\right) = 2xe^{x^2-y^2}\cdot\cos(2xy) - 2ye^{x^2-y^2}\cdot\sin(2xy)$

$$\boxed{\frac{\partial u}{\partial x} = 2e^{x^2-y^2}\left(x\cdot\cos(2xy) - y\cdot\sin(2xy)\right)}$$

2) $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^2-y^2}\cdot\sin(2xy)\right. = -2ye^{x^2-y^2}\cdot\sin(2xy) + 2xe^{x^2-y^2}\cdot\cos(2xy)$

$$\boxed{\frac{\partial v}{\partial y} = 2e^{x^2-y^2}\left(x\cdot\cos(2xy) - y\cdot\sin(2xy)\right)}$$

3) $\frac{\partial u}{\partial y} = \frac{d}{dy} = \left[e^{x^2-y^2}\cdot\cos(2xy)\right] = -2ye^{x^2-y^2}\cdot\cos(2xy) - 2xe^{x^2-y^2}\cdot\sin(2xy)$

$$\boxed{\frac{\partial u}{\partial y} = -2e^{x^2-y^2}\left(y\cdot\cos(2xy) + x\sin(2xy)\right)}$$

a) $\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left[e^{x^2-y^2}\cdot \sin(2xy)\right] = 2xe^{x^2-y^2}\cdot \sin 2xy + 2ye^{x^2-y^2}\cdot \cos(2xy)$

$$\boxed{\frac{\partial v}{\partial x} = 2e^{x^2-y^2}\left(y\cdot\cos(2xy) + x\sin(2xy)\right)}$$

Thus $\frac{\partial v}{\partial x} = 2e^{x^2-y^2}\left(x\cdot\cos(2xy) - y\sin(2xy)\right) = \frac{\partial v}{\partial y} \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial v}{\partial x} = -2e^{x^2-y^2}\left(y\cdot\cos(2xy) + x\cdot\sin(2xy)\right) = -\frac{\partial v}{\partial y} \Leftrightarrow \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$

Function $f(z)$ is analytic all points $z \in \mathbb{C}$

b) if $f(z) = u(x,y) + iv(x,y)$ is analytic then

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{(z=z_0)} = \left(\frac{\partial v}{\partial x} + i\frac{-\partial v}{\partial y}\right)_{(x_0,y_0)}$$

In our case

$z_0 = i = 0+1i \Leftrightarrow (x_0,y_0) = (0,1)$

7b) $z_0 = i = 0 + i0 \Leftrightarrow (x_0, y_0) = (0,1)$

$$f(z) = e^{x^2-y^2}\left[\cos(2xy) + i\sin(2xy)\right] = u(x,y) + iv(x,y) \Leftrightarrow$$

$$\begin{cases} u(x,y) = e^{x^2-y^2} \cdot \cos(2xy) \\ v(x,y) = e^{x^2-y^2} \cdot \sin(2xy) \end{cases}$$

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}\left(x \cdot \cos(2xy) - y \cdot \sin(2xy)\right)$$

$$\frac{\partial v}{\partial x} = 2e^{x^2-y^2}\left(y \cdot \cos(2xy) + x \cdot \sin(2xy)\right)$$

Then $f'(i) = \left(\frac{\partial u}{\partial x}\right) + i\left(\frac{\partial v}{\partial x}\right)_{z=i} = \frac{\partial u}{\partial x}(0,1) + i\,\frac{\partial v}{\partial x}(0,1)$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,1)} = 2e^{0^2-1^2}\left(0 \cdot \cos(2\cdot 0\cdot 1) - 1\sin(2\cdot 1\cdot 0)\right)$$

$$\boxed{\left(\frac{\partial u}{\partial x}\right)_{(0,1)} = 2e^{-1}}$$

$$\frac{\partial v}{\partial x}_{(0,1)} = 2e^{0^2-1^2}\left(1 \cdot \cos(2\cdot 0\cdot 1) + 0 \cdot \sin(2\cdot 1\cdot 0)\right)$$

$$\boxed{\left(\frac{\partial v}{\partial x}\right)_{(0,1)} = 2e^{-1}}$$

Conclusion, $f'(i) = \left(\frac{\partial v}{\partial x}\right)_{(0,1)} + i\left(\frac{\partial v}{\partial x}\right)_{(0,1)} = 2e^{-1} + i\,2e^{-1}$

$$\boxed{f'(i) = 2e^{-1}(1+i)}$$

9. $f(z) = e^{x} + ie^{2y}$, $\begin{cases} v(x,y) = e^{x} \\ v(x,y) = e^{2y} \end{cases}$

$\frac{\partial v}{\partial x} = e^{x}$, $\partial v/\partial y = 2e^{2y} \Rightarrow \frac{\partial v}{\partial x} \neq \frac{\partial v}{\partial y}$

The function is not analytic. However it is differentiable at

$e^{x} = 2e^{2y}$

$\ln e^{x} = \ln 2 \ln e^{2y}$

$\frac{\ln e^{x}}{\ln 2} = \ln e^{2y}$

$$\boxed{\frac{x - \ln 2}{2} = y \quad \text{(real solution)}}$$

$$\boxed{y = \frac{1}{2} i \left(2\pi n - i \ln\left(\frac{e^{x}}{2}\right)\right), \quad n \in \mathbb{Z}}$$

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9. By definition, rules for taking derivatives of complex functions

$$(f(z) \pm g(z))' = f'(z) \pm g'(z)$$

$$(f(z)\,g(z))' = f'(z)\,g(z) + f(z)\,g'(z)$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)\,g(z) - f(z)\,g'(z)}{g^2(z)}, \quad g(z) \neq 0$$

$$f(g(z))' = f'(g(z)) \cdot g'(z)$$

In our case,

a) $f(z) = (z^2 - 3i)^{-6}$

$$f'(z) = ((z^2-3i)^{-6})' = -6 \cdot (z^2-3i)^{-6-1} \cdot (z^2-3i)' =$$

$$(-6) \cdot (z^2-3i)^{-7} \cdot 2z = -12z(z^2-3i)^{-7}$$

$$\boxed{f(z) = (z^2-3i)^{-6} \Leftrightarrow f'(z) = -12z(z^2-3i)^{-7}}$$

b) $f(z) = \dfrac{(z+2)^3}{(z^2+iz+1)^4}$

$$f'(z) = \frac{((z+2)^3)'\,(z^2+iz+1)^4 - (z+2)^3\,((z^2+iz+1)^4)'}{((z^2+iz+1)^4)^2}$$

$$\frac{(z+2)^2 (z^2+iz+1)^3 \left(3(z^2+iz+1) - 8z(z+2)\right)}{(z^2+iz+1)^8}$$

$$\frac{(z+2)^2 \left(-5z^2 + z(-16+3i) + 3\right)}{(z^2+iz+1)^5}$$

$$\boxed{f(z) = \frac{(z+2)^3}{(z^2+iz+1)^4} \Rightarrow f'(z) = \frac{(z+2)^2 \left(-5z^2 + z(-16+3i) + 3\right)}{(z^2+iz+1)^5}}$$

10. Let $f_1(z) = \operatorname{Re} z$, $f_2(z) = i \operatorname{Im} z$ which are not analytic

Then $f_1(z) + f_2(z) = z$ which is analytic

11. Since $f(z)$ is ~~[illegible]~~ analytic, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

For some $g(z)$ to be analytic, we need $p(x,y) - iq(x,y)$ to be

$$\frac{dp}{dx} = d(-q)/dy$$

$$\frac{d-q}{dx}$$

With first equations in mind, we need

$$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$$

$$-\frac{\partial v}{\partial x} = \partial v/\partial x$$

In terms of u, v $(p=u, q=v)$

$$\partial v/\partial x = -\partial v/\partial y$$

$$\partial u/\partial y = \partial v/\partial x$$

and both these are 0. So u and v must be constants

f(z) must be constant function

rthogonal
x, y0).

12. Suppose $z_0 = x_0 + iy_0$ is a point in D and $c_1 = u(x_0, y_0)$ and $c_2 = v(x_0, y_0)$. Define

$$\vec{n}_1 = \begin{pmatrix} u_x(x_0, y_0) \\ u_y(x_0, y_0) \end{pmatrix} \quad \text{and} \quad \vec{n}_2 = \begin{pmatrix} v_x(x_0, y_0) \\ v_y(x_0, y_0) \end{pmatrix}$$

Since f is analytic and $f'(z_0) \neq 0$ (by assumption) and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = u_x(x_0, y_0) - iu_y(x_0, y_0) = v_y(x_0, y_0) + iv_x(x_0, y_0)$$

Using C-R, we see $\vec{n}_1 \neq \vec{0}$ and $\vec{n}_2 \neq \vec{0}$. The vector $\vec{n}_1$ is orthogonal to tangent line I_1 to the level curve $u(x, y) = c_1$ at point (x_0, y_0). Similarly, vector $\vec{n}_2$ is orthogonal to tangent line I_2 to level curve $v(x, y) = c_2$ at point (x_0, y_0). The lines

I_1, I_2 will be orthogonal if and only if the normal vectors $\vec{n}_1$, $\vec{n}_2$ are orthogonal ie $\vec{n}_1 \cdot \vec{n}_2 = 0$

Computing

$$\vec{n}_1 \cdot \vec{n}_2 = \begin{pmatrix} u_x(x_0, y_0) \\ u_y(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} v_x(x_0, y_0) \\ v_y(x_0, y_0) \end{pmatrix} =$$

$$= u_x(x_0, y_0)\, v_x(x_0, y_0) + u_y(x_0, y_0)\, v_y(x_0, y_0) =$$

$$= -u_x(x_0, y_0)\, u_y(x_0, y_0) + u_y(x_0, y_0)\, u_x(x_0, y_0) = 0$$

by the CR equations $v_x = -u_y$, $v_y = u_x$. Thus I_1, I_2 are orthogonal at (x_0, y_0).