MATH 366 Assignment 2 P1

1. a) Note $e^z = e^x(\cos y + i\sin y)$ for $z = x+iy$

If $x=c$, then $w = e^z = e^x(\cos y + i\sin y) = e^c\cos y + ie^c\sin y$. Then

$$|w| = \sqrt{(e^c\cos y)^2 + (e^c\sin y)^2} = \sqrt{(e^c)^2\cos^2 y + (e^c)^2\sin^2 y} =$$
$$\boxed{= \sqrt{(e^c)^2 \cdot 1} = e^c}$$

So the function $w=e^z$ maps line $x=c$, with c a constant onto the circles $|w| = e^c$

b) If $y=c$, then $w=e^z = e^x(\cos y + i\sin y) = e^x\cos c + ie^x\sin c$. Then for each $x \in \mathbb{R}$, we have $\tan\theta = \frac{e^x\sin c}{e^x\cos c} = \frac{\sin c}{\cos c} = \tan c$, $\theta = c$.

Note $w = e^z = e^x(\cos y + i\sin y) \neq 0$ because ~~ex~~ $e^x > 0$ for each x and there is no y such that ~~cos~~ $\cos y = \sin y = 0$.

Hence, function $w=e^z$ maps line $y=c$ with c constant onto half rays $\theta = c$ from the origin to infinity excluding origin.

c) Consider $w = \mathbb{C} - \{0\}$. Then $w = |w|(\cos\theta + i\sin\theta)$ where $|w| \neq 0$ and $0 \leq \theta \leq 2\pi$

~~If~~ If $w = e^z$, then $z = \log(w) = \ln|w| + i(\theta + 2\pi n)$, $n \in \mathbb{Z}$

For $n=0$, we have $z = \ln|w| + i(\theta + 2\pi \cdot 0) = \ln|w| + i\theta$. This means $z \in \Pi$

Hence, function $w = e^z$ maps strip $\Pi = \{z = x+iy \mid 0 \leq y \leq 2\pi\}$ onto $\mathbb{C} - \{0\}$

a) Recall $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$. Let $z = x + iy$. Then

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y+ix} - e^{y-ix}}{2i} = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \frac{\cos x\,(e^{-y} - e^{y}) + i\sin x\,(e^{-y} + e^{y})}{2i} = \frac{\sin x\,(e^{-y} + e^{y})}{2} + \frac{i\cos x\,(e^{y} - e^{-y})}{2}$$

$$= \sin x \cosh y + i\cos x \sinh y$$

So $|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$= \sin^2 x \cosh^2 y + (1 - \sin^2 x)\sinh^2 y = \sin^2 x \cosh^2 y - \sin^2 x \sinh^2 y + \sinh^2 y$

$= \sin^2 x\,(\cosh^2 y - \sinh^2 y) + \sinh^2 y = \sin^2 x + \sinh^2 y$ because $\cosh^2 y - \sinh^2 y = 1$

b) Since $\sinh = \frac{e^y - e^{-y}}{2}$, $\lim_{y \to \infty} \sinh y = \lim_{y\to\infty} \frac{e^y - e^{-y}}{2} = \frac{\infty - 0}{2} = \infty$

This means for any positive constant M, there exists y such that

$\sinh > M$. Consider $z = 0 + iy$. Then

$|\sin z|^2 = \sin^2 0 + \sinh^2 y = \sinh^2 y$ and so

$$|\sin z| = \sinh y > M$$

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3\. a) By definition, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. Then

$$\sin z = 0 \iff \frac{e^{iz} - e^{-iz}}{2i} = 0 \iff e^{iz} - e^{-iz} = 0 \iff e^{i2z} - 1 = 0$$

$$\iff e^{i2z} = 1 \iff i2z = \operatorname{Log} 1 \iff i2z = \ln|1| + i(0 + 2\pi n), n \in \mathbb{Z}$$

$$\iff \cancel{i2z}\ z = \pi n, n \in \mathbb{Z}.$$

So $\boxed{z = \pi n, n \in \mathbb{Z}.}$

b) By definition, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. Then

$$\cos \pi z = 0 \iff \frac{e^{i\pi z} + e^{-i\pi z}}{2} = 0 \iff e^{i\pi z} + e^{-i\pi z} = 0$$

$$\iff e^{i2\pi z} + 1 = 0 \iff e^{i2\pi z} = -1 \iff i2\pi = \operatorname{Log}(-1) \iff i2\pi z = \ln|-1| + i(\pi + 2\pi n), n \in \mathbb{Z}$$

$$\iff i2\pi z = \ln 1 + i(\pi + 2\pi n), n \in \mathbb{Z} \iff i2\pi z = i(\pi + 2\pi n), n \in \mathbb{Z}$$

$$\iff z = \tfrac{1}{2} + n, n \in \mathbb{Z} \iff z = \frac{2n+1}{2}, n \in \mathbb{Z}$$

So $\boxed{z = \dfrac{2n+1}{2}, n \in \mathbb{Z}}$

c) $1 + e^{2z} = 0 \Leftrightarrow e^{2z} = -1 \Leftrightarrow 2z = \mathrm{Log}(-1) \Leftrightarrow 2z = \ln|-1| + i(\pi + 2\pi n)$, $n \in \mathbb{Z}$

$\Leftrightarrow 2z = \ln 1 + i(\pi + 2\pi n), n \in \mathbb{Z} \Leftrightarrow 2z = i(\pi + 2\pi n), n \in \mathbb{Z}$

$\Leftrightarrow z = i\left(\frac{\pi}{2} + \pi n\right), n \in \mathbb{Z}$

So $\boxed{z = i\left(\frac{\pi}{2} + \pi n\right), n \in \mathbb{Z}}$

1. a) Let ~~$e^z \in \mathbb{Z}$~~ $e^z \in \mathbb{R}$ and $z = x + iy$. Since $e^z = e^{x+iy}$
$= e^x(\cos y + i \sin y) = e^x \cos y + i e^x \sin y$ and $e^z \in \mathbb{R}$, $e^x \sin y = 0$.
Note that $e^x > 0$ for each $x \in \mathbb{R}$. Then $\sin y = 0$ and so $y = \pi n$ where $n \in \mathbb{Z}$. This means $\mathrm{Im}\, z = n\pi$, ~~$n \in \mathbb{R}$~~ $n \in \mathbb{Z}$.

b) Let $e^z \in i \cdot \mathbb{R}$ and $z = x + iy$. Since $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$
$= e^x \cos y + i e^x \sin y$ and $e^z \in i\mathbb{R}$, $e^x \cos y = 0$. Note $e^x > 0$ for $x \in \mathbb{R}$.
Then $\cos y = 0$ and so $y = \frac{\pi}{2} + \pi n$, where $n \in \mathbb{Z}$. This means that $\boxed{\mathrm{Im}\, z = \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}.}$

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5. Let $z = x+iy$. Then

$$e^{5z+3i+2} = e^{5x+5iy+3i+2} = e^{5x+2+i(5y+3)} = e^{5x+2}\left(\cos(5y+3)+i\sin(5y+3)\right),$$

$$e^{iz^2} = e^{i(x+iy)^2} = e^{i(x^2+2ixy+i^2y^2)} = e^{i(x^2+2ixy-y^2)} = e^{-2xy+i(x^2-y^2)}$$

$$= e^{-2xy}\left(\cos(x^2-y^2)+i\sin(x^2-y^2)\right),$$

and so $\left|e^{5z+3i+2}\right| = e^{5x+2}$ and $\left|e^{iz^2}\right| = e^{-2xy}$.

Therefore

$$\left|e^{5z+3i+2}+e^{iz^2}\right| \leq \left|e^{5z+3i+2}\right| + \left|e^{iz^2}\right| \leq e^{5x+2}+e^{-2xy}$$

6) $\mathrm{Log}(z^2-1) = i\frac{\pi}{2}$,

$z^2-1 = e^{i\pi/2}$, $z = e^{\frac{i\pi}{2}}+1$, $z^2 = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}+1$, $z^2 = 0+i1+1$

$z^2 = 1+i$

So $|1+i| = \sqrt{1+1} = \sqrt{2}$ and

$$z = \sqrt{1+i} = \sqrt{2}\left(\cos\frac{\frac{\pi}{4}+2\pi k}{2}+i\sin\frac{\frac{\pi}{4}+2\pi k}{2}\right), k=0,1$$

If $k=0$, then $z_0 = \sqrt{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$ →

If $k=1$, then $z_1 = \sqrt{2}\left(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}\right)$

Hence $\boxed{z_0 = \sqrt{2}\left(\cos \frac{\pi}{8} + i\sin \frac{\pi}{8}\right) \text{ and } z_1 = \sqrt{2}\left(\cos \frac{9\pi}{8} + i\sin \frac{9\pi}{8}\right)}$

b) $e^{2z} + e^{z} + 1 = 0$

Let $t = e^{z}$. Then $t^2 + t + 1 = 0$

$t = \dfrac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot 1}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

So $t_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $t_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

If $t_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$, then $e^{z} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$,

$z = \operatorname{Log}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$, $z = \ln\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + i\left(\frac{2\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$

$z = \ln\sqrt{\frac{1}{4} + \frac{3}{4}} + i\left(\frac{2\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$

$z = \ln 1 + i\left(\frac{2\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$

$\boxed{z = i\left(\frac{2\pi}{3} + 2\pi n\right), n \in \mathbb{Z} \text{ for } t_1.}$

6 b) For $t_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$, $e^z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$z = \operatorname{Log}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$, $z = \ln\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} + i\left(\frac{4\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$

$z = \ln\sqrt{\frac{1}{4} + \frac{3}{4}} + i\left(\frac{4\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$

$\vdots$

$z = \ln\sqrt{1} + i\left(\frac{4\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$

Hence $z = i\left(\frac{4\pi}{3} + 2\pi n\right), n \in \mathbb{Z}$ for t_2

Conclusion, $\boxed{z = i\left(\pi \pm \frac{\pi}{3} + 2\pi n\right), n \in \mathbb{Z}}$

7. Recall that the composition of analytic functions is analytic

a) $f(z) = \sin(e^z)$. Since $h(z) = \sin z$ and $g(z) = e^z$ are analytic on $\mathbb{C}$,

$\boxed{f(z) = h(g(z)) \text{ is analytic on } \mathbb{C}.}$

b) $f(z) = \sqrt{e^z + 1}$. Let $h(z) = \sqrt{z}$ and $g(z) = e^z + 1$. The function $h(z) = \sqrt{z}$ is analytic on $\mathbb{C} - \{0\}$. The function $g(z) = e^z + 1$ is analytic on $\mathbb{C}$. So $e^z + 1 = 0$, $e^z = -1$, $z = \ln|-1| + i(\pi + 2\pi n), n \in \mathbb{Z}$; $z = i(\pi + 2\pi n), n \in \mathbb{Z}$.

Therefore, function $f(z) = h(g(z))$ is analytic on $\boxed{\mathbb{C} - \{i(\pi + 2\pi n) \mid n \in \mathbb{Z}\}}$

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c) $f(z) = \frac{1}{e^z - 1}$. Let $h(z) = 1/z$ and $g(z) = e^z - 1$. The function

$h(z) = 1/z$ is analytic on $\mathbb{C} - \{0\}$. The function $g(z) = e^z - 1$ is analytic on $\mathbb{C}$.

So $e^z - 1 = 0$, $e^z = 1$, $z = \mathrm{Log}(1) = \ln 1 + i(0 + 2\pi n)$, $n \in \mathbb{Z}$

$z = i2\pi n$, $n \in \mathbb{Z}$.

Therefore, the function $f(z) = h(g(z))$ is analytic on $\mathbb{C} - \{i2\pi n \mid n \in \mathbb{Z}\}$

9. Recall $\cos z = \frac{e^{iz} + e^{-iz}}{2}$. So $\frac{e^{iz} + e^{-iz}}{2} = \sqrt{3}$

$e^{iz} + e^{-iz} = 2\sqrt{3}$, $\frac{e^{i2z} - 2\sqrt{3}e^{iz} + 1}{e^{iz}} = 0$, $\boxed{e^{i2z} - 2\sqrt{3}e^{iz} + 1 = 0}$

Let $t = e^{iz}$. Then $t^2 - 2\sqrt{3}t + 1 = 0$

$t = \frac{2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{2\sqrt{3} \pm \sqrt{12 - 4}}{2} = \boxed{\sqrt{3} \pm \sqrt{2}}$

If $t = \sqrt{3} + \sqrt{2}$, then $e^{iz} = \sqrt{3} + \sqrt{2}$

$iz = \mathrm{Log}(\sqrt{3} + \sqrt{2})$, $iz = \ln(\sqrt{3} + \sqrt{2}) + i(0 + 2\pi n)$, $n \in \mathbb{Z}$,

$z = -i\ln(\sqrt{3} + \sqrt{2}) + 2\pi n$, $n \in \mathbb{Z}$, $\boxed{z = 2\pi n - i\ln(\sqrt{3} + \sqrt{2}),\ n \in \mathbb{Z}}$

If $t = \sqrt{3} - \sqrt{2}$, then $e^{iz} = \sqrt{3} - \sqrt{2}$. $iz = \mathrm{Log}(\sqrt{3} - \sqrt{2})$, $iz = \ln(\sqrt{3} - \sqrt{2}) + i(0 + 2\pi n)$

$iz = \ln(\sqrt{3} - \sqrt{2}) + i2\pi n$, $n \in \mathbb{Z}$. $z = -i\ln(\sqrt{3} - \sqrt{2}) + 2\pi n$, $n \in \mathbb{Z}$.

Hence, $z = 2\pi n - i\ln(\sqrt{3} \pm \sqrt{2})$, $n \in \mathbb{Z}$