# puter Systems: A Programmer's Perspective

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# Contents

| IV | Differential Calculus |
|---|---|

# Reference

# 1. Numbers

## 1.1 Number Systems

**Definition 1.1.1 – Natural Numbers.**

$$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$$

R It is not uncommon for zero to be excluded from the natural numbers. In fact, some exclude zero from the natural numbers and then describe the set of natural numbers that include zero the whole numbers.

$$\mathbb{W} = \{0, 1, 2, 3, \ldots\}$$

For the purposes of these notes, zero will be included within the set of natural numbers.

**Definition 1.1.2 – Integers.**

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

**Definition 1.1.3 – Positive Integers.**

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

**Definition 1.1.4 – Rational Numbers.**

$$\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$$

**Definition 1.1.5 – Proper Fraction.** Given $m < n$, then the fraction $m/n$ is called proper.

**Definition 1.1.6 – Improper Faction.** Given $m > n$, then the fraction $m/n$ is called improper.

## 1.2 Prime Numbers

**Definition 1.2.1 – Greatest Common Divisor.** Suppose that $m$ and $n$ are positive integers. The greatest common divisor is the largest divisor (factor) common to both $m$ and $n$.

**Definition 1.2.2 – Relatively Prime.** Two integers $m$ and $n$ are relatively prime to each other, $m \perp n$, if they share no common positive integer divisors (factors) except 1.

$m \perp n$ if $\gcd(m, n) = 1$.

### 1.2.1 Listing of Prime Numbers 2-997

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 |
| 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 | 73 | 79 | 83 | 89 |
| 97 | 101 | 103 | 107 | 109 | 113 | 127 | 131 | 137 | 139 | 149 | 151 |
| 157 | 163 | 167 | 173 | 179 | 181 | 191 | 193 | 197 | 199 | 211 | 223 |
| 227 | 229 | 233 | 239 | 241 | 251 | 257 | 263 | 269 | 271 | 277 | 281 |
| 283 | 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 | 353 | 359 |
| 367 | 373 | 379 | 383 | 389 | 397 | 401 | 409 | 419 | 421 | 431 | 433 |
| 439 | 443 | 449 | 457 | 461 | 463 | 467 | 479 | 487 | 491 | 499 | 503 |
| 509 | 521 | 523 | 541 | 547 | 557 | 563 | 569 | 571 | 577 | 587 | 593 |
| 599 | 601 | 607 | 613 | 617 | 619 | 631 | 641 | 643 | 647 | 653 | 659 |
| 661 | 673 | 677 | 683 | 691 | 701 | 709 | 719 | 727 | 733 | 739 | 743 |
| 751 | 757 | 761 | 769 | 773 | 787 | 797 | 809 | 811 | 821 | 823 | 827 |
| 829 | 839 | 853 | 857 | 859 | 863 | 877 | 881 | 883 | 887 | 907 | 911 |
| 919 | 929 | 937 | 941 | 947 | 953 | 967 | 971 | 977 | 983 | 991 | 997 |

# 2. Operations

## 2.1 Dyadic Operations

**Definition 2.1.1 – Operation of Addition (OOA).**

$$\underbrace{\underbrace{a}_{\text{Augend}} + \underbrace{b}_{\text{Addend}}}_{\text{Sum}} \tag{2.1}$$

More generally,

$$\underbrace{\underbrace{a}_{\text{Summand}} + \underbrace{b}_{\text{Summand}}}_{\text{Sum}} \tag{2.2}$$

**Definition 2.1.2 – Operation of Multiplication (OOM).**

$$\underbrace{\underbrace{a}_{\text{Multiplicand}} \times \underbrace{b}_{\text{Multiplier}}}_{\text{Product}} \tag{2.3}$$

More generally,

$$\underbrace{\underbrace{a}_{\text{Factor}} \times \underbrace{b}_{\text{Factor}}}_{\text{Product}} \tag{2.4}$$

**Definition 2.1.3 – Operation of Exponentiation (OOE).**

$$\underbrace{\underbrace{b}_{base}^{\overbrace{m}^{Exponent}}}_{Power} \tag{2.5}$$

**Definition 2.1.4 – Common Denominator (CD).**

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \tag{2.6a}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \tag{2.6b}$$

**Rule 2.1.1 – Fraction Operation of Addition (FOOA).**

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \tag{2.7a}$$

$$\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d} \tag{2.7b}$$

# 3. Notation

## 3.1 Negation Notation

**Notation 3.1.1 – Operation of Negation (ONeg).**

$$-a = \neg a \tag{3.1a}$$
$$\neg a = -a \tag{3.1b}$$

I have used a different symbol, $\neg$, as the prefix negation operator only to differentiate it from the minus sign infix operator symbol, $-$, which is also used as the infix operator for the dyadic operation of subtraction. I will refer to this change of symbol as ONeg. This is used only as a teaching tool and should not be confused with the logic negation operator. Another advantage of using this symbol is that it reduces the number of delimiters used in an expression for example, $\neg a$ versus $(-a)$.

- Negative five: $-5$
- Negative five: $\neg 5$
- Four minus five: $4 - 5$
- Four minus negative five: $4 - -5$
- Four minus negative five: $4 - (-5)$
- Four minus negative five: $4 - \neg 5$
- Negative four minus five: $-4 - 5$
- Negative four minus five: $\neg 4 - 5$

## 3.2 Multiplication Notation

**Notation 3.2.1 – Multiplication Center-Dot (MC).**

$$a \cdot b \tag{3.2}$$

**Notation 3.2.2 – Multiplication Juxtaposition (MJ).**

$$ab, a(b), (a)b, (a)(b), a[b], [a]b, [a][b] \tag{3.3}$$

**Notation 3.2.3 – Multiplication Times (MT).**

$$a \times b \tag{3.4}$$

**Notation 3.2.4 – Juxtaposition to Center-Dot (JTC).**

$$ab = a \cdot b \tag{3.5}$$

**Notation 3.2.5 – Center-Dot to Justapostion (CTJ).**

$$a \cdot b = ab \tag{3.6}$$

## 3.3 Power Notation

**Notation 3.3.1 – Power Exponent Negative Exponent (PoNegE).**

$$b^{-k} = \frac{1}{b^k} \tag{3.7}$$

$$\frac{1}{b^k} = b^{-k} \tag{3.8}$$

**Notation 3.3.2 – Power To Factor (PoTF).**

$$a^n = a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot a_n \tag{3.9}$$

**Notation 3.3.3 – Power To Logarithm (PoTL).**

$$y = b^x \Rightarrow x = \log_b y \tag{3.10}$$

**Notation 3.3.4 – Factor To Power (FTPo).**

$$a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot a_n = a^n \tag{3.11}$$

**Notation 3.3.5 – Radical To Power (RTPo).**

$$\sqrt[m]{b^n} = b^{\frac{n}{m}} \tag{3.12}$$

## 3.4 Logarithm Notation

**Notation 3.4.1 – Logarithm Exponent Visible (LEV).**

$$\log_b y \Rightarrow \log_b y = x \tag{3.13}$$

**Notation 3.4.2 – Logarithm to Power (LTPo).**

$$x = \log_b y \Rightarrow y = b^x \tag{3.14}$$

## 3.5 Derivative Notation

**Notation 3.5.1 – Leibniz's first derivative.**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{d\,[f(x)]}{\mathrm{d}x} = \frac{d}{\mathrm{d}x}\,[f(x)] \tag{3.15}$$

**Notation 3.5.2 – Leibniz's second derivative.**

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \tag{3.16}$$

**Notation 3.5.3 – Leibniz's nth derivative.**

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} \tag{3.17}$$

**Notation 3.5.4 – Leibniz's Evaluate derivative.**

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = \frac{\mathrm{d}y}{\mathrm{d}x}(a) \tag{3.18}$$

**Notation 3.5.5 – LaGrange's first derivative.**

$$f'(x) \tag{3.19}$$

**Notation 3.5.6 – LaGrange's second derivative.**

$$f''(x) \tag{3.20}$$

**Notation 3.5.7 – LaGrange's nth derivative.**

$$f^{(n)}(x) \tag{3.21}$$

**Notation 3.5.8 – LaGrange's Evaluate derivative.**

$$f'(a) \tag{3.22}$$

**Notation 3.5.9 – Euler's first derivative.**

$$Df = D_x f \tag{3.23}$$

**Notation 3.5.10 – Euler's second derivative.**

$$D^2 f = D_x^2 f \tag{3.24}$$

**Notation 3.5.11 – Euler's nth derivative.**

$$D^n f = D_x^n \tag{3.25}$$

# 4. Properties

## 4.1 Summary of Field Properties

| Name | Addition | Multiplication |
|---|---|---|
| Commutative | $a+b=b+a$ | $a\cdot b=b\cdot a$ |
| Associative | $(a+b)+c=a+(b+c)$ | $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ |
| Distributive | $a(b+c)=ab+ac$ | $(a+b)c=ac+bc$ |
| Identity | $a+0=a=0+a$ | $a\cdot 1=a=1\cdot a$ |
| Inverse | $a+(-a)=0=(-a)+a$ | $a\cdot a^{-1}=1=a^{-1}\cdot a$ |

Table 4.1: Summary of the Field Properties

## 4.2 Properties of Addition

**Property 4.2.1 – Commutative Property of Addition (CPA).**

$$ab=ba \tag{4.1}$$

**Property 4.2.2 – Associative Property of Addition (APA).**

$$a+b+c=(a+b)+c \tag{4.2a}$$
$$a+b+c=a+(b+c) \tag{4.2b}$$

**Property 4.2.3 – Distributive Property Factoring (DPF).**

$$ba+ca=(b+c)a \tag{4.3a}$$
$$ab+ac=a(b+c) \tag{4.3b}$$

**Property 4.2.4 – Additive Identity (AId).**

$$a+0=a \tag{4.4a}$$
$$a=a+0 \tag{4.4b}$$

**Property 4.2.5 – Additive Inverse (AI).**

$$a + (-a) = 0 \tag{4.5a}$$

## 4.3 Properties of Multiplication

**Property 4.3.1 – Commutative Property of Multiplication (CPM).**

$$a \cdot b = b \cdot a \tag{4.6}$$

**Property 4.3.2 – Associative Property of Multiplication (APM).**

$$a \cdot b \cdot c = (a \cdot b) \cdot c \tag{4.7a}$$

$$a \cdot b \cdot c = a \cdot (b \cdot c) \tag{4.7b}$$

**Property 4.3.3 – Distributive Property Expanding (DPE).**

$$a(b + c) = ab + ac \tag{4.8a}$$

$$(b + c)a = ba + ca \tag{4.8b}$$

**Property 4.3.4 – Multiplicative Identity (MId).**

$$1a = a \tag{4.9a}$$

$$a = 1a \tag{4.9b}$$

R If the coefficient of a univariate monomial is the multiplicative identity 4.9a, 1, then it is not shown in it's canonical form.

$$\begin{aligned} C_k x^k &= C_k x^k \\ &= 1x^k \\ &= x^k \end{aligned}$$

**Property 4.3.5 – Multiplicative Inverse (MI).**

$$a \cdot \frac{1}{a} = 1 \tag{4.10a}$$

$$a \cdot a^{-1} = 1 \tag{4.10b}$$

**Property 4.3.6 – Zero Product (ZPr).**

$$\text{if} a \cdot b = 0, \text{ then} a = 0 \,\text{or} b = 0 \tag{4.11a}$$

## 4.4 Properties of Subtraction

**Definition 4.4.1 – Definition of Subtraction (DOS).**

$$a - b = a + \neg b \tag{4.12a}$$
$$a + \neg b = a - b \tag{4.12b}$$

## 4.5 Properties of Powers

**Property 4.5.1 – Power Inverse (PoId).**

$$1 = b^0 \tag{4.13a}$$
$$b^0 = 1 \tag{4.13b}$$

## 4.6 Properties of Equality

**Property 4.6.1 – Reflexive Property of Equality (RPE).**

$$a = a \tag{4.14a}$$

**Property 4.6.2 – Substitution Property of Equality (SPE).**

Given $a = b$, then

$$E(a) = E(b) \tag{4.15}$$

$E(x)$ represents any expression.

**Property 4.6.3 – Symmetric Property of Equality (SyPE).**

$$a = b \quad \text{then} \quad b = a \tag{4.16a}$$

**Property 4.6.4 – Transitive Property of Equality (TPE).**

$$\text{if} \quad a = b \quad \text{and} \quad b = c \quad \text{then} \quad a = c \tag{4.17a}$$

**Property 4.6.5 – Zero Factor Property (ZFP).**

$$\text{if} \quad a \cdot b = 0 \quad \text{then} \quad a = 0 \quad \text{or} \quad b = 0 \tag{4.18a}$$

## 4.7 Properties of Inequality

**Property 4.7.1 – Substitution Property of Inequality (SPIn).**

$$a < b \quad \text{then} \quad a + c < b + c \tag{4.19a}$$

$$a < b \quad \text{and} c > 0, \quad \text{then} \quad ca < cb \tag{4.19b}$$

$$a < b \quad \text{and} c < 0, \quad \text{then} \quad ca > cb \tag{4.19c}$$

**Property 4.7.2 – Transitive Property of Inequality (TPIn).**

$$\text{if} \quad a < b \quad \text{and} \quad b < c \quad \text{then} \quad a > c \tag{4.20a}$$

# 5. Identities

## 5.1 Power Identities

**Identity 5.1.1 – Power of Power (PoPo).**

$$(b^m)^k = b^{m \cdot k} \tag{5.1a}$$
$$b^{m \cdot k} = (b^m)^k \tag{5.1b}$$

**Identity 5.1.2 – Power of a Product (PoPr).**

$$(a \cdot b)^k = a^k \cdot b^k \tag{5.2a}$$
$$a^k \cdot b^k = (a \cdot b)^k \tag{5.2b}$$

**Identity 5.1.3 – Product Common Base Powers (PrCBPo).**

$$b^m \cdot b^n = b^{m+n} \tag{5.3a}$$
$$b^{m+n} = b^m \cdot b^n \tag{5.3b}$$

**Identity 5.1.4 – Quotient Common Base Powers (QCBPo).**

$$\frac{b^m}{b^n} = b^{m-n} \tag{5.4a}$$
$$b^{m-n} = \frac{b^m}{b^n} \tag{5.4b}$$

**Identity 5.1.5 – Power of a Quotient of Powers (PoQPo).**

$$\left(\frac{a^m}{b^n}\right)^k = \frac{a^{m \cdot k}}{b^{n \cdot k}} \tag{5.5a}$$
$$\frac{a^{m \cdot k}}{b^{n \cdot k}} = \left(\frac{a^m}{b^n}\right)^k \tag{5.5b}$$

**Identity 5.1.6 – Power of a Product of Powers (PoPrPo).**

$$(a^m \cdot b^n)^k = a^{m \cdot k} \cdot b^{n \cdot k} \tag{5.6a}$$
$$a^{m \cdot k} \cdot b^{n \cdot k} = (a^m \cdot b^n)^k \tag{5.6b}$$

## 5.2 Logarithm Identities

**Identity 5.2.1 – Logarithm Power of a Power (LPoPo).**

$$\log_b x^n = n \log_b x \tag{5.7a}$$
$$n \log_b x = \log_b x^n \tag{5.7b}$$

**Identity 5.2.2 – Logarithm Product of Common Base Powers (LPrCBPo).**

$$\log_b(mn) = \log_b m + \log_b n \tag{5.8a}$$
$$\log_b m + \log_b n = \log_b(mn) \tag{5.8b}$$

**Identity 5.2.3 – Logarithm Quotient of Common Base Powers (LQCBPo).**

$$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n \tag{5.9a}$$
$$\log_b m - \log_b n = \log_b \left(\frac{m}{n}\right) \tag{5.9b}$$

## 5.3 Trigonometric Identities

**Identity 5.3.1 – Trigonometric Reciprocal Identities (TRId).**

$$\sin\theta = \frac{1}{\csc\theta} \tag{5.10a}$$
$$\cos\theta = \frac{1}{\sec\theta} \tag{5.10b}$$
$$\cot\theta = \frac{1}{\tan\theta} \tag{5.10c}$$
$$\csc\theta = \frac{1}{\sin\theta} \tag{5.10d}$$
$$\sec\theta = \frac{1}{\cos\theta} \tag{5.10e}$$
$$\tan\theta = \frac{1}{\cot\theta} \tag{5.10f}$$

**Identity 5.3.2 – Trigonometric Pythagorean Identities (TPythagId).**

$$\sin^2\theta + \cos^2\theta = 1 \tag{5.11a}$$
$$\sec^2\theta = \tan^2\theta + 1 \tag{5.11b}$$
$$\csc^2\theta = 1 + \cot^2\theta \tag{5.11c}$$

**Identity 5.3.3 – Trigonometric Tangent Identity (TanId).**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{5.12a}$$

**Identity 5.3.4 – Trigonometric Cotangent Identity (CotId).**

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{5.13a}$$

**Identity 5.3.5 – Sine Double Angle Identity (SinDAId).**

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{5.14a}$$

**Identity 5.3.6 – Cosine Double Angle Identity (CosDAId).**

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta && (5.15a)\\ &= 1 - 2\sin^2\theta && (5.15b)\\ &= 2\cos^2\theta - 1 && (5.15c)\end{aligned}$$

**Identity 5.3.7 – Tangent Double Angle Identity (TanDAId).**

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{5.16a}$$

**Identity 5.3.8 – Sine Sum of Angles Identity (SinSAId).**

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{5.17a}$$

**Identity 5.3.9 – Sine Difference of Angles Identity (SinDiffAId).**

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \tag{5.18a}$$

**Identity 5.3.10 – Cosine Sum of Angles Identity (CosSAId).**

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{5.19a}$$

**Identity 5.3.11 – Cosine Difference of Angles Identity (CosDiffAId).**

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \tag{5.20a}$$

**Identity 5.3.12 – Tangent Sum of Angles Identity (TanSAId).**

$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} \tag{5.21a}$$

**Identity 5.3.13 – Tangent Difference of Angles Identity (TanDiffAId).**

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} \tag{5.22a}$$

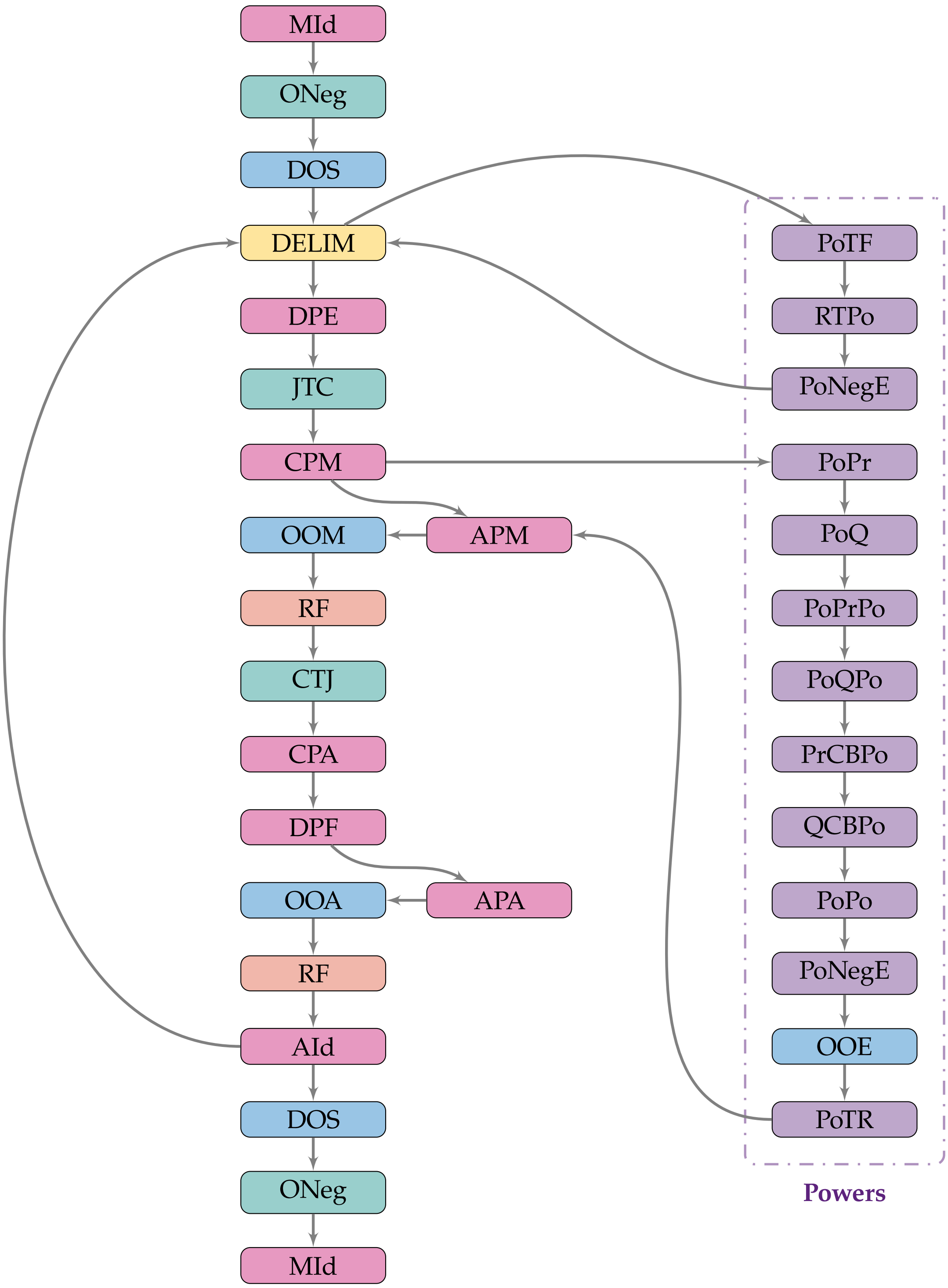



Figure 5.1: Simplifying Expressions Workflow:
Property, Operation, Notation, Powers, Delimiters, Process, Not Used

# 6. Limit Properties

## 6.1 Algebraic Limit Theorem

**Rule 6.1.1 – Algebraic Limit Theorem of a Constant(ALTC).** If $g(x) = a$, where $A$ is a constant, then

$$\lim_{x \to c} [A] = A \tag{6.1}$$

**Rule 6.1.2 – Algebraic Limit Theorem of a Sum (ALTS).** If both the limits $\lim_{x \to c} g(x) = L_1$ and $\lim_{x \to c} h(x) = L_2$ exist, then

$$\lim_{x \to c} [g(x) + h(x)] = \lim_{x \to c} g(x) + \lim_{x \to c} h(x) \tag{6.2}$$

**Rule 6.1.3 – Algebraic Limit Theorem of a Difference (ALTD).** If both the limits $\lim_{x \to c} g(x) = L_1$ and $\lim_{x \to c} h(x) = L_2$ exist, then

$$\lim_{x \to c} [g(x) - h(x)] = \lim_{x \to c} g(x) - \lim_{x \to c} h(x) \tag{6.3}$$

**Rule 6.1.4 – Algebraic Limit Theorem of a Product (ALTPr).** If both the limits $\lim_{x \to c} g(x) = L_1$ and $\lim_{x \to c} h(x) = L_2$ exist , then

$$\lim_{x \to c} [g(x) \cdot h(x)] = \lim_{x \to c} g(x) \cdot \lim_{x \to c} h(x) \tag{6.4}$$

**Rule 6.1.5 – Algebraic Limit Theorem of a Quotient (ALTQ).** If both the limits $\lim_{x \to c} g(x) = L_1$ and $\lim_{x \to c} h(x) = L_2$ exist and $L_2 \neq 0$ , then

$$\lim_{x \to c} \left[ \frac{g(x)}{h(x)} \right] = \frac{\lim_{x \to c} g(x)}{\lim_{x \to c} h(x)} \tag{6.5}$$

# 7. Calculus Rules

## 7.1 Monomial Derivative Rules

**Rule 7.1.1 – Derivative of a Constant (DC).**

$$[c]' = 0 \tag{7.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0 \tag{7.2}$$

**Rule 7.1.2 – Derivative of a Constant Multiple (DCM).**

$$[c\,f(x)]' = c\,[f(x)]' \tag{7.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[c\,f(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \tag{7.4}$$

**Rule 7.1.3 – Derivative of a Power (DPo).**

$$[x^n]' = nx^{n-1} \tag{7.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1} \tag{7.6}$$

## 7.2 Derivative Structural Rules

**Rule 7.2.1 – Derivative of a Sum (DS).**

$$[f(x) + g(x)]' = f'(x) + g'(x) \tag{7.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + \frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{7.8}$$

**Rule 7.2.2 – Derivative of a Difference (DD).**

$$[f(x) - g(x)]' = f'(x) - g'(x) \tag{7.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) - g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] - \frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{7.10}$$

**Rule 7.2.3 – Derivative of a Product (DPr).**

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \tag{7.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)]g(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{7.12}$$

**Rule 7.2.4 – Derivative of a Quotient (DQ).**

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \tag{7.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]\,g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{[g(x)]^2} \tag{7.14}$$

**Rule 7.2.5 – Derivative of a Composite Function (DComp).**

$$[f(g(x))]' = [g(x)]'\,[f(g(x))]' \tag{7.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = \frac{\mathrm{d}}{\mathrm{d}x}[g(x)]\,\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] \tag{7.16}$$

## 7.3 Trigonometric Derivative Rules

**Rule 7.3.1 – Derivative of Sine (DSin).**

$$[\sin(x)]' = \cos(x) \tag{7.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(x)] = \cos(x) \tag{7.18}$$

**Rule 7.3.2 – Derivative of Cosine (DCos).**

$$[\cos(x)]' = -\sin(x) \tag{7.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)] = -\sin(x) \tag{7.20}$$

**Rule 7.3.3 – Derivative of Tangent (DTan).**

$$[\tan x]' = \sec^2 \tag{7.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 \tag{7.22}$$

**Rule 7.3.4 – Derivative of Cosecant (DCsc).**

$$[\csc x]' = -\csc x \cot x \tag{7.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x \tag{7.24}$$

**Rule 7.3.5 – Derivative of Secant (DSec).**

$$[\sec x]' = \sec x \tan x \tag{7.25}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x \tag{7.26}$$

**Rule 7.3.6 – Derivative of Cotangent (DCot).**

$$[\cot x]' = -\csc^2 x \tag{7.27}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x \tag{7.28}$$

## 7.4 Logarithm Derivative Rules

**Rule 7.4.1 – Derivative of a Logarithm (DL).**

$$[\log_a x]' = \frac{1}{x \ln a} \tag{7.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a x] = \frac{1}{x \ln a} \tag{7.30}$$

**Rule 7.4.2 – Derivative of a Natural Logarithm (DNL).**

$$[\ln x]' = \frac{1}{x} \tag{7.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x} \tag{7.32}$$

## 7.5 Exponential Derivative Rules

**Rule 7.5.1 – Derivative of an Exponential(DExp).**

$$[a^x]' = a^x \ln a \tag{7.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[a^x] = a^x \ln a \tag{7.34}$$

**Rule 7.5.2 – Derivative of a Natural Exponential(DNExp).**

$$[e^x]' = e^x \tag{7.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x \tag{7.36}$$

II

# Algebra

# 8. Simplifying Univariate Polynomials

The definition of a univariate polynomial expression is based on the expanded canonical form of some polynomial expression. It might be that the original expression might not be in the expanded canonical form, so a process called simplifying by expanding will be introduced to manipulate the expression such that it can be written in its expanded canonical form.

This process of simplifying by expanding polynomial expressions will be developed to the extent that it will be used to simplify multivariate polynomials. We will start by simplifying univariate monomial expressions.

## 8.1 Simplifying Degree 0 Univariate Monomials

**Definition 8.1.1 – Indeterminate.**

$$x$$

An indeterminate is a symbol that is treated as a variable, but does not stand for anything else but itself and is used as a placeholder.

- it does **not** designate a constant or a parameter
- it is **not** an unknown that could be solved for
- it is **not** a variable designating a function argument

**[wikipedia:indeterminate]**

**Definition 8.1.2 – Degree of the Indeterminate.**

$$x^k$$

The exponent of an indeterminate power, $k$ is called the degree of the indeterminate.

**wikipedia:polynomial**

**Definition 8.1.3 – Coefficient.**

$$Cx^k$$

A coefficient, $C$ is a real number multiplicative factor.

**Definition 8.1.4 – Univariate Monomial.**

$$C_k x^k$$

A univariate monomial is made up of two factors. The first factor of a monomial, $C_k$, is the coefficient. The second factor of each monomial, $x^k$, is an indeterminate raised to a non-negative integer power $k$.

Degree 0 univariate polynomial expressions are made up of univariate monomials, $C_0$, called constants. The power identity is an indeterminate raised to a power of 0 has a value of 1. Thus, $x^0 = 1$ and results in the monomial $C_0 \cdot 1$. The canonical form of a product does not show the multiplicative identity factor, so what remains of this monomial product is only the coefficient factor $C_0$ and from now on will be referred to as a constant.

Degree 0 univariate polynomial expressions are usually a monomial in their canonical form if $C_0$ is a non-zero real number. The exception is if $C_0 = 0$, the additive identity, then the result is the zero polynomial, which can be considered a degree -1 polynomial.

The expression can be manipulated into its monomial canonical form by simplifying the expression. Simplifying the expression can be defined as evaluating the expression by following order of operations, which is the same as evaluating an arithmetic expression.

**Definition 8.1.5 – Univariate Like Terms.**

$$C_1 x^k = C_2 x^k$$

Two or more univariate monomials are defined as having like terms if each monomial has the same term, which will be the same indeterminate raised to the same positive integer power.

Sometimes the word term is used to describe monomials (including both the coefficient and the term), which may be confusing when trying to define like terms. For this reason, we will refer to the summands of a polynomial as monomials.

The monomials $5x^1$ and $3x^1$ can be described as having like terms because they share the common term $x^1$. One could also say that $5x^1$ and $3x^1$ are like terms by definition and consequenlty giving the reader an impression that $5x^1$ and $3x^1$ are terms themselves.

R A degree 1 indeterminate does not display the multiplicative identity in the exponent when its in canonical form.

**Example 8.1 – id:20141121-093747.**

Express $13x^0$ in canonical form.

S

**Solution:**

13 PoID(4.13b)

■

## 8.2 Simplifying Degree 1 Univariate Monomials

**Example 8.2 – id:20141120-202042.**

Express $5x^1$ in canonical form

S

**Solution:**

$5x$ MId(4.9b)

**Example 8.3 – id:20141121-093439.**

Express $7x^1 + 5$ in canonical form.

S

**Solution:**

$7x + 5$ MId(4.9b)

If the constant monomial is $0$, the additive identity, then the canonical form of a degree 1 univariate polynomial is a degree 1 monomial.

**Example 8.4 – id:20141120-203846.**

Simplify by expanding $6x + 7x$

S

**Solution:**
Notice that the indeterminate of each monomial is of degree 1; however, the exponent 1 is not shown. The monomials $6x$ and $7x$ have a like term of $x$.

$(6 + 7)x$ DPF(4.3a)
$13x$ OOA(2.1)

Notice that the sum of two monomials that have like terms can be found by adding the coefficients of the monomials. The distributive property in the factoring direction provides some insight to why we can add the coefficients of monomials that have like terms.

S

**Less Steps Solution:**

$13x$ OOA(2.1)

**Example 8.5 – id:20141027-075159.**

Simplify by expanding $7\,\text{cm} + 8\,\text{cm}$

S

**Solution:**

$$
\begin{aligned}
(7+8)\,\text{cm} & \quad \text{DPF(4.3a)} \\
15\,\text{cm} & \quad \text{OOA(2.1)}
\end{aligned}
$$

R Remember, if a monomial does not have a coefficient factor, then it's implied that the coefficient factor is 1, the multiplicative identity, and consequently its not explicitly shown.

**Example 8.6 – id:20141121-185558.**

Simplify by expanding $x + 5x$

S

**Solution:**
It can be useful when simplifying expressions to make the multiplicative identity (MId) factor explicit.

$$
\begin{aligned}
1x + 5x & \quad \text{MId(4.9a)} \\
(1+5)x & \quad \text{DPF(4.3a)} \\
6x & \quad \text{OOA(2.1)}
\end{aligned}
$$

S

**Less Steps Solution:**

$$
\begin{aligned}
1x + 5x & \quad \text{MId(4.9a)} \\
6x & \quad \text{OOA(2.1)}
\end{aligned}
$$

As one becomes more experienced, there is no reason to make the multiplicative identity coefficient explicit.

S

**Less Steps Solution:**

| | |
|---|---|
| $6x$ | OOA(2.1) |

■

**Example 8.7 – id:20141121-190857.**

Simplify by expanding $8x - 6x$

S

**Solution:**

| | |
|---|---|
| $8x + \neg 6x$ | DOS(4.12a) |
| $(8 + \neg 6)x$ | DPF(4.3a) |
| $2x$ | OOA(2.1) |

S

**Less Steps Solution:**

| | |
|---|---|
| $8x + \neg 6x$ | DOS(4.12a) |
| $2x$ | OOA(2.1) |

■

**Example 8.8 – id:20141121-193636.**

Simplify by expanding $3x - 5x$

S

**Solution:**

| | |
|---|---|
| $3x + \neg 5x$ | DOS(4.12a) |
| $(3 + \neg 5)x$ | DPF(4.3a) |
| $\neg 2x$ | OOA(2.1) |
| $-2x$ | ONeg(3.1b) |

S

**Less Steps Solution:**

$$\begin{aligned} 3x + \neg 5x & \quad \text{DOS(4.12a)} \\ \neg 2x & \quad \text{OOA(2.1)} \\ -2x & \quad \text{ONeg(3.1b)} \end{aligned}$$

S

**Less Steps Solution:**

$$-2x \quad \text{OOA(2.1)}$$

■

**Example 8.9 – id:20141106-150622.**

Simplify by expanding $13x - x$

S

**Solution:**

$$\begin{aligned} 13x - 1x & \quad \text{MId(4.9a)} \\ 13x + \neg 1x & \quad \text{DOS(4.12a)} \\ (13 + \neg 1)x & \quad \text{DPF(4.3a)} \\ 12x & \quad \text{OOA(2.1)} \end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned} 13x + \neg x & \quad \text{DOS(4.12a)} \\ 12x & \quad \text{OOA(2.1)} \end{aligned}$$

■

It is possible for a univariate monomial to have more than two terms in its non-canonical form. The associative property of addition will be used to help simplify these expressions.

**Example 8.10 – id:20141121-184652.**

Simplify by expanding $3x + 7x + 8x$

S

| | |
|---|---|
| $(3x + 7x) + 8x$ | APA(4.2a) |
| $(3 + 7)x + 8x$ | DPF(4.3a) |
| $10x + 8x$ | OOA(2.1) |
| $(10 + 8)x$ | DPF(4.3a) |
| $18x$ | OOA(2.1) |

S

**Less Steps Solution:**

| | |
|---|---|
| $(3x + 7x) + 8x$ | APA(4.2a) |
| $10x + 8x$ | OOA(2.1) |
| $18x$ | OOA(2.1) |

You might have noticed that this expression could be simplified in one step by adding the coefficient of the three monomials $3x$, $7x$ and $8x$, which have the like term $x$.

S

**Less Steps Solution:**

| | |
|---|---|
| $18x$ | OOA(2.1) |

**Example 8.11 – id:20141106-152020.**

Simplify by expanding $4x - 2x - x$

S

**Solution:**

| | |
|---|---|
| $4x - 2x - 1x$ | MId(4.9a) |
| $4x + \neg 2x + \neg 1x$ | DOS(4.12a) |
| $(4 + \neg 2)x + \neg 1x$ | DPF(4.3a) |
| $2x + \neg 1x$ | OOA(2.1) |
| $(2 + \neg 1)x$ | DPF(4.3a) |
| $1x$ | OOA(2.1) |
| $x$ | MId(4.9b) |

S

**Less Steps Solution:**

$$\begin{aligned} 4x + \neg 2x + \neg x && \text{DOS(4.12a)} \\ x && \text{OOA(2.1)} \end{aligned}$$

**Example 8.12 – id:20141108-194431.**

Simplify by expanding $-3 \cdot 7x - 2x \cdot 4$

S

**Solution:**

$$\begin{aligned} \neg 3 \cdot 7x - 2x \cdot 4 && \text{ONeg(3.1a)} \\ \neg 3 \cdot 7x + \neg 2x \cdot 4 && \text{DOS(4.12a)} \\ \neg 3 \cdot 7 \cdot x + \neg 2 \cdot x \cdot 4 && \text{JTC(3.5)} \\ \neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x && \text{CPM(4.6)} \\ (\neg 3 \cdot 7) \cdot x + (\neg 2 \cdot 4) \cdot x && \text{APM(4.7a)} \\ \neg 21 \cdot x + \neg 8 \cdot x && \text{OOM(2.3)} \\ \neg 21x + \neg 8x && \text{CTJ(3.6)} \\ (\neg 21 + \neg 8)x && \text{DPF(4.3a)} \\ \neg 29x && \text{OOA(2.1)} \\ -29x && \text{ONeg(3.1b)} \end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned} -3 \cdot 7x + \neg 2x \cdot 4 && \text{DOS(4.12a)} \\ \neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x && \text{CPM(4.6)} \\ \neg 21x + \neg 8x && \text{OOM(2.3)} \\ -29x && \text{OOA(2.1)} \end{aligned}$$

**Example 8.13 – id:20141108-194156.**

Simplify by expanding $3 \cdot 5x + 3x \cdot 4$

S

**Solution:**

$$\begin{aligned}
3\cdot 5\cdot x+3\cdot x\cdot 4 &\qquad \text{JTC(3.5)}\\
3\cdot 5\cdot x+3\cdot 4\cdot x &\qquad \text{CPM(4.6)}\\
(3\cdot 5)\cdot x+(3\cdot 4)\cdot x &\qquad \text{APM(4.7a)}\\
15\cdot x+12\cdot x &\qquad \text{OOM(2.3)}\\
15x+12x &\qquad \text{CTJ(3.6)}\\
(15+12)x &\qquad \text{DPF(4.3a)}\\
27x &\qquad \text{OOA(2.1)}
\end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned}
3\cdot 5\cdot x+3\cdot 4\cdot x &\qquad \text{CPM(4.6)}\\
15x+12x &\qquad \text{OOM(2.3)}\\
27x &\qquad \text{OOA(2.1)}
\end{aligned}$$

**Example 8.14 – id:20141108-173613.**

Simplify by expanding $8x\cdot 5$

S

**Solution:**

$$\begin{aligned}
8\cdot x\cdot 5 &\qquad \text{JTC(3.5)}\\
8\cdot 5\cdot x &\qquad \text{CPM(4.6)}\\
(8\cdot 5)\cdot x &\qquad \text{APM(4.7a)}\\
40\cdot x &\qquad \text{OOM(2.3)}\\
40x &\qquad \text{CTJ(3.6)}
\end{aligned}$$

## 8.3 Simplifying Degree 2 Univariate Monomials

**Example 8.15 – id:20151018-184931.**
Simplify $x\cdot x$

S

**Solution:**

| | |
|---|---|
| $1x^1 \cdot 1x^1$ | MId(4.9a) |
| $1x^{(1+1)}$ | PrCBPo(5.3a) |
| $1x^2$ | OOA(2.1) |
| $x^2$ | MId(4.9a) |

**Example 8.16 – id:20141120-202842.**

Express by expanding $1x^2$ in canonical form.

S

**Solution:**

| | |
|---|---|
| $x^2$ | MId(4.9b) |

**Example 8.17 – id:20141106-151138.**

Simplify by expanding $4x^2 + 12x^2$

S

**Solution:**

| | |
|---|---|
| $(4 + 12)x^2$ | DPF(4.3a) |
| $16x^2$ | OOA(2.1) |

S

**Less Steps Solution:**

| | |
|---|---|
| $16x^2$ | OOA(2.1) |

**Example 8.18 – id:20141106-154547.**

Simplify by expanding $x^2 - x + x^2 + x$

S

**Solution:**

$$\begin{aligned}
1x^2 - 1x + 1x^2 + 1x && \text{MId(4.9a)}\\
1x^2 + \neg 1x + 1x^2 + 1x && \text{DOS(4.12a)}\\
1x^2 + 1x^2 + \neg 1x + 1x && \text{CPA(4.1)}\\
(1+1)x^2 + (\neg 1 + 1)x && \text{DPF(4.3a)}\\
2x^2 + 0x && \text{OOA(2.1)}\\
2x^2 && \text{MId(4.9b)}
\end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned}
x^2 + \neg x + x^2 + x && \text{DOS(4.12a)}\\
x^2 + x^2 + \neg x + x && \text{CPA(4.1)}\\
2x^2 && \text{OOA(2.1)}
\end{aligned}$$

**Example 8.19 – id:20141108-194709.**

Simplify by expanding $-2x4x - x \cdot -x3$

S

**Solution:**

$$\begin{aligned}
-2x4x - 1x \cdot 1x3 && \text{MId(4.9a)}\\
\neg 2x4x - 1x \cdot 1x3 && \text{ONeg(3.1a)}\\
\neg 2 \cdot x \cdot 4 \cdot x + \neg 1 \cdot x \cdot 1 \cdot x \cdot 3 && \text{JTC(3.5)}\\
\neg 2 \cdot 4 \cdot x \cdot x + \neg 1 \cdot \neg 1 \cdot 3 \cdot x \cdot x && \text{CPM(4.6)}\\
\neg 2 \cdot 4 \cdot x^2 + \neg 1 \cdot \neg 1 \cdot 3 \cdot x^2 && \text{PrCBPo(5.3a)}\\
(\neg 2 \cdot 4) \cdot x^2 + (\neg 1 \cdot \neg 1 \cdot 3)x^2 && \text{APM(4.7a)}\\
\neg 8x^2 + 3x^2 && \text{OOM(2.3)}\\
(\neg 8 + 3)x^2 && \text{DPF(4.3a)}\\
\neg 5x^2 && \text{OOA(2.1)}\\
-5x^2 && \text{ONeg(3.1b)}
\end{aligned}$$

S

**Less Steps Solution:**

| | |
|---|---|
| $-2x4x + \neg x \cdot -x3$ | DOS(4.12a) |
| $\neg 2 \cdot 4 \cdot x \cdot x + 3 \cdot \neg x \cdot \neg x$ | CPM(4.6) |
| $\neg 2 \cdot 4 \cdot x^2 + 3 \cdot x^2$ | PrCBPo(5.3a) |
| $\neg 8x^2 + 3x^2$ | OOM(2.3) |
| $-5x^2$ | OOA(2.1) |

**Example 8.20 – id:20141108-191616.**

Simplify by expanding $-5x \cdot 4x$

S

**Solution:**

| | |
|---|---|
| $\neg 5x \cdot 4x$ | ONeg(3.1a) |
| $\neg 5 \cdot x \cdot 4 \cdot x$ | JTC(3.5) |
| $\neg 5 \cdot 4 \cdot x \cdot x$ | CPM(4.6) |
| $\neg 5 \cdot 4 \cdot x^2$ | PrCBPo(5.3a) |
| $(\neg 5 \cdot 4) \cdot x^2$ | APM(4.7a) |
| $\neg 20 \cdot x^2$ | OOM(2.3) |
| $\neg 20x^2$ | CTJ(3.6) |
| $-20x^2$ | ONeg(3.1b) |

S

**Less Steps Solution:**

| | |
|---|---|
| $\neg 5 \cdot 4 \cdot x \cdot x$ | CPM(4.6) |
| $\neg 5 \cdot 4 \cdot x^2$ | PrCBPo(5.3a) |
| $-20x^2$ | OOM(2.3) |

## 8.4 Simplifying Degree 3 Univariate Monomials

## 8.5 Simplifying Degree 1 Univariate Binomials

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

Degree 1 univariate polynomial expressions can be expressed with at most two different terms and consequently this expression in its canonical form has at most two monomial summands – called a binomial.

**Example 8.21 – id:20141109-090809.**

Simplify by expanding $5(x+4)$

**Solution:**

| | |
|---|---|
| $5(1x+4)$ | MId(4.9a) |
| $5\cdot 1x+5\cdot 5$ | DPF(4.3a) |
| $5\cdot 1\cdot x+5\cdot 5$ | JTC(3.5) |
| $5\cdot x+25$ | OOM(2.3) |
| $5x+25$ | CTJ(3.6) |

**Less Steps Solution:**

| | |
|---|---|
| $5x+20$ | DPE(4.8a) |

**Example 8.22 – id:20141109-091015.**

Simplify by expanding $5(3x-9)$

**Solution:**

| | |
|---|---|
| $5(3x+\neg 9)$ | DOS(4.12a) |
| $5\cdot 3x+5\cdot\neg 9$ | DPE(4.8a) |
| $5\cdot 3\cdot x+5\cdot\neg 9$ | JTC(3.5) |
| $15\cdot x+\neg 45$ | OOM(2.3) |
| $15x+\neg 45$ | CTJ(3.6) |
| $15x-45$ | DOS(4.12b) |

**Less Steps Solution:**

| | |
|---|---|
| $5(3x+\neg 9)$ | DOS(4.12a) |
| $15x+\neg 40$ | DPE(4.8a) |
| $15x-40$ | DOS(4.12b) |

**Example 8.23 – id:20141109-092448.**

Simplify by expanding $-(5x+7)$

**Solution:**

| | |
|---|---|
| $\neg 1(5x+7)$ | MId(4.9a) |
| $\neg 1 \cdot 5x + \neg 1 \cdot 7$ | DPE(4.8a) |
| $\neg 1 \cdot 5 \cdot x + \neg 1 \cdot 7$ | JTC(3.5) |
| $\neg 5 \cdot x + \neg 7$ | OOM(2.3) |
| $\neg 5x + \neg 7$ | CTJ(3.6) |
| $\neg 5x - 7$ | DOS(4.12b) |
| $-5x - 7$ | ONeg(3.1b) |

**Less Steps Solution:**

| | |
|---|---|
| $-5x - 7$ | DPE(4.8a) |

**Example 8.24 – id:20141109-092651.**

Simplify by expanding $-13(7x-9)$

**Solution:**

| | |
|---|---|
| $\neg 13(7x + \neg 9)$ | DOS(4.12a) |
| $\neg 13 \cdot 7x + \neg 13 \cdot \neg 9$ | DPE(4.8a) |
| $\neg 13 \cdot 7 \cdot x + \neg 13 \cdot \neg 9$ | JTC(3.5) |
| $\neg 91 \cdot x + 117$ | OOM(2.3) |
| $\neg 91x + 117$ | CTJ(3.6) |
| $-91x + 117$ | ONeg(3.1b) |

**Less Steps Solution:**

$$\begin{aligned} -13(7x + \neg 9) && \text{DOS(4.12a)} \\ -91x + 117 && \text{DPE(4.8a)} \end{aligned}$$

**Example 8.25 – id:20141109-092910.**

Simplify by expanding $a(x + b)$ , where $a, b \in \mathbb{Z}$

S

**Solution:**

$$\begin{aligned} a(1x + b) && \text{MId(4.9a)} \\ a \cdot 1x + a \cdot b && \text{DPE(4.8a)} \\ a \cdot 1 \cdot x + a \cdot b && \text{JTC(3.5)} \\ 1 \cdot a \cdot x + a \cdot b && \text{CPM(4.6)} \\ 1ax + ab && \text{JTC(3.5)} \\ ax + ab && \text{MId(4.9b)} \end{aligned}$$

S

**Less Steps Solution:**

$$ax + ab \qquad \text{DPE(4.8a)}$$

**Example 8.26 – id:20141109-093220.**

Simplify by expanding $5(x + 2) + 4$

S

**Solution:**

$$\begin{aligned} 5(1x + 2) + 4 && \text{MId(4.9a)} \\ 5 \cdot 1x + 5 \cdot 2 + 4 && \text{DPE(4.8a)} \\ 5 \cdot 1 \cdot x + 5 \cdot 2 + 4 && \text{JTC(3.5)} \\ 5 \cdot x + 10 + 4 && \text{OOM(2.3)} \\ 5x + 10 + 4 && \text{CTJ(3.6)} \\ 5x + 14 && \text{OOA(2.1)} \end{aligned}$$

**Less Steps Solution:**

$$\begin{aligned} 5x + 10 + 4 & \qquad \text{DPE(4.8a)} \\ 5x + 14 & \qquad \text{OOA(2.1)} \end{aligned}$$

**Example 8.27 – id:20141109-093419.**

Simplify by expanding $7x + 5(4x + 8)$

**Solution:**

$$\begin{aligned} 7x + 5 \cdot 4x + 5 \cdot 8 & \qquad \text{DPE(4.8a)} \\ 7 \cdot x + 5 \cdot 4 \cdot x + 5 \cdot 8 & \qquad \text{JTC(3.5)} \\ 7 \cdot x + 20 \cdot x + 40 & \qquad \text{OOM(2.3)} \\ 7x + 20x + 40 & \qquad \text{CTJ(3.6)} \\ (7 + 20)x + 40 & \qquad \text{DPF(4.3a)} \\ 27x + 40 & \qquad \text{OOA(2.1)} \end{aligned}$$

**Example 8.28 – id:20141109-094928.**

Simplify by expanding $4(3x + 4) + x + 6$

**Solution:**

$$\begin{aligned} 4(3x + 4) + 1x + 6 & \qquad \text{MId(4.9a)} \\ 4 \cdot 3x + 4 \cdot 4 + 1x + 6 & \qquad \text{DPE(4.8a)} \\ 4 \cdot 3 \cdot x + 4 \cdot 4 + 1 \cdot x + 6 & \qquad \text{JTC(3.5)} \\ 12 \cdot x + 16 + 1 \cdot x + 6 & \qquad \text{OOM(2.3)} \\ 12x + 16 + 1x + 6 & \qquad \text{CTJ(3.6)} \\ 12 + 1x + 16 + 6 & \qquad \text{CPA(4.1)} \\ (12 + 1)x + 16 + 6 & \qquad \text{DPF(4.3a)} \\ 13x + 22 & \qquad \text{OOA(2.1)} \end{aligned}$$

**Less Steps Solution:**

| | |
|---|---|
| $12x + 16 + x + 6$ | DPE(4.8a) |
| $12x + x + 16 + 6$ | CPA(4.1) |
| $13x + 22$ | OOA(2.1) |

**Example 8.29 – id:20141109-095151.**

Simplify by expanding $5(x - 4) + 3x - 5$

S

**Solution:**

| | |
|---|---|
| $5(1x - 4) + 3x - 5$ | MId(4.9a) |
| $5(1x + \neg 4) + 3x + \neg 5$ | DOS(4.12a) |
| $5 \cdot 1x + 5 \cdot \neg 4 + 3x + \neg 5$ | DPE(4.8a) |
| $5 \cdot 1 \cdot x + 5 \cdot \neg 4 + 3 \cdot x + \neg 5$ | JTC(3.5) |
| $5 \cdot x + \neg 20 + 3 \cdot x + \neg 5$ | OOM(2.3) |
| $5x + \neg 20 + 3x + \neg 5$ | JTC(3.5) |
| $5x + 3x + \neg 20 + \neg 5$ | CPA(4.1) |
| $(5 + 3)x + \neg 20 + \neg 5$ | DPF(4.3a) |
| $8x + \neg 25$ | OOA(2.1) |
| $8x - 25$ | DOS(4.12b) |

S

**Less Steps Solution:**

| | |
|---|---|
| $5(x + \neg 4) + 3x + \neg 5$ | DOS(4.12a) |
| $5x + \neg 20 + 3x + \neg 5$ | DPE(4.8a) |
| $5x + 3x + \neg 20 + \neg 5$ | CPA(4.1) |
| $8x + \neg 25$ | OOA(2.1) |
| $8x - 25$ | DOS(4.12b) |

**Example 8.30 – id:20141109-095536.**

Simplify by expanding $8x - 5 - 4(x - 3)$

S

**Solution:**

$$\begin{aligned}
8x - 5 - 4(1x - 3) &\qquad \text{MId(4.9a)} \\
8x + \neg 5 + \neg 4(1x + \neg 3) &\qquad \text{DOS(4.12a)} \\
8x + \neg 5 + \neg 4 \cdot 1x + \neg 4 \cdot \neg 3 &\qquad \text{DPE(4.8a)} \\
8 \cdot x + \neg 5 + \neg 4 \cdot 1 \cdot x + \neg 4 \cdot \neg 3 &\qquad \text{JTC(3.5)} \\
8 \cdot x + \neg 5 + \neg 4 \cdot x + 12 &\qquad \text{OOM(2.3)} \\
8x + \neg 5 + \neg 4x + 12 &\qquad \text{CTJ(3.6)} \\
8x + \neg 4x + \neg 5 + 12 &\qquad \text{CPA(4.1)} \\
(8 + \neg 4)x + \neg 5 + 12 &\qquad \text{DPF(4.3a)} \\
4x + 7 &\qquad \text{OOA(2.1)}
\end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned}
8x + \neg 5 + \neg 4(x + \neg 3) &\qquad \text{DOS(4.12a)} \\
8x + \neg 5 + \neg 4x + 12 &\qquad \text{DPE(4.8a)} \\
8x + \neg 4x + \neg 5 + 12 &\qquad \text{CPA(4.1)} \\
4x + 7 &\qquad \text{OOA(2.1)}
\end{aligned}$$

■

**Example 8.31 – id:20141109-095842.**

Simplify by expanding $5(x+3) + 3(x+2)$

S

**Solution:**

$$\begin{aligned}
5 \cdot x + 5 \cdot 3 + 3 \cdot x + 3 \cdot 2 &\qquad \text{DPE(4.8a)} \\
5 \cdot x + 15 + 3 \cdot x + 6 &\qquad \text{OOM(2.3)} \\
5x + 15 + 3x + 6 &\qquad \text{CTJ(3.6)} \\
5x + 3x + 15 + 6 &\qquad \text{CPA(4.1)} \\
(3 + 5)x + 15 + 6 &\qquad \text{DPF(4.3a)} \\
8x + 21 &\qquad \text{OOA(2.1)}
\end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned} 5x + 15 + 3x + 6 &\quad \text{DPE(4.8a)} \\ 5x + 3x + 15 + 6 &\quad \text{CPA(4.1)} \\ 8x + 21 &\quad \text{OOA(2.1)} \end{aligned}$$

## 8.6 Simplifying Degree 2 Univariate Binomials

**Example 8.32 – id:20141106-152339.**

Simplify by expanding $3x^2 + 2x + 5x^2 + 4x$

S

**Solution:**

$$\begin{aligned} 3x^2 + 5x^2 + 2x + 4x &\quad \text{CPA(4.1)} \\ (3+5)x^2 + (2+4)x &\quad \text{DPF(4.3a)} \\ 8x^2 + 6x &\quad \text{OOA(2.1)} \end{aligned}$$

*If needed we could continue and express it in the simplified factored form using the distributive property*

$$(4x + 3)2x \quad \text{DPF(4.3a)}$$

S

**Less Steps Solution:**

$$\begin{aligned} 3x^2 + 5x^2 + 2x + 4x &\quad \text{CPA(4.1)} \\ 8x^2 + 6x &\quad \text{OOA(2.1)} \end{aligned}$$

**Example 8.33 – id:20141107-121834.**

Simplify by expanding $\left(\sqrt{9 - x^2}\right)^2$

S

**Solution:**

| | |
|---|---|
| $\left(\sqrt{9 - 1x^2}\right)^2$ | MId(4.9a) |
| $\left(\sqrt{9 + \neg 1x^2}\right)^2$ | DOS(4.12a) |
| $\left[\left(9 + \neg x^2\right)^{\frac{1}{2}}\right]^2$ | RTPo(3.12) |
| $9 + \neg 1x^2$ | PoPo(5.1a) |
| $\neg 1x^2 + 9$ | CPA(4.1) |
| $\neg x^2 + 9$ | MId(4.9a) |
| $-x^2 + 9$ | ONeg(3.1b) |

S

**Less Steps Solution:**

| | |
|---|---|
| $9 - x^2$ | PoPo(5.1a) |

*It might be easier to view this using a variable substitution for the radicand,* $9 - x^2$. *Let* $k = 9 + \neg 1x^2$.

| | |
|---|---|
| $\left(\sqrt{k}\right)^2$ | MId(4.9a) |
| $\left(\sqrt{k}\right)^2$ | DOS(4.12a) |
| $\left[(k)^{\frac{1}{2}}\right]^2$ | RTPo(3.12) |
| $k$ | PoPo(5.1a) |
| $9 + \neg 1x^2$ | CPA(4.1) |
| $\neg 1x^2 + 9$ | CPA(4.1) |
| $\neg x^2 + 9$ | MId(4.9a) |
| $-x^2 + 9$ | ONeg(3.1b) |

D

**Dependencies:**example **??**-20141105-144223

**Example 8.34 – id:20141209-145211.**

Simplify by expanding $2x(2x + 4) + x^2 \cdot 2 \cdot 1 + 0$

S

**Solution:**

$$
\begin{aligned}
2x \cdot 2x + 2x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0 && \text{DPE(4.8a)}\\
2 \cdot x \cdot 2 \cdot x + 2 \cdot x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0 && \text{JTC(3.5)}\\
2 \cdot 2 \cdot x \cdot x + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0 && \text{CPM(4.6)}\\
2 \cdot 2 \cdot x^2 + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0 && \text{PrCBPo(5.3a)}\\
4 \cdot x^2 + 8 \cdot x + 2 \cdot x^2 + 0 && \text{OOM(2.3)}\\
4x^2 + 8x + 2x^2 + 0 && \text{CTJ(3.6)}\\
4x^2 + 2x^2 + 8x && \text{APA(4.2a)}\\
(4+2)x^2 + 8x && \text{DPF(4.3b)}\\
6x^2 + 8x && \text{OOA(2.1)}
\end{aligned}
$$

D

**Dependencies:**example 15.7-20141209-144203 ▪

## 8.7 Simplifying Degree 2 Univariate Trinomials

**Example 8.35 – id:20141109-133008.**

Simplify by expanding $(x+5)(x-8)$

S

**Solution:**

$$
\begin{aligned}
(1x+5)(1x-8) && \text{MId(4.9a)}\\
(1x+5)(1x+\neg 8) && \text{DOS(4.12a)}\\
1x(1x+\neg 8) + 5(1x+\neg 8) && \text{DPE(4.8b)}\\
1x \cdot 1x + 1x \cdot \neg 8 + 5 \cdot 1x + 5 \cdot \neg 8 && \text{DPE(4.8a)}\\
1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot \neg 8 + 5 \cdot 1 \cdot x + 5 \cdot \neg 8 && \text{JTC(3.5)}\\
1 \cdot 1 \cdot x \cdot x + \neg 8 \cdot 1 \cdot x + 1 \cdot 5 \cdot x + \neg 8 \cdot 5 && \text{CPM(4.6)}\\
1 \cdot 1 \cdot x^2 + \neg 8 \cdot 1 \cdot x + 1 \cdot 5 \cdot x + \neg 8 \cdot 5 && \text{PrCBPo(5.3a)}\\
1 \cdot x^2 + \neg 8 \cdot x + 5 \cdot x + \neg 40 && \text{OOM(2.3)}\\
1x^2 + \neg 8x + 5x + \neg 40 && \text{CTJ(3.6)}\\
1x^2 + \neg 3x + \neg 40 && \text{OOA(2.1)}\\
1x^2 - 3x - 40 && \text{DOS(4.12b)}\\
x^2 - 3x - 40 && \text{MId(4.9a)}
\end{aligned}
$$

S

**Less Steps Solution:**

| | |
|---|---|
| $(x+5)(x+\neg 8)$ | DOS(4.12a) |
| $x(x+\neg 8)+5(x+\neg 8)$ | DPE(4.8b) |
| $x^2+\neg 8x+5x+\neg 40$ | DPE(4.8a) |
| $x^2-3x-40$ | OOA(2.1) |

**Example 8.36 – id:20141109-133316.**

Simplify by expanding $(x+a)(x+b)$, where $a, b \in \mathbb{Z}$

S

**Solution:**

| | |
|---|---|
| $(1x+a)(1x+b)$ | MId(4.9a) |
| $1x(1x+b)+a(1x+b)$ | DPE(4.8b) |
| $1x\cdot 1x+1x\cdot b+a\cdot 1x+a\cdot b$ | DPE(4.8a) |
| $1\cdot x\cdot 1\cdot x+1\cdot x\cdot b+a\cdot 1\cdot x+a\cdot b$ | JTC(3.5) |
| $1\cdot 1\cdot x\cdot x+1\cdot b\cdot x+1\cdot a\cdot x+a\cdot b$ | CPM(4.6) |
| $1\cdot 1\cdot x^2+1\cdot b\cdot x+1\cdot a\cdot x+a\cdot b$ | PrCBPo(5.3a) |
| $1\cdot x^2+1\cdot b\cdot x+1\cdot a\cdot x+a\cdot b$ | OOM(2.3) |
| $1x^2+1bx+1ax+ab$ | CTJ(3.6) |
| $1x^2+(1b+1a)x+ab$ | DPF(4.3a) |
| $x^2+(b+a)x+ab$ | MId(4.9b) |

S

**Less Steps Solution:**

| | |
|---|---|
| $x(x+b)+a(x+b)$ | DPE(4.8b) |
| $x^2+(b+a)x+ab$ | DPE(4.8a) |

**Example 8.37 – id:20141109-140659.**

Simplify by expanding $(2x+3)(5x+13)$

S

**Solution:**

| | |
|---|---|
| $2x(5x+13)+3(5x+13)$ | DPE(4.8b) |
| $2x\cdot 5x+2x\cdot 13+3\cdot 5x+3\cdot 13$ | DPE(4.8a) |
| $2\cdot x\cdot 5\cdot x+2\cdot x\cdot 13+3\cdot 5\cdot x+3\cdot 13$ | JTC(3.5) |
| $2\cdot 5\cdot x\cdot x+2\cdot 13\cdot x+5\cdot 3\cdot x+3\cdot 13$ | CPM(4.6) |
| $2\cdot 5\cdot x^2+2\cdot 13\cdot x+5\cdot 3\cdot x+3\cdot 13$ | PrCBPo(5.3a) |
| $10\cdot x^2+26\cdot x+16\cdot x+39$ | OOM(2.3) |
| $10x^2+26x+15x+39$ | CTJ(3.6) |
| $10x^2+41x+39$ | OOA(2.1) |

S

**Less Steps Solution:**

| | |
|---|---|
| $2x(5x+13)+3(5x+13)$ | DPE(4.8b) |
| $10x^2+26x+15x+39$ | DPE(4.8a) |
| $10x^2+41x+39$ | OOA(2.1) |

**Example 8.38 – id:20141109-141019.**

Simplify by expanding $(-3x-5)(7x+8)$

S

**Solution:**

| | |
|---|---|
| $(\neg 3x-5)(7x+8)$ | ONeg(3.1a) |
| $(\neg 3x+\neg 5)(7x+8)$ | DOS(4.12a) |
| $\neg 3x(7x+8)+\neg 5(7x+8)$ | DPE(4.8b) |
| $\neg 3x\cdot 7x+\neg 3x\cdot 8+\neg 5\cdot 7x+\neg 5\cdot 8$ | DPE(4.8a) |
| $\neg 3\cdot x\cdot 7\cdot x+\neg 3\cdot x\cdot 8+\neg 5\cdot 7\cdot x+\neg 5\cdot 8$ | JTC(3.5) |
| $\neg 3\cdot 7\cdot x\cdot x+\neg 3\cdot 8\cdot x+\neg 5\cdot 7\cdot x+\neg 5\cdot 8$ | CPM(4.6) |
| $\neg 3\cdot 7\cdot x^2+\neg 3\cdot 8\cdot x+\neg 5\cdot 7\cdot x+\neg 5\cdot 8$ | PrCBPo(5.3a) |
| $\neg 21\cdot x^2+\neg 24\cdot x+\neg 35\cdot x+\neg 40$ | OOM(2.3) |
| $\neg 21x^2+\neg 24x+\neg 35x+\neg 40$ | CTJ(3.6) |
| $\neg 21x^2+\neg 59x+\neg 40$ | OOA(2.1) |
| $\neg 21x^2-59x-40$ | DOS(4.12b) |
| $-21x^2-59x-40$ | ONeg(3.1b) |

S

**Less Steps Solution:**

| | |
|---|---|
| $(-3x + \neg 5)(7x + 8)$ | DOS(4.12a) |
| $\neg 3x(7x + 8) + \neg 5(7x + 8)$ | DPE(4.8b) |
| $\neg 21x^2 + \neg 24x + \neg 35x + \neg 40$ | CTJ(4.8a) |
| $-21x^2 - 59x - 40$ | OOA(2.1) |

**Example 8.39 – id:20141109-141347.**

Simplify by expanding $(ax + b)(cx + d)$, where $a, b, c, d \in \mathbb{Z}$

S

**Solution:**

| | |
|---|---|
| $ax(cx + d) + b(cx + d)$ | DPE(4.8b) |
| $ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d$ | DPE(4.8a) |
| $a \cdot x \cdot c \cdot x + a \cdot x \cdot d + b \cdot c \cdot x + b \cdot d$ | JTC(3.5) |
| $a \cdot c \cdot x \cdot x + a \cdot d \cdot x + b \cdot c \cdot x + b \cdot d$ | CPM(4.6) |
| $a \cdot c \cdot x^2 + a \cdot d \cdot x + b \cdot c \cdot x + b \cdot d$ | PrCBPo(5.3a) |
| $acx^2 + adx + bcx + bd$ | CTJ(3.6) |
| $acx^2 + (ad + bc)x + bd$ | DPF(4.3a) |

S

**Less Steps Solution:**

| | |
|---|---|
| $ax(cx + d) + b(cx + d)$ | DPE(4.8b) |
| $acx^2 + (ad + bc)x + bd$ | DPE(4.8a) |

**Example 8.40 – id:20141105-161225.**

Simplify by expanding $\left(2 - \frac{x}{2}\right)^2$.

S

**Solution:**

$$
\begin{aligned}
&\left(2 - \frac{1x}{2}\right)^2 && \text{MId(4.9a)}\\
&\left(2 + \neg\frac{1x}{2}\right)^2 && \text{DOS(4.12a)}\\
&\left(2 + \neg\frac{1x}{2}\right)\left(2 + \neg\frac{1x}{2}\right) && \text{PoTF(3.9)}\\
&2\left(2 + \neg\frac{1x}{2}\right) + \neg\frac{1x}{2}\left(2 + \neg\frac{1x}{2}\right) && \text{DPE(4.8b)}\\
&2\cdot 2 + 2\cdot\neg\frac{1x}{2} + \neg\frac{1x}{2}\cdot 2 + \neg\frac{1x}{2}\cdot\neg\frac{1x}{2} && \text{DPE(4.8a)}\\
&2\cdot 2 + 2\cdot\neg\frac{1}{2}\cdot x + \neg\frac{1}{2}\cdot x\cdot 2 + \neg\frac{1}{2}\cdot x\cdot\neg\frac{1}{2}\cdot x && \text{JTC(3.5)}\\
&2\cdot 2 + 2\cdot\neg\frac{1}{2}\cdot x + \neg\frac{1}{2}\cdot 2\cdot x + \neg\frac{1}{2}\cdot\neg\frac{1}{2}\cdot x\cdot x && \text{CPM(4.6)}\\
&2\cdot 2 + 2\cdot\neg\frac{1}{2}\cdot x + \neg\frac{1}{2}\cdot 2\cdot x + \neg\frac{1}{2}\cdot\neg\frac{1}{2}\cdot x^2 && \text{PrCBPo(5.3a)}\\
&4 + \neg 1\cdot x + \neg 1\cdot x + \frac{1}{4}\cdot x^2 && \text{OOM(2.3)}\\
&4 + \neg 1x + \neg 1x + \frac{1}{4}x^2 && \text{CTJ(3.6)}\\
&\frac{1}{4}x^2 + \neg 1x + \neg 1x + 4 && \text{CPA(4.1)}\\
&\frac{1}{4}x^2 + \neg 2x + 4 && \text{OOA(2.1)}\\
&\frac{1}{4}x^2 - 2x + 4 && \text{DOS(4.12b)}
\end{aligned}
$$

S

**Less Steps Solution:**

$$
\begin{aligned}
&7x + 20x + 40 && \text{DPE(4.8a)}\\
&27x + 40 && \text{OOA(2.1)}
\end{aligned}
$$

## 8.8 Simplifying Degree $n$ Univariate Polynomials

**Definition 8.8.1 – Univariate Polynomial Expression.**

$$\sum_{k=0}^{n} C_k x^n = C_n x^n + C_{n-1}x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0} \tag{8.1}$$

A univariate polynomial in an indeterminate $x$ is an expression made up of one or more summands of the form $C_k x^k$, which are called monomials. The first factor of each monomial, $C_k$, is a numerical factor called the coefficient where $C_k \in$. The second factor of each monomial, $x^k$, is an indeterminate raised to a non-negative integer power

$i$.

**wikipedia:polynomial**

**Definition 8.8.2 – Degree of the Univaritate Polynomial.**

$$C_n x^{n} + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

The degree of the univariate polynomial is determined by the monomial with the largest degree of the indeterminate.

# 9. Simplifying Multivariate Monomials

**Definition 9.0.1 – Zero Polynomial.** A constant polynomial with a value of zero whose coefficients are all equal to zero is called a zero polynomial. The zero polynomial, 0, can be classified as a degree -1 polynomial. The zero polynomial is the additive identity, AId (4.4a), for polynomials.

**[mathworld:zeropolynomial]**

## 9.1 Simplifying Degree -1 Multivariate Monomials

**Example 9.1 – id:20151018-165132.**
Simplify $xy - xy$

S

**Solution:**

| | |
|---|---|
| $1xy - 1xy$ | MId(4.9a) |
| $1xy + \neg 1xy$ | DOS(4.12a) |
| $(1 + \neg 1)xy$ | DPF(4.3a) |
| $0xy$ | OOA(2.1) |
| $0$ | AId(4.4b) |

D

**Dependencies:**example 9.7-20151018-164638

R This is a special case of a polynomial that is made up of one monomial whose coefficient is equal to zero. This corresponds to a constant with a value of zero and called the zero polynomial9.0.1. The zero polynomial is the additive identity.

## 9.2 Simplifying Degree 2 Multivariate Monomials

**Example 9.2 – id:20151018-161315.**
Simplify by expanding $4x2y$

S

**Solution:**

| | |
|---|---|
| $4 \cdot x \cdot 2 \cdot y$ | JTC(3.5) |
| $4 \cdot 2 \cdot x \cdot y$ | CPM(4.6) |
| $8 \cdot x \cdot y$ | OOM(2.3) |
| $8xy$ | CTJ(3.6) |

■

## 9.3 Simplifying Degree 2 Multivariate Binomials

**Example 9.3 – id:20151018-164351.**

Simplify $10xy + 21xy$

S

**Solution:**

| | |
|---|---|
| $(10+21)xy$ | DPF(4.3a) |
| $31xy$ | OOA(2.1) |

D

**Dependencies:**example 9.4-20151018-161544

■

## 9.4 Simplifying Degree 2 Multivariate Trinomials

**Example 9.4 – id:20151018-161544.**
Simplify by expanding $(2x+3y)(7x+5y)$

S

**Solution:**

| | |
|---|---|
| $2x(7x+5y)+3y(7x+5y)$ | DPE(4.8b) |
| $2x \cdot 7x + 2x \cdot 5y + 3y \cdot 7x + 3y \cdot 5y$ | DPE(4.8a) |
| $2 \cdot x \cdot 7 \cdot x + 2 \cdot x \cdot 5 \cdot y + 3 \cdot y \cdot 7 \cdot x + 3 \cdot y \cdot 5 \cdot y$ | JTC(3.5) |
| $2 \cdot 7 \cdot x \cdot x + 2 \cdot 5 \cdot x \cdot y + 3 \cdot 7 \cdot x \cdot y + 3 \cdot 5 \cdot y \cdot y$ | CPM(4.6) |
| $2 \cdot 7 \cdot x^2 + 2 \cdot 5 \cdot x \cdot y + 3 \cdot 7 \cdot x \cdot y + 3 \cdot 5 \cdot y^2$ | PrCBPo(5.3a) |
| $14 \cdot x^2 + 10 \cdot x \cdot y + 21 \cdot x \cdot y + 15 \cdot y^2$ | OOM(2.3) |
| $14x^2 + 10xy + 21xy + 15y^2$ | CTJ(3.6) |
| $14x^2 + 31xy + 15y^2$ | OOA(2.1) goto 9.3 |

■

**Example 9.5 – id:20151018-183900.**
Simplify by expanding $(x + y)^2$

S

**Solution:**

$$
\begin{array}{rr}
(1x + 1y)^2 & \text{MId(4.9a)} \\
(1x + 1y)(1x + 1y) & \text{PoTF(3.9)} \\
1x(1x + 1y) + 1y(1x + 1y) & \text{DPE(4.8b)} \\
1x \cdot 1x + 1x \cdot 1y + 1y \cdot 1x + 1y \cdot 1y & \text{DPE(4.8a)} \\
1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot 1 \cdot y + 1 \cdot y \cdot 1 \cdot x + 1 \cdot y \cdot 1 \cdot y & \text{JTC(3.5)} \\
1 \cdot 1 \cdot x \cdot x + 1 \cdot 1 \cdot x \cdot y + 1 \cdot 1 \cdot x \cdot y + 1 \cdot 1 \cdot y \cdot y & \text{CPM(4.6)} \\
1 \cdot 1 \cdot x^2 + 1 \cdot 1 \cdot x \cdot y + 1 \cdot 1 \cdot x \cdot y + 1 \cdot 1 \cdot y^2 & \text{PrCBPo(5.3a)} \\
1 \cdot x^2 + 1 \cdot x \cdot y + 1 \cdot x \cdot y + 1 \cdot y^2 & \text{OOM(2.3)} \\
1x^2 + 1xy + 1xy + 1y^2 & \text{CTJ(3.6)} \\
1x^2 + (1 + 1)xy + 1y^2 & \text{DPF(4.3a)} \\
1x^2 + 2xy + 1y^2 & \text{OOA(2.1)} \\
x^2 + 2xy + y^2 & \text{MId(4.9b)}
\end{array}
$$

■

**Example 9.6 – id:20151018-182202.**
Simplify by expanding $(x - y)^2$

S

**Solution:**

| | |
|---|---|
| $(1x - 1y)^2$ | MId(4.9a) |
| $(1x - 1y)(1x - 1y)$ | PoTF(3.9) |
| $(1x - 1y)(1x - 1y)$ | MId(4.9a) |
| $(1x + \neg 1y)(1x + \neg 1y)$ | DOS(4.12a) |
| $1x(1x + \neg 1y) + \neg 1y(1x + \neg 1y)$ | DPE(4.8b) |
| $1x \cdot 1x + 1x \cdot \neg 1y + \neg 1y \cdot 1x + \neg 1y \cdot \neg 1y$ | DPE(4.8a) |
| $1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot \neg 1 \cdot y + \neg 1 \cdot y \cdot 1 \cdot x + \neg 1 \cdot y \cdot \neg 1 \cdot y$ | JTC(3.5) |
| $1 \cdot 1 \cdot x \cdot x + 1 \cdot \neg 1 \cdot x \cdot y + \neg 1 \cdot 1 \cdot x \cdot y + \neg 1 \cdot \neg 1 \cdot y \cdot y$ | CPM(4.6) |
| $1 \cdot 1 \cdot x^2 + 1 \cdot \neg 1 \cdot x \cdot y + \neg 1 \cdot 1 \cdot x \cdot y + \neg 1 \cdot \neg 1 \cdot y^2$ | PrCBPo(5.3a) |
| $1 \cdot x^2 + \neg 1 \cdot x \cdot y + \neg 1 \cdot x \cdot y + 1 \cdot y^2$ | OOM(2.3) |
| $1x^2 + \neg 1xy + \neg 1xy + 1y^2$ | CTJ(3.6) |
| $1x^2 + (\neg 1 + \neg 1)xy + 1y^2$ | DPF(4.3a) |
| $1x^2 + \neg 2xy + 1y^2$ | OOA(2.1) |
| $1x^2 - 2xy + 1y^2$ | DOS(4.12b) |
| $x^2 - 2xy + y^2$ | AId(4.4b) |

■

**Example 9.7 – id:20151018-164638.**
Simplify by expanding $(x - y)(x + y)$

S

**Solution:**

| | |
|---|---|
| $(1x - 1y)(1x + 1y)$ | MId(4.9a) |
| $(1x + \neg 1y)(1x + 1y)$ | DOS(4.12a) |
| $1x(1x + 1y) + \neg 1y(1x + 1y)$ | DPE(4.8b) |
| $1x \cdot 1x + 1x \cdot 1y + \neg 1y \cdot 1x + \neg 1y \cdot 1y$ | DPE(4.8a) |
| $1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot 1 \cdot y + \neg 1 \cdot y \cdot 1 \cdot x + \neg 1 \cdot y \cdot 1 \cdot y$ | JTC(3.5) |
| $1 \cdot 1 \cdot x \cdot x + 1 \cdot 1 \cdot x \cdot y + \neg 1 \cdot 1 \cdot x \cdot y + \neg 1 \cdot 1 \cdot y \cdot y$ | CPM(4.6) |
| $1 \cdot 1 \cdot x^2 + 1 \cdot 1 \cdot x \cdot y + \neg 1 \cdot 1 \cdot x \cdot y + \neg 1 \cdot 1 \cdot y^2$ | PrCBPo(5.3a) |
| $1 \cdot x^2 + 1 \cdot x \cdot y + \neg 1 \cdot x \cdot y + \neg 1 \cdot y^2$ | OOM(2.3) |
| $1x^2 + 1xy + \neg 1xy + \neg 1y^2$ | CTJ(3.6) |
| $1x^2 + \neg 1y^2$ | OOA(2.1) goto 9.1 |
| $1x^2 - 1y^2$ | DOS(4.12b) |
| $x^2 - y^2$ | MId(4.9b) |

■

# 10. Factoring Univariate Trinomials

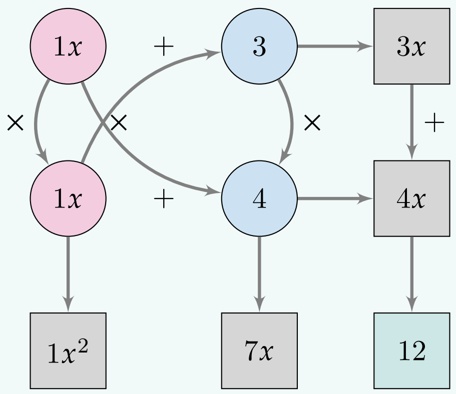
**Example 10.1 – id:20151015-184212.**
Simplify by factoring $x^2 + 7x + 12$

S

**Solution:**
Using the distributive property organizer

The factors of $(x^2 + 7x + 12)$ are $(x + 3)(x + 4)$

$$1x^2 + 7x + 12 \qquad \text{MId(4.9a)}$$




$$(1x + 3)(1x + 4)$$
$$(x + 3)(x + 4) \qquad \text{MId(4.9b)}$$

**Example 10.2 – id:20151016-063338.**

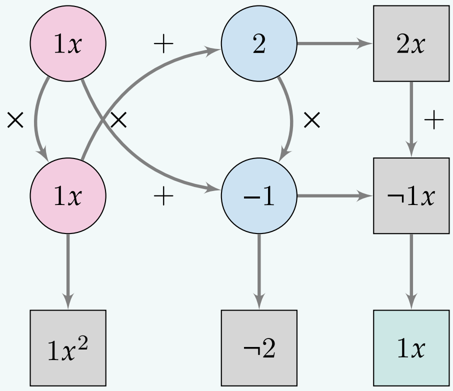
Simplify by factoring $x^2 + x - 2$

S

**Solution:**
Using the distributive property organizer

The factors of $(x^2 + x - 2)$ are $(x + 2)(x - 1)$

$$
\begin{aligned}
1x^2 + 1x - 2 & \qquad \text{AId(4.4a)} \\
1x^2 + 1x + \neg 2 & \qquad \text{DOS(4.12a)}
\end{aligned}
$$




$$
\begin{aligned}
(1x + 2)(1x + \neg 1) & \\
(1x + 2)(1x - 1) & \qquad \text{DOS(4.12b)} \\
(x + 2)(x - 1) & \qquad \text{MId(4.9b)}
\end{aligned}
$$

# 11. Solving Linear Equations

## 11.1 Power Inverse

**Example 11.1 – id:20141206-102142.**

Solve the equation $x + a = b$ for $x$

S

**Solution:**

$$
\begin{aligned}
[x + a] + \neg a &= [b] + \neg a && \text{SPE(4.15)} + \text{AI(4.5a)} \\
x + (a + \neg a) &= b + \neg a && \text{APA(4.2b)} \\
x + 0 &= b + \neg a && \text{OOA(2.1)} \\
x &= b + \neg a && \text{AId(4.4b)} \\
x &= b - a && \text{DOS(4.12b)}
\end{aligned}
$$

▪

**Example 11.2 – id:20141111-222931.**

Solve the equations $x + 8 = 0$

S

**Solution:**

$$
\begin{aligned}
[x + 8] + \neg 8 &= [0] + \neg 8 && \text{SPE(4.15)} + \text{AI(4.5a)} \\
x + (8 + \neg 8) &= 0 + \neg 8 && \text{APA(4.2b)} \\
x + 0 &= \neg 8 && \text{OOA(2.1)} \\
x &= \neg 8 && \text{AId(4.4b)} \\
x &= -8 && \text{ONeg(3.1b)}
\end{aligned}
$$

S

**Less Steps Solution:**

$$
\begin{aligned}
[x + 8] + \neg 8 &= [0] + \neg 8 && \text{SPE(4.15)} + \text{AI(4.5a)} \\
x &= -8 && \text{OOA(2.1)}
\end{aligned}
$$

**Dependencies:**
example ??-20141111-190212

**Example 11.3 – id:20141206-101632.**

Solve the equation $x + 4 = 7$

**Solution:**

$$\begin{aligned} [x+4] + \neg 4 &= [7] + \neg 4 && \text{SPE(4.15)} + \text{AI(4.5a)} \\ x + (4 + \neg 4) &= 7 + \neg 4 && \text{APA(4.2b)} \\ x + 0 &= 3 && \text{OOA(2.1)} \\ x &= 3 && \text{AId(4.4b)} \end{aligned}$$

**Less Steps Solution:**

$$\begin{aligned} [x] + 4 + \neg 4 &= [7] + \neg 4 && \text{SPE(4.15)} + \text{AI(4.5a)} \\ x &= 3 && \text{OOA(2.1)} \end{aligned}$$

**Example 11.4 – id:20141206-101107.**

Solve the equation $x - 8 = 15$ for $x$

**Solution:**

$$\begin{aligned} x + \neg 8 &= 15 && \text{DOS(4.12a)} \\ [x + \neg 8] + 8 &= [15] + 8 && \text{SPE(4.15)} + \text{AI(4.5a)} \\ x + (\neg 8 + 8) &= 15 + 8 && \text{APA(4.2b)} \\ x + 0 &= 23 && \text{OOA(2.1)} \\ x &= 23 && \text{AId(4.4b)} \end{aligned}$$

**Less Steps Solution:**

$$\begin{aligned} [x + \neg 8] + 8 &= [15] + 8 && \text{SPE(4.15)} + \text{AI(4.5a)} \\ x &= 23 && \text{OOA(2.1)} \end{aligned}$$

**Example 11.5 – id:20141206-102404.**

Solve the equation $5x = 9$ for $x$.

**Solution:**

$$\begin{aligned} \frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(4.15)} + \text{MI(4.10a)} \\ \frac{1}{5} \cdot [5 \cdot x] &= \frac{1}{5} \cdot 9 && \text{JTC(3.5)} \\ \left(\frac{1}{5} \cdot 5\right) \cdot x &= \frac{1}{5} \cdot 9 && \text{APM(4.7b)} \\ 1 \cdot x &= \frac{9}{5} && \text{OOM(2.3)} \\ x &= \frac{9}{5} && \text{MId(4.9b)} \end{aligned}$$

**Less Steps Solution:**

$$\begin{aligned} \frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(4.15)} + \text{MI(4.10a)} \\ x &= \frac{9}{5} && \text{OOM(2.3)} \end{aligned}$$

**Example 11.6 – id:20141206-104404.**

Solve the equation $ax = b$ for $x$.

**Solution:**

$$\begin{aligned} \frac{1}{a}\left[ax\right] &= \frac{1}{a}\left[b\right] && \text{SPE(4.15)} + \text{MI(4.10a)} \\ \frac{1}{a}\cdot(a\cdot x) &= \frac{1}{a}\cdot b && \text{JTC(3.5)} \\ \left(\frac{1}{a}\cdot a\right)\cdot x &= \frac{1}{a}\cdot b && \text{APM(4.7b)} \\ 1\cdot x &= \frac{b}{a} && \text{OOM(2.3)} \\ x &= \frac{b}{a} && \text{MId(4.9b)} \end{aligned}$$

**Example 11.7 – id:20141206-102723.**

Solve the equation $-2x = 7$ for $x$

S

**Solution:**

$$\begin{aligned} \neg 2x &= 7 && \text{ONeg(3.1a)} \\ \neg\frac{1}{2}\left[\neg 2x\right] &= \neg\frac{1}{2}\left[7\right] && \text{SPE(4.15)} + \text{MI(4.10b)} \\ \neg\frac{1}{2}\cdot(\neg 2\cdot x) &= \neg\frac{1}{2}\cdot 7 && \text{JTC(3.5)} \\ \left(\neg\frac{1}{2}\cdot\neg 2\right)\cdot x &= \neg\frac{1}{2}\cdot 7 && \text{APM(4.7b)} \\ 1\cdot x &= \neg\frac{7}{2} && \text{OOM(2.3)} \\ 1\cdot x &= -\frac{7}{2} && \text{ONeg(3.1b)} \\ x &= -\frac{7}{2} && \text{MId(4.9b)} \end{aligned}$$

S

**Less Steps Solution:**

$$\begin{aligned} \neg\frac{1}{2}\left[\neg 2x\right] &= \neg\frac{1}{2}\left[7\right] && \text{SPE(4.15)} + \text{MI(4.10b)} \\ x &= -\frac{7}{2} && \text{OOM(2.3)} \end{aligned}$$

**Example 11.8 – id:20141111-215726.**

Solve the equation $2x + 5 = 0$ for $x$

S

**Solution:**

$$
\begin{aligned}
[\mathbf{2x+5}] + \neg 5 &= [\mathbf{0}] + \neg 5 && \text{SPE(4.15)} + \text{AI(4.5a)} \\
2x + (5 + \neg 5) &= 0 + \neg 5 && \text{APA(4.2a)} \\
2x + 0 &= \neg 5 && \text{OOA(2.1)} \\
2x &= \neg 5 && \text{AId(4.4a)} \\
\frac{1}{2}[\mathbf{2x}] &= \frac{1}{2}[\mathbf{\neg 5}] && \text{SPE(4.15)} + \text{MI(4.10a)} \\
\frac{1}{2} \cdot 2 \cdot x &= \frac{1}{2} \cdot \neg 5 && \text{JTC(3.5)} \\
\left(\frac{1}{2} \cdot 2\right) \cdot x &= \frac{1}{2} \cdot \neg 5 && \text{APM(4.7a)} \\
1 \cdot x &= \frac{\neg 5}{2} && \text{OOM(2.3)} \\
1x &= -\frac{5}{2} && \text{ONeg(3.1b)} \\
x &= -\frac{5}{2} && \text{MId(4.9b)}
\end{aligned}
$$

S

**Less Steps Solution:**

$$
\begin{aligned}
[\mathbf{2x+5}] + \neg 5 &= [\mathbf{0}] + \neg 5 && \text{SPE(4.15)} + \text{AI(4.5a)} && (11.1) \\
2x &= \neg 5 && \text{OOA(2.1)} && (11.2) \\
\frac{1}{2}[\mathbf{2x}] &= \frac{1}{2}[\mathbf{\neg 5}] && \text{SPE(4.15)} + \text{MI(4.10a)} && (11.3) \\
x &= -\frac{5}{2} && \text{OOM(2.3)} && (11.4)
\end{aligned}
$$

D

**Dependencies:**
example ??-20141111-192213 ▪

**Example 11.9 – id:20151015-104754.**
Solve the equation $2ax + b = 0$ for $x$.

S

**Solution:**

$$
\begin{aligned}
[\mathbf{2ax+b}] + \neg b &= [\mathbf{0}] + \neg b && \text{AI(4.5a), SPE(4.15)}\\
2ax + (b + \neg b) &= 0 + \neg b && \text{APA(4.2b)}\\
2ax + 0 &= 0 + \neg b && \text{OOA(2.1)}\\
2ax &= \neg b && \text{AId(4.4a)}\\
\frac{1}{2a}[\mathbf{2ax}] &= \frac{1}{2a}[\mathbf{\neg b}] && \text{MI(4.5a), SPE(4.15)}\\
\frac{1}{2a}\cdot(2ax) &= \frac{1}{2a}\cdot(\neg b) && \text{JTC(3.5)}\\
\left(\frac{1}{2a}\cdot 2a\right)\cdot x &= \frac{1}{2a}\cdot\neg b && \text{APA(4.2a)}\\
\left(\frac{1}{2a}\cdot\frac{2a}{1}\right)x &= \frac{1}{2a}\cdot\frac{negb}{1} && \text{MId(4.9a)}\\
\frac{2a}{2a}\cdot x &= \frac{\neg b}{2a} && \text{OOM(2.3)}\\
1\cdot x &= \frac{\neg b}{2a}x & = \frac{\neg b}{2a} && \text{MId(4.9b)}\\
x &= -\frac{b}{2a} && \text{ONeg(3.1b)}
\end{aligned}
$$

D

**Dependencies:**example 18.1-20151008-110208

# 12. Solving Quadratic Equations

## 12.1 Mutiliplicative Inverse

**Example 12.1 – id:20141107-131748.**

Solve the equation $2 - x^2 = 0$ for $x$

S

**Solution:**

$$\begin{aligned}
2 - 1x^2 &= 0 && \text{MId(4.9a)} \\
2 + \neg 1x^2 &= 0 && \text{DOS(4.12a)} \\
\left[2 + \neg 1x^2\right] + 1x^2 &= [0] + 1x^2 && \text{SPE(4.15)} + \text{AI(4.5a)} \\
2 + \left(\neg 1x^2 + 1x^2\right) &= 0 + 1x^2 && \text{APA(4.2a)} \\
2 + 0 &= 0 + 1x^2 && \text{OOA(2.1)} \\
2 &= 1x^2 && \text{AId(4.4a)} \\
2 &= x^2 && \text{MId(4.9b)} \\
\pm [2]^{\frac{1}{2}} &= \left[x^2\right]^{\frac{1}{2}} && \text{SPE(4.15)} + \text{MI(4.10a)} \\
\pm 2^{\frac{1}{2}} &= x && \text{PoPo(5.1a)} \\
\pm\sqrt{2} &= x && \text{PoTR(\textbf{??})} \\
x &= \pm\sqrt{2} && \text{SyPE(4.16a)}
\end{aligned}$$

■

**Example 12.2 – id:20151012-192313.**

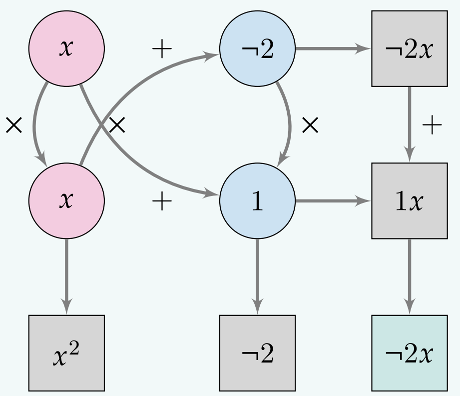
Solve the equation $6x^2 - 6x - 12 = 0$

S

**Solution:**

$$\begin{aligned}
6x^2 + \neg 6x + \neg 12 &= 0 && \text{DOS(4.12a)} \\
6(1x^2 \neg 1x + \neg 2) &= 0 && \text{DPF(4.3b)}
\end{aligned}$$

Using the factor orgnanizer,




Solving the linear equations using ZPr (4.11a)

$$\begin{aligned}
x_1 + \neg 2 &= 0 && \text{Case I}\\
[x_1 + \neg 2] + 2 &= [0] + \neg 2 && \text{AI(4.5a), SPE(4.15)}\\
x_1 + (\neg 2 + 2) &= 0 + \neg 2 && \text{APA(4.2a)}\\
x_1 + 0 &= \neg 2 && \text{OOA(2.1)}\\
x_1 &= \neg 2 && \text{AId(4.4b)}\\
x_1 &= -2 && \text{ONeg(3.1b)}
\end{aligned}$$

$$\begin{aligned}
x_2 + 1 &= 0 && \text{Case II}\\
[x_2 + 1] + \neg 1 &= [0] + \neg 1 && \text{AI(4.5a), SPE(4.15)}\\
x_2 + (1 + \neg 1) &= 0 + \neg 1 && \text{APA(4.2a)}\\
x_2 + 0 &= \neg 2 && \text{OOA(2.1)}\\
x_2 &= \neg 2 && \text{AId(4.4b)}\\
x_2 &= -2 && \text{ONeg(3.1b)}
\end{aligned}$$

D

**Dependencies:**example 17.1-20151012-190708

## 12.2 Completing The Square

Completing the square is an algebraic process used to find the roots of quadratic equations of the form, $ax^2 + bx + c = 0$. Essentially, we want to manipulate this equation such that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

*Proof.* Let's begin with a quadratic equation in the general form: $ax^2 + bx + c = 0$. Since we are trying to manipulate the equation $ax^2 + bx + c = 0$ such that $x =$ some value(s), we first want the coefficient factor $a$ to be equal to 1.

$$\begin{aligned}
\frac{1}{a}\left[\boldsymbol{ax^2+bx+c}\right] &= \frac{1}{a}\,[\boldsymbol{0}] && \text{SPE(4.15) + MI(4.10a)}\\
\frac{1}{a}\cdot ax^2+\frac{1}{a}\cdot bx+\frac{1}{a}\cdot c &= \frac{1}{a}\,[0] && \text{DPE(4.8a)}\\
\frac{1}{a}\cdot a\cdot x^2+\frac{1}{a}\cdot b\cdot x+\frac{1}{a}\cdot c &= \frac{1}{a}\,[0] && \text{JTC(3.5)}\\
x^2+\frac{b}{a}\cdot x+\frac{c}{a} &= 0 && \text{OOM(2.3)}\\
x^2+\frac{b}{a}x+\frac{c}{a} &= 0 && \text{CTJ(3.6)}
\end{aligned}$$

We now have three summands in the left hand expression where the first two summands have $x^2$ and $x$ terms respectively. The goal is to have $x =$ some value, so the next step is focused on removing the $\dfrac{c}{a}$ summand.

$$\begin{aligned}
\left[\boldsymbol{x^2+\frac{b}{a}x+\frac{c}{a}}\right]+\neg\frac{c}{a} &= [\boldsymbol{0}]+\neg\frac{c}{a} && \text{SPE(4.15) + AI(4.5a)}\\
x^2+\frac{b}{a}x+\left(\frac{c}{a}+\neg\frac{c}{a}\right) &= 0+\neg\frac{c}{a} && \text{APA(4.2a)}\\
x^2+\frac{b}{a}x+0 &= \neg\frac{c}{a} && \text{OOA(2.1)}\\
x^2+\frac{b}{a}x &= \neg\frac{c}{a} && \text{AId(4.4b)}
\end{aligned}$$

The next step is called completing the square - the creative step. The idea is to add a *NeW* constant, $k$, to the left-hand expression, $x^2+\frac{b}{a}x+k$, such that the quadratic expression can then be factored as two identical factors, $(x+m)(x+m) = (x+m)^2$, where $k = m\cdot m$.

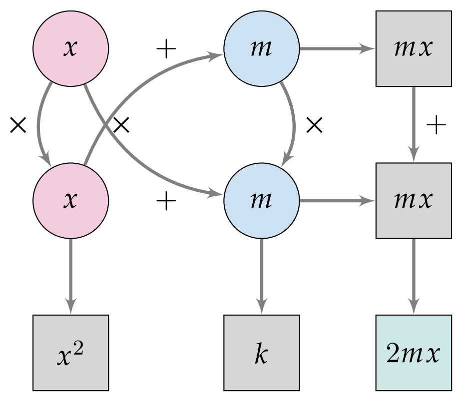



Figure 12.1: The Organization of the Distributive Property

Adding $k$ to the right-hand expression is a consequence of adding $k$ to the left-hand expression to get what we want (a perfect square), $\left[\boldsymbol{x^2+\frac{b}{a}x}\right]+k = \left[\boldsymbol{\neg\frac{c}{a}}\right]+k$.

To determine the values of both $m$ and $k$ we should refer to the the organisation of the two factors, $(x+m)^2$, that make up the product of the quadratic expression, $x^2+\frac{b}{a}x+k$.

Since both factors of this new quadratic expression are the same, both terms that make up the middle term, must also be the same. We know that $mx + mx = \frac{b}{a}x$, so we should be able to determine the value of $m$ from this equation. If we can determine the value of $m$, then we can determine the value of $k$.

$$\begin{aligned} \frac{b}{a}x &= mx + mx & \\ &= 2mx & \text{OOA(2.1)} \end{aligned}$$

Solving for $m$,

$$\begin{aligned} 2mx &= \frac{b}{a}x & \\ \frac{1}{2}\left[\mathbf{2mx}\right] &= \frac{1}{2}\left[\frac{\mathbf{b}}{\mathbf{a}}\mathbf{x}\right] & \text{SPE(4.15)} + \text{MI(4.10a)} \\ \left(\frac{1}{2}\cdot 2\right)mx &= \left(\frac{1}{2}\frac{b}{a}\right)x & \text{APA(4.2a)} \\ 1mx &= \frac{b}{2a}x & \text{OOM(2.3)} \\ mx &= \frac{b}{2a}x & \text{MId(4.9b)} \\ \left[\mathbf{mx}\right]\frac{1}{x} &= \left[\frac{\mathbf{b}}{\mathbf{2a}}\mathbf{x}\right]\frac{1}{x} & \text{SPE(4.15)} + \text{MI(4.10a)} \\ m\left(x\cdot\frac{1}{x}\right) &= \frac{b}{2a}\left(x\cdot\frac{1}{x}\right) & \text{APM(4.7b)} \\ m\cdot 1 &= \frac{b}{2a}\cdot 1 & \text{OOM(2.3)} \\ m &= \frac{b}{2a} & \text{MId(4.9b)} \end{aligned}$$

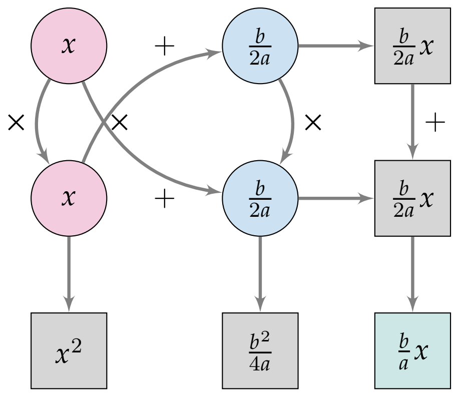



Figure 12.2: The Organization of the Distributive Property

$$\left[x^2 + \frac{b}{a}x\right] + \left(\frac{b}{2a}\right)^2 = \left[\neg\frac{c}{a}\right] + \left(\frac{b}{2a}\right)^2 \qquad \text{SPE(4.15) + Completing the Square}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{APA(4.2a)}$$

$$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{DPF(4.3b)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{PoTF(3.9)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \frac{(b)^2}{(2a)^2} \qquad \text{PoQPo(5.5a)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \frac{b^2}{2^2a^2} \qquad \text{PoPrPo(5.6a)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{OOE(2.5)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} \qquad \text{MId(4.9a)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c \cdot 4 \cdot a}{a \cdot 4 \cdot a} + \frac{b^2}{4a^2} \qquad \text{JTC(3.5)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{4 \cdot a \cdot c}{4 \cdot a \cdot a} + \frac{b^2}{4a^2} \qquad \text{CPM(4.6)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{4 \cdot a \cdot c}{4 \cdot a^2} + \frac{b^2}{4a^2} \qquad \text{PrCBPo(5.3a)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \text{CTJ(3.6)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{\neg 4ac + b^2}{4a^2} \qquad \text{CD(2.6a)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 + \neg 4ac}{4a^2} \qquad \text{CPM(4.6)}$$

$$\left[\left(x + \frac{b}{2a}\right)^2\right]^{\frac{1}{2}} = \pm\left[\frac{b^2 + \neg 4ac}{4a^2}\right]^{\frac{1}{2}} \qquad \text{SPE(4.15)}$$

$$x + \frac{b}{2a} = \pm\left[\frac{b^2 + \neg 4ac}{4a^2}\right]^{\frac{1}{2}} \qquad \text{PoPrPo(5.6a)}$$

$$x + \frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{\left[4a^2\right]^{\frac{1}{2}}} \qquad \text{PoQPo(5.5a)}$$

$$x + \frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{4^{\frac{1}{2}}a} \qquad \text{PoPrPo(5.6a)}$$

$$x + \frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{OOE(2.5)}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{RTPo(3.12)}$$

$$
\begin{aligned}
\left[\boldsymbol{x+\frac{b}{2a}}\right]+\neg\frac{b}{2a} &= \left[\boldsymbol{\pm\frac{\sqrt{b^2-4ac}}{2a}}\right]+\neg\frac{b}{2a} && \text{SPE(4.15)} + \text{AI(4.5a)} \\
x+\left(\frac{b}{2a}+\neg\frac{b}{2a}\right) &= \pm\frac{\sqrt{b^2-4ac}}{2a}+\neg\frac{b}{2a} && \text{APA(4.2b)} \\
x+\left(\frac{b}{2a}+\neg\frac{b}{2a}\right) &= \neg\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} && \text{CPM(4.6)} \\
x+0 &= \neg\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} && \text{OOA(2.1)} \\
x &= \neg\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} && \text{AId(4.4b)} \\
x &= \frac{\neg b\pm\sqrt{b^2-4ac}}{2a} && \text{CD(2.6a)} \\
x &= \frac{\neg b\pm\sqrt{b^2-4ac}}{2a} && \text{DOS(4.12b)} \\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} && \text{ONeg(3.1b)}
\end{aligned}
$$

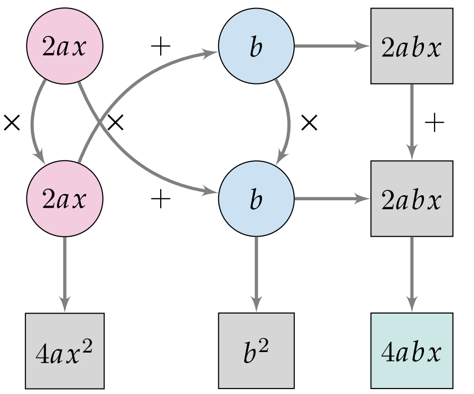
■

The previous proof starts with choosing to multiply both expressions by the multiplicative inverse of the coefficient of the degree two term such that $ax^2$ becomes $x^2$. It is easier to manually complete the square, guess the two binomial factors, when the coefficient of the degree 2 term is 1. However, as a consequence, the coefficients of the degree one and degree zero terms become fractions $\frac{b}{a}$ and $\frac{c}{a}$ respectively. All that it means is that we have to work with fractions throughout the procedure.

*Proof.*

$$
\begin{aligned}
ax^2+bx+c &= 0 \\
4a\left[\boldsymbol{ax^2+bx+c}\right] &= 4a\,[\boldsymbol{0}] && \text{SPE(4.15)} \\
4ax^2+4abx+4ac &= 4a(0) && \text{DPE(4.8a)} \\
4ax^2+4abx+4ac &= 0 && \text{OOM(2.3)} \\
\left[\boldsymbol{4ax^2+4abx+4ac}\right]+\neg 4ac &= [\boldsymbol{0}]+\neg 4ac && \text{AI(4.5a), SPE(4.15)} \\
4ax^2+4abx+(4ac+\neg 4ac) &= 0+\neg 4ac && \text{APA(4.2a)} \\
4ax^2+4abx+0 &= 0+\neg 4ac && \text{OOA(2.1)} \\
4ax^2+4abx &= \neg 4ac && \text{AId(4.4b)}
\end{aligned}
$$

Completing the square




$$
\begin{aligned}
\left[\mathbf{4ax^2 + 4abx}\right] + b^2 &= [\mathbf{-4ac}] + b^2 && \text{SPE(4.15)}\\
4ax^2 + 4abx + b^2 &= b^2 + \neg 4ac && \text{CPA(4.1)}\\
(2ax+b)(2ax+b) &= b^2 + \neg 4ac && \text{DPF(4.3a)}\\
(2ax+b)^2 &= b^2 + \neg 4ac && \text{PoTF(3.9)}\\
\left[\mathbf{(2ax+b)^2}\right]^{\frac{1}{2}} &= \pm\left[\mathbf{b^2 + \neg 4ac}\right]^{\frac{1}{2}} && \text{SPE(4.15)}\\
2ax + b &= \pm(b^2 + \neg 4ac)^{\frac{1}{2}} && \text{PoPo(5.1a)}\\
2ax + b &= \pm\sqrt{b^2 + \neg 4ac} && \text{PoTR(\textbf{??})}\\
[\mathbf{2ax + b}] + \neg b &= \left[\mathbf{\pm\sqrt{b^2 + \neg 4ac}}\right] + \neg b && \text{AI(4.5a), SPE(4.15)}\\
(2ax+b) + \neg b &= \neg b \pm \sqrt{b^2 + \neg 4ac} && \text{CPA(4.1)}\\
2ax + (b + \neg b) &= \neg b \pm \sqrt{b^2 + \neg 4ac} && \text{APA(4.2b)}\\
2ax + 0 &= \neg b \pm \sqrt{b^2 + \neg 4ac} && \text{OOA(2.1)}\\
2ax &= \neg b \pm \sqrt{b^2 + \neg 4ac} && \text{AId(4.4b)}\\
\frac{1}{2a}\,[\mathbf{2ax}] &= \frac{1}{2a}\left[\mathbf{\neg b \pm \sqrt{b^2 + \neg 4ac}}\right] && \text{MI(4.5a), SPE(4.15)}\\
\left(\frac{1}{2a}\cdot 2a\right)x &= \frac{1}{2a}\left(\neg b \pm \sqrt{b^2 + \neg 4ac}\right) && \text{APM(4.7a)}\\
\frac{2a}{2a}x &= \frac{\neg b \pm \sqrt{b^2 + \neg 4ac}}{2a} && \text{OOM(2.3)}\\
1x &= \frac{\neg b \pm \sqrt{b^2 + \neg 4ac}}{2a} && \text{RF(\textbf{??})}\\
1x &= \frac{\neg b \pm \sqrt{b^2 - 4ac}}{2a} && \text{DOS(4.12b)}\\
1x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{ONeg(3.1b)}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{MId(4.9b)}
\end{aligned}
$$

■

# III Functions

# 13. Functions

**Property 13.0.1 – Function Value Argument Substitution (FVAS).**

$$f(a) \tag{13.1a}$$

**Example 13.1 – id:20151012-201647.**
Find $f(2)$ given $f(x) = 2x^3 - 3x^2 - 12x + 1$

S

**Solution:**

| | |
|---|---|
| $f(x) = 2x^3 + \neg 3x^2 + \neg 12x + 1$ | DOS(4.12a) |
| $f(2) = 2\,[\mathbf{2}]^3 + \neg 3\,[\mathbf{2}]^2 + \neg 12\,[\mathbf{2}] + 1$ | SPE(4.15) |
| $f(2) = 2(8) + \neg 3(4) + \neg 12(2) + 1$ | OOE(2.5) |
| $f(2) = 16 + \neg 12 + \neg 24 + 1$ | OOM(2.3) |
| $f(2) = -19$ | OOA(2.1) |

D

**Dependencies:**example 17.1-20151012-190708

**Example 13.2 – id:20151012-203549.**
Find $f(-1)$ given $f(x) = 2x^3 - 3x^2 - 12x + 1$

S

**Solution:**

$$
\begin{aligned}
f(x) &= 2x^3 + \neg 3x^2 + \neg 12x + 1 && \text{DOS(4.12a)} \\
f(\neg 1) &= 2\left[\neg \mathbf{1}\right]^3 + \neg 3\left[\neg \mathbf{1}\right]^2 + \neg 12\left[\neg \mathbf{1}\right] + 1 && \text{SPE(4.15)} \\
f(\neg 1) &= 2(\neg 1) + \neg 3(1) + \neg 12(\neg 1) + 1 && \text{OOE(2.5)} \\
f(\neg 1) &= \neg 2 + \neg 3 + 12 + 1 && \text{OOM(2.3)} \\
f(\neg 1) &= 8 && \text{OOA(2.1)}
\end{aligned}
$$

D

**Dependencies:**example 17.1-20151012-190708

## 13.1 Inverse Functions

## 13.2 Inverses

**Property 13.2.1 – Cosine Inverse (ArcCos).**

$$\cos^{-1}(\cos\theta) = \theta \tag{13.2a}$$

**Property 13.2.2 – Sine Inverse (ArcSin).**

$$\sin^{-1}(\sin\theta) = \theta \tag{13.3a}$$

**Property 13.2.3 – Tangent Inverse (ArcTan).**

$$\tan^{-1}(\tan\theta) = \theta \tag{13.4a}$$

**Property 13.2.4 – Exponential Inverse (EI).**

$$\log_a(a^x) = x \tag{13.5a}$$

**Property 13.2.5 – Logarithmic Inverse (LI).**

$$a^{\log_a x} = x \tag{13.6a}$$

**Property 13.2.6 – Power Inverse (PoI).**

$$(b^m)^{\frac{1}{m}} = b \tag{13.7a}$$

# IV Differential Calculus

# 14. Derivative by First Principles

## 14.1 Limit of the Difference Quotient

**Definition 14.1.1 – Derivative.** The derivative of a function $f(x)$ with respect to the variable $x$ is defined as

$$f'(x) \equiv \lim_{\Delta x \to 0} \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{Difference Quotient}} \tag{14.1}$$

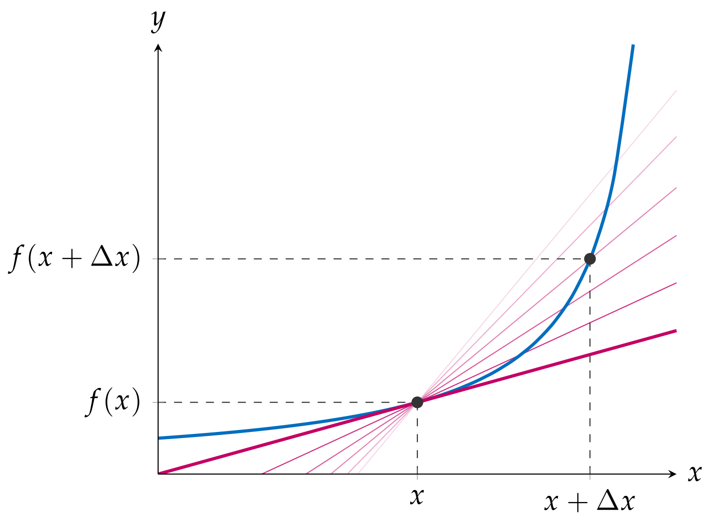



Figure 14.1: **[mooculus:textbook]**

**Example 14.1 – id:20141219-212546.**

Differentiate the function $f(x) = 5$

S

**Solution:**

$$
\begin{aligned}
f(x) &= 5x^0 && \text{PoID(4.13a)} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{5\left[\boldsymbol{x + \Delta x}\right]^0 - 5\left[\boldsymbol{x}\right]^0}{\Delta x} && \text{SPE(4.15)\&DBFP(14.1)} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{5(1) - 5(1)}{\Delta x} && \text{PoID(4.13b)} \\
f'(x) &= \lim_{\Delta x \to 0} 0 && \text{OOM(2.3)} \\
f'(x) &= 0
\end{aligned}
$$

■

# 15. Derivative Rules

## 15.1 Derivative of a Monomial Functions

**Example 15.1 – id:20141124-153017.**

Differentiate $f(x) = -3$

S

**Solution:**

$$\begin{aligned} f'(x) &= [-3]' && \text{SPE(4.15)} \\ f'(x) &= 0 && \text{DC(7.1)} \end{aligned}$$

D

**Dependencies:**
example 15.3-20141124-152503

**Example 15.2 – id:20141124-141850.**

Differentiate $f(x) = x^2$

S

**Solution:**

$$\begin{aligned} f'(x) &= \left[x^2\right]' && \text{SPE(4.15)} \\ f'(x) &= 2x^{2-1} && \text{DPo(7.5)} \\ f'(x) &= 2x^1 && \text{OOA(2.1)} \\ f'(x) &= 2x && \text{MId(4.9b)} \end{aligned}$$

S

**Less Steps Solution:**

$$f'(x) = 2x \qquad \text{DPo(7.5)}$$

**Dependencies:**
example 15.2-20141124-141850
example 15.3-20141124-152503

## 15.2 Derivative of Polynomial Functions

**Example 15.3 – id:20141124-152503.**

Differentiate $f(x) = x^2 - 3$

**Solution:**

$$
\begin{aligned}
f(x) &= x^2 + \neg 3 && \text{DOS(4.12a)}\\
f'(x) &= \left[\boldsymbol{x^2 + \neg 3}\right]' && \text{SPE(4.15)}\\
f'(x) &= \left[\boldsymbol{x^2}\right]' + [\boldsymbol{\neg 3}]' && \text{DS(7.7)}\\
f'(x) &= \left[\boldsymbol{x^2}\right]' + 0 && \text{DC(7.1)}\\
f'(x) &= \left[\boldsymbol{x^2}\right]' && \text{AId(4.4a)}\\
f'(x) &= 2x && \text{DPo(7.5) goto 15.2}
\end{aligned}
$$

**Less Steps Solution:**

$$f(x) = 2x^2 \qquad \text{DPo(7.5)\&DC(7.5)}$$

**Dependencies:**
example 15.14-20141124-205219

**Example 15.4 – id:20151015-155838.**
Differentiate $y = x^3 + 1$

**Solution:**

$$
\begin{aligned}
[y]' &= \left[x^3 + 1\right] && \text{SPE(4.15)}\\
y' &= [x^3]' + [1]' && \text{DS(7.7)}\\
y' &= 3x^2 + [1]' && \text{DPo(7.5)}\\
y' &= 3x^2 + 0 && \text{DC(7.1)}\\
y' &= 3x^2 && \text{MId(4.9b)}
\end{aligned}
$$

D

**Dependencies:**example 15.16-20151015-153507

**Example 15.5 – id:20151011-195002.**
Differentiate $f(x) = 2x^2 + 3x + 7$

S

**Solution:**

$$
\begin{aligned}
[f'(x)]' &= \left[2x^2 + 3x + 7\right]' && \text{SPE(4.15)}\\
f'(x) &= \left[2x^2\right]' + [3x]' + [7]' && \text{DS(7.7)}\\
f'(x) &= 2\left[x^2\right]' + 3\,[x]' + [7]' && \text{DCM(7.3)}\\
f'(x) &= 2\left[x^2\right]' + 3\,[x]' + 0 && \text{DC(7.1)}\\
f'(x) &= 2\left[x^2\right]' + 3\,[x]' && \text{AId(4.4a)}\\
f'(x) &= 2(2x) + 3 && \text{DPo(7.5)}\\
f'(x) &= 4x + 3 && \text{OOM(2.3)}
\end{aligned}
$$

D

**Dependencies:**example 16.1-20151011-154209

**Example 15.6 – id:20141128-151834.**

Differentiate $f(x) = 3x^2 - 6x + 4$

S

**Solution:**

$$
\begin{aligned}
f(x) &= 3x^2 + \neg 6x + 4 && \text{DOS(4.12a)}\\
f'(x) &= \left[\mathbf{3x^2 + \neg 6x + 4}\right]' && \text{SPE(4.15)}\\
f'(x) &= \left[\mathbf{3x^2}\right]' + [\mathbf{6x}]' + [\mathbf{4}]' && \text{DS(7.7)}\\
f'(x) &= \left[\mathbf{3x^2}\right]' + [\mathbf{6x}]' + 0 && \text{DC(7.1)}\\
f'(x) &= \left[\mathbf{3x^2}\right]' + [\mathbf{6x}]' && \text{AId(4.4a)}\\
f'(x) &= 3\left[\mathbf{x^2}\right]' + 6\,[\mathbf{x}]' && \text{DCM(7.3)}\\
f'(x) &= 3\,(2x) + 6\,(1) && \text{DPo(7.5)}\\
f'(x) &= 6x + 6 && \text{OOM(2.3)}
\end{aligned}
$$

S

$$f'(x) = 6x + 6 \qquad \text{DS(7.7)}$$

■

**Example 15.7 – id:20141209-144203.**

Differentiate $f(x) = x^2(2x+4)$

S

**Solution:**

$$
\begin{aligned}
f'(x) &= \left[\mathbf{x^2(2x+4)}\right]' && \text{SPE(4.15)}\\
f'(x) &= \left[\mathbf{x^2}\right]'(2x+4) + x^2\,[\mathbf{2x+4}]' && \text{DPr(7.11)}\\
f'(x) &= \left[\mathbf{x^2}\right]'(2x+4) + x^2\,[\mathbf{2x}] + [\mathbf{4}]' && \text{DS(7.7)}\\
f'(x) &= \left[\mathbf{x^2}\right]'(2x+4) + x^2\cdot 2\,[\mathbf{x}] + [\mathbf{4}]' && \text{DCM(7.3)}\\
f'(x) &= 2x(2x+4) + x^2\cdot 2\cdot 1 + [\mathbf{4}]' && \text{DPo(7.5)}\\
f'(x) &= 2x(2x+4) + x^2\cdot 2\cdot 1 + 0 && \text{DC(7.1)}\\
f'(x) &= 6x^2 + 8x && \text{simplify goto } 8.34
\end{aligned}
$$

■

**Example 15.8 – id:20141209-142321.**

Differentiate $f(x) = x^2\cos(x)$

S

**Solution:**

| | |
|---|---|
| $f'(x) = \left[x^2 \cos(x)\right]'$ | SPE(4.15) |
| $f'(x) = \left[x^2\right]' \cos(x) + x^2 \left[\cos(x)\right]'$ | DPr(7.11) |
| $f'(x) = 2x \cos(x) + x^2 \left[\cos(x)\right]'$ | DPo(7.5) |
| $f'(x) = 2x \cos(x) + x^2(-1 \sin(x))$ | DCos(7.19) |
| $f'(x) = 2x \cos(x) - x^2 \sin x$ | OOM(2.3) |

**Example 15.9 – id:20150910-115935.**

Differentiate $f(x) = \sin(x) \cos(x)$

S

**Solution:**

| | |
|---|---|
| $f'(x) = [\sin(x) \cos(x)]'$ | SPE(4.15) |
| $f'(x) = [\sin(x)]' \cos(x) + \sin(x) [\cos(x)]'$ | DPr(7.11) |
| $f'(x) = \cos(x) \cos(x) + \sin(x) [\cos(x)]'$ | DSin(7.17) |
| $f'(x) = \cos(x) \cos(x) + \sin(x)(-\sin(x))$ | DCos(7.19) |
| $f'(x) = \cos^2(x) - \sin^2(x)$ | simplify goto **??** |

**Example 15.10 – id:20141209-151354.**

Differentiate $f(x) = \sin(x) \sin(x)$

S

**Solution:**

| | |
|---|---|
| $f'(x) = [\sin(x) \sin(x)]'$ | SPE(4.15) |
| $f'(x) = [\sin(x)]' \sin(x) + \sin(x) [\sin(x)]$ | DPr(7.11) |
| $f'(x) = \cos(x) \sin(x) + \sin(x) \cos(x)$ | DSin(7.17) |
| $f'(x) = \cos(x) \sin(x) + \cos(x) \sin(x)$ | CPM(4.6) |
| $f'(x) = 2 \cos(x) \sin(x)$ | OOA(2.1) |

**Example 15.11 – id:20141124-203850.**

Differentiate $y = \ln(3x)$

S

**Solution:**
After identifying that $y = \ln(3x)$ is a composite function, we let $u = 3x$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(7.16)}$$

We need to find the factors $\frac{dy}{du}$ and $\frac{du}{dx}$.

$$\begin{aligned} y &= \ln(u) & u &= 3x \\ \frac{dy}{du} &= \frac{1}{u} & \frac{du}{dx} &= 3 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{DComp(7.16)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} && \text{DNL(7.31)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot 3 && \text{DPo(7.5)} \\ \frac{dy}{dx} &= \frac{1}{3x} \cdot 3 \\ \frac{dy}{dx} &= \frac{3}{3x} && \text{OOM(2.3)} \\ \frac{dy}{dx} &= \frac{1}{x} \end{aligned}$$

D

**Dependencies:**
example 15.14-20141124-205219

**Example 15.12 – id:20141128-160248.**

Differentiate $y = \ln(3x^2 - 6x + 4)$

S

**Solution:** After identifying that $y = \ln(3x^2 - 6x + 4)$ is a composite function, we let $u = 3x^2 - 6x + 4$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(7.16)}$$

We need to find the factors $\frac{dy}{du}$ and $\frac{du}{dx}$.

$$\begin{aligned} y &= \ln(u) & u &= 3x^2 - 6x + 4 \\ \frac{dy}{du} &= \frac{1}{u} & \frac{du}{dx} &= 6x - 6 & \text{goto } 15.6 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{DComp(7.16)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} && \text{DNL(7.31)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot (6x - 6) && \text{DPo(7.5)} \\ \frac{dy}{dx} &= \frac{1}{3x^2 - 6x + 4} \cdot (6x - 6) \\ \frac{dy}{dx} &= \frac{6x - 6}{3x^2 - 6x + 4} && \text{OOM(2.3)} \end{aligned}$$

■

**Example 15.13 – id:20141128-155506.**

Differentiate $y = \ln(\cos x)$

S

**Solution:** After identifying that $y = \ln(\cos x)$ is a composite function, we let $u = \cos x$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(7.16)}$$

We need to find the factors $\frac{dy}{du}$ and $\frac{du}{dx}$.

$$\begin{aligned} y &= \ln(u) & u &= \cos x \\ \frac{dy}{du} &= \frac{1}{u} & \frac{du}{dx} &= -\sin x \end{aligned}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ DComp(7.16)

$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$ DNL(7.31)

$$\frac{dy}{dx} = \frac{1}{u} \cdot -\sin x$$ DPo(7.5)

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x}$$ OOM(2.3)

$$\frac{dy}{dx} = -\tan x$$

■

**Example 15.14 – id:20141124-205219.**

Differentiate $y = \left(x^2 - 1\right) \ln(3x)$

S

**Solution:**

$$y' = \left[\boldsymbol{x^2 - 3}\right]' \cdot \ln x + \left(x^2 - 3\right) \cdot [\ln \boldsymbol{3x}]'$$ DPr(7.11)

$$y' = 2x \cdot \ln x + \left(x^2 - 3\right) \cdot [\ln \boldsymbol{3x}]'$$ differentiate goto 15.3

$$y' = 2x \cdot \ln x + \left(x^2 - 3\right) \cdot \frac{1}{x}$$ differentiate goto 15.11

$$y' = 2x \cdot \ln x + \frac{x^2 - 3}{x}$$ OOM(2.3)

$$y' = 2x^2 \ln x + \frac{x^2 - 3}{x}$$ JTC(3.5)

$$y' = \frac{2x^2 \ln x + \left(x^2 - 3\right)}{x}$$ OOA(2.1)

■

## 15.3 Derivative of a Quotient

**Example 15.15 – id:20151015-165037.**

Differentiate $y = \dfrac{\sin x}{e^x}$

S

**Solution:**

$$\frac{\mathrm{d}\,[y]}{\mathrm{d}x} = \frac{\mathrm{d}\left[\frac{\sin x}{e^x}\right]}{\mathrm{d}x} \qquad \text{SPE(4.15)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]\, e^x - \sin x \frac{\mathrm{d}}{\mathrm{d}x}[e^x]}{[e^x]^2} \qquad \text{DQ(??)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]\, e^x - \sin x \frac{\mathrm{d}}{\mathrm{d}x}[e^x]}{e^{2x}} \qquad \text{PoPo(5.1a)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x e^x - \sin x \frac{\mathrm{d}}{\mathrm{d}x}[e^x]}{e^{2x}} \qquad \text{DSin(7.18)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x e^x - \sin x e^x}{e^{2x}} \qquad \text{DExp(7.34)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x(\cos x - \sin x)}{e^{2x}} \qquad \text{DPF(4.3b)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x(\cos x - \sin x)}{e^x e^x} \qquad \text{PrCBPo(5.3b)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cancel{e^x}(\cos x - \sin x)}{\cancel{e^x} e^x} \qquad \text{MId(4.9a)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x - \sin x}{e^x}$$

■

## 15.4 Derivative of a Rational Function

**Example 15.16 – id:20151015-153507.**

Differentiate $f(x) = \dfrac{x^3 + 1}{x^2}$

S

**Solution:**

Method 1: Derivative of a quotient

$$[f(x)]' = \left[\frac{x^3+1}{x^2}\right]' \qquad \text{SPE(4.15)}$$

$$f'(x) = \frac{\left[x^3+1\right]'(x^2) + \neg 1\left((x^3+1)\left[x^2\right]'\right)}{\left[x^2\right]^2} \qquad \text{DQ(??)}$$

$$f'(x) = \frac{(3x^2)(x^2)\neg 1\left((x^3+1)\left[x^2\right]'\right)}{\left[x^2\right]^2} \qquad \text{differentiate goto } 15.4$$

$$f'(x) = \frac{(3x^2)(x^2)\neg 1\left((x^3+1)(2x)\right)}{\left[x^2\right]^2} \qquad \text{differentiate goto } 15.2$$

$$f'(x) = \frac{3x^3-2x^2-2}{x^3} \qquad \text{Simplify goto } \textbf{??}$$

■

# 16. Equations of Tangent & Secant Lines

## 16.1 Essential Questions

**Essential Questions 16.1**

1. How do we find the equation of the tangent line of a given function at the point $P(a, b)$?
2. How do we find the equation of the tangent line of a given function at $x = a$?

## 16.2 Finding the Equation of the Tangent Line

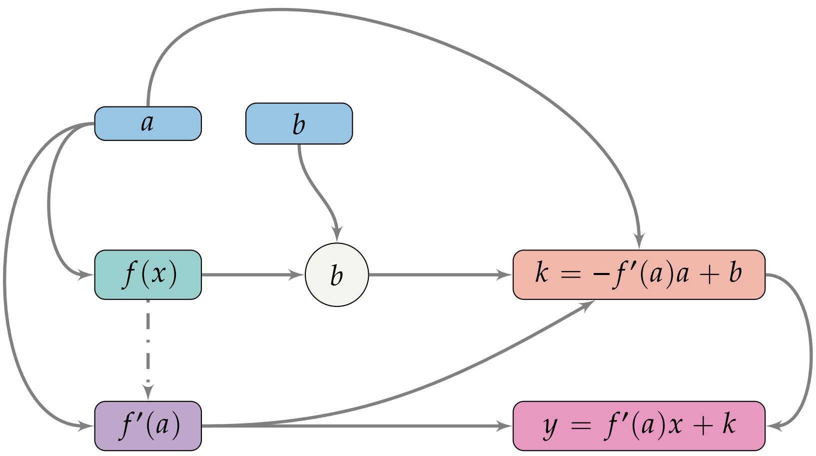



Figure 16.1: Finding the Equation of a Tangent Line Workflow

For a given function, $f(x)$, our goal is to find the equation of the tangent line at the point $P(a, b)$, which can be expressed in slope-intercept form as:

$$y = f'(a)x + k$$

Where $f'(a)$ is the value of the derivative, slope, at $x = a$ and $k$ is the $y$-intercept. We therefore need to:

1. Find the derivative of the function $f(x)$: $f'(x)$
2. Find the value of the derivative at $x = a$: $f'(a)$
3. Find the value of the $y$-intercept: $k = -f'(a) + b$:
4. If the ordinate $b$ is not explicitly given, then find $f(a) = b$

**Example 16.1 – id:20151011-154209.**
Find the equation of the line tangent to the curve of the function $f(x) = 2x^2 + 3x + 7$ at the point $P(2, 21)$.

S

**Solution:**
Find the derivative of $f(x)$

$$f'(x) = 4x + 3 \text{ goto } 15.5$$

Evaluate the derivative at $x = 2$

$$\begin{aligned} f'(2) &= 4\,[2] + 3 && \text{SPE(4.15)} \\ f'(2) &= 8 + 3 && \text{OOM(2.3)} \\ f'(2) &= 11 && \text{OOA(2.1)} \end{aligned}$$

Find the $y$-intercept, $k$, of the equation of the tangent line.

$$\begin{aligned} y &= f'(x)x + k && \\ [21] &= [11]\,[2] + k && \text{SPE(4.15)} \\ 21 &= 22 + k && \text{OOM(2.3)} \\ \neg 22 + [21] &= \neg 22 + [22 + k] && \text{SPE+AI} \\ \neg 22 + 21 &= (\neg 22 + 22) + k && \text{APA(4.2a)} \\ \neg 1 &= 0 + k && \text{OOA(2.1)} \\ \neg 1 &= k && \text{AId(4.4a)} \\ -1 &= k && \text{ONeg(3.1b)} \\ k &= -1 && \text{SyPE(4.16a)} \end{aligned}$$

The equation of the tangent line is

$$\begin{aligned} y &= f'(2)x + k && \\ y &= 11x - 1 && \text{SPE(4.15)} \end{aligned}$$

■

**Example 16.2 – id:20151015-171630.**
Given $f(x) = e^{3x}$, find the the equation of the tangent line $L$ to the curve $f$ at the point $(2, e^2)$.

S

**Solution:**

Find the derivative of $f(x)$

$$
\begin{aligned}
[f(x)]' &= \left[e^{3x}\right]' && \text{SPE(4.15)}\\
f'(x) &= [3x]'\left[e^{3x}\right]' && \text{DCF(7.15)}\\
f'(x) &= 3\,[x]'\left[e^{3x}\right]' && \text{DCM(7.3)}\\
f'(x) &= 3\cdot 1\left[e^{3x}\right]' && \text{DPo(7.5)}\\
f'(x) &= 3\left[e^{3x}\right]' && \text{OOM(2.3)}\\
f'(x) &= 3e^{3x} && \text{DExp(7.33)}
\end{aligned}
$$

Evaluate the derivative at $x = 2$

$$
\begin{aligned}
f'(2) &= 3e^{3[2]} && \text{SPE(4.15)}\\
f'(2) &= 3e^{6} && \text{OOM(2.3)}
\end{aligned}
$$

Find the $y$-intercept, $k$, of the equation of the tangent line.

$$
\begin{aligned}
y &= f'(x)x + k && \\
\left[e^{2}\right] &= \left[3e^{6}\right][2] + k && \text{SPE(4.15)}\\
e^{2} &= 3\cdot e^{6}\cdot 2 + k && \text{JTC(3.5)}\\
e^{2} &= 3\cdot 2\cdot e^{6} + k && \text{CPM(4.6)}\\
e^{2} &= 6\cdot e^{6} + k && \text{OOM(2.3)}\\
e^{2} &= 6e^{6} + k && \text{CTJ(3.6)}\\
\neg 6e^{6} + \left[e^{2}\right] &= \neg 6e^{6} + \left[6e^{6} + k\right] && \text{AI(4.5a), SPE(4.15)}\\
e^{2}(\neg 6e^{4} + 1) &= \neg 6e^{6} + \left[6e^{6} + k\right] && \text{DPF(4.3b)}\\
\neg 6e^{6} + e^{2} &= (\neg 6e^{6} + 6e^{6}) + k && \text{APA(4.2a)}\\
e^{2}(\neg 6e^{4} + 1) &= 0 + k && \text{OOA(2.1)}\\
e^{2}(\neg 6e^{4} + 1) &= k && \text{AId(4.4b)}\\
e^{2}(-6e^{4} + 1) &= k && \text{ONeg(3.1b)}
\end{aligned}
$$

The equation of the tangent line, $L$ is

$$
\begin{aligned}
y &= f'(2)x + k && \\
y &= 3e^{6}x + e^{2}(-6e^{4} + 1) && \text{SPE(4.15)}
\end{aligned}
$$

■

# 17. First Derivative Test

**Example 17.1 – id:20151012-190708.**

Given the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ find the critical points, classify the critical points and find the intervals of increasing/decreasing.

S

**Solution:**

Differentiate the function:

$$\begin{aligned}
f(x) &= 2x^3 + \neg 3x^2 + \neg 12x + 1 && \text{DOS(4.12a)}\\
[f(x)]' &= \left[\mathbf{2x^2 + \neg 3x^2 + \neg 12x + 1}\right]' && \text{SPE(4.15)}\\
f'(x) &= \left[\mathbf{2x^3}\right]' + \left[\mathbf{\neg 3x^2}\right]' + [\mathbf{\neg 12x}]' + [\mathbf{1}]' && \text{DS(7.7)}\\
f'(x) &= 2\left[\mathbf{x^3}\right]' + \neg 3\left[\mathbf{x^2}\right]' + \neg 12\,[\mathbf{x}]' + [\mathbf{1}]' && \text{DCM(7.3)}\\
f'(x) &= 2(3x^2) + \neg 3(2x) + \neg 12(1) + [\mathbf{1}]' && \text{DPo(7.5)}\\
f'(x) &= 6x^2 + \neg 6x + \neg 12 + [\mathbf{1}]' && \text{OOM(2.3)}\\
f'(x) &= 6x^2 + \neg 6x + \neg 12 + 0 && \text{DC(7.1)}\\
f'(x) &= 6x^2 + \neg 6x + \neg 12 && \text{AId(4.4a)}
\end{aligned}$$

Solving the equation $6x^2 + \neg 6x + \neg 12 = 0$ to find the $x$ value(s) of the critical points, goto 12.2, we find that $x = 2$ and $x = -1$ are critical values of the function.

Since we are looking for critical **points**, we need to find the ordinates, $y$-values, of the critical points by evaluating the function for the given critical $x$-values.

$$\begin{aligned}
f(2) &= -19 \;\text{ goto }\; 13.1\\
f(-1) &= 8 \;\text{ goto }\; 13.2
\end{aligned}$$

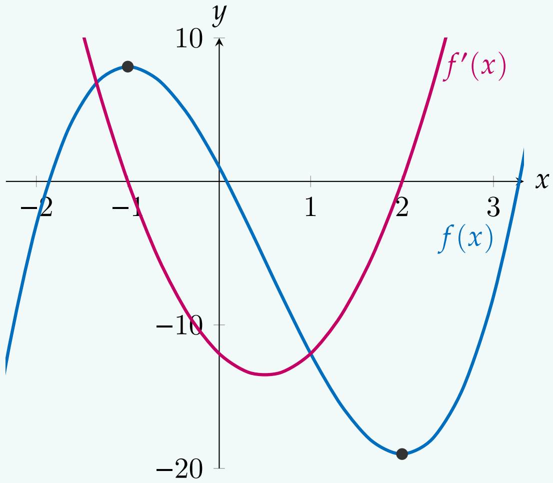
Therefore the critical points are $(2, -19)$ and $(-1, 8)$.

The first derivative test can be used to determine the intervals of increasing/decreasing and consequently we will be able to classify the critical points. Since we are only interested in the values of the derivative, it will be easier to used the factored form of the derivative, $f'(x)$=6(x-2)(x+1)

**1st Derivative Test Table**

| $x$ | $x < -1$ | $x = -1$ | $-1 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\nearrow$ | $\longrightarrow$ | $\searrow$ | $\longrightarrow$ | $\nearrow$ |
| C.P. | | Max | | Min | |

From the first derivative table we can classify the critical point $(2, -19)$ as a local maximum and $(-1, 8)$ as a local minimum.




■

# 18. Curve Sketching

### 18.0.1 Finding the vertex of a quadratic function using differentiation.

We can find the vertex of a quadratic function, $f(x)$ using differentiation by:

1. Differentiate the function: Find $f'(x)$.
2. Set the derivative equal to zero: $f'(x) = 0$.
3. Find the abscissa of the vertex by solving the equation $f'(x) = 0$ for $x$ to find the critical $x$ value: $x = k$.
4. Find the ordinate of the vertex by substituting the value of critical value $x = k$ into the function $f(x)$: Evaluate $f(k)$

**Example 18.1 – id:20151008-110208.**
Find the vertex of the quadratic function, $f(x) = ax^2 + bx + c$, using differentiation.

S

**Solution:**
1. Find the derivative of $f(x)$

$$
\begin{aligned}
[f(x)]' &= \left[ax^2 + bx + c\right]' && \text{SPE(4.15)}\\
f'(x) &= \left[ax^2\right]' + [bx]' + [c]' && \text{DS(7.7)}\\
f'(x) &= a\left[x^2\right]' + b\,[x]' + [c]' && \text{DCM(7.3)}\\
f'(x) &= 2\cdot a\cdot x + b\cdot 1 + [c]' && \text{DPo(7.5)}\\
f'(x) &= 2\cdot a\cdot x + 1\cdot b + [c]' && \text{CPM(4.6)}\\
f'(x) &= 2ax + 1b + [c]' && \text{CTJ(3.6)}\\
f'(x) &= 2ax + b + [c]' && \text{MId(4.9b)}\\
f'(x) &= 2ax + b + 0 && \text{DC(7.1)}\\
f'(x) &= 2ax + b && \text{AId(4.4b)}
\end{aligned}
$$

2. Set the derivative equal to zero and solve for $x$.

$$
\begin{aligned}
f'(x) &= 0\\
2ax + b &= 0\\
x &= -\frac{b}{2a} \quad \text{goto} \quad 11.9
\end{aligned}
$$

The abscissa of the vertex is $x = -\frac{b}{2a}$.

3. Find the ordinate of the vertex by substituting the argument $x = -\frac{b}{2a}$ into $f(x)$

**Example 18.2 – id:20150923-152515.**
Find the vertex of the parabola $y = x^2 - 2x - 6$ using differentiation.

S

1. Differentiate the function.

$$\begin{aligned}
f(x) &= x^2 - 2x - 6 \\
f(x) &= x^2 + \neg 2x + \neg 6 && \text{DOS(4.12a)} \\
[f(x)]' &= \left[x^2 + \neg 2x + \neg 6\right]' && \text{SPE(4.15)} \\
f'(x) &= \left[x^2\right]' + [\neg 2x]' + [\neg 6]' && \text{DS(7.7)} \\
f'(x) &= \left[x^2\right]' + \neg 2\,[x]' + [\neg 6]' && \text{DCM(7.3)} \\
f'(x) &= 2x + \neg 2 + [\neg 6]' && \text{DPo(7.5)} \\
f'(x) &= 2x + \neg 2 + 0 && \text{DC(7.1)} \\
f'(x) &= 2x + \neg 2 && \text{AId(4.4b)} \\
f'(x) &= 2x - 2 && \text{DOS(4.12b)}
\end{aligned}$$

2 and 3. Set the derivative equal to zero and solve for $x$

$$\begin{aligned}
2x - 2 &= 0 \\
x &= 1
\end{aligned}$$

4. Find the value of $f(1)$

$$\begin{aligned}
f(x) &= x^2 - 2x - 6 \\
f(1) &= [1]^2 - 2\,[1] - 6 && \text{SPE(4.15)} \\
f(1) &= -7 && \text{Evaluate}
\end{aligned}$$

The vertex of this parabola is the point $(1, -7)$

# 19. Second Derivative Test

# 20. Curve Sketching

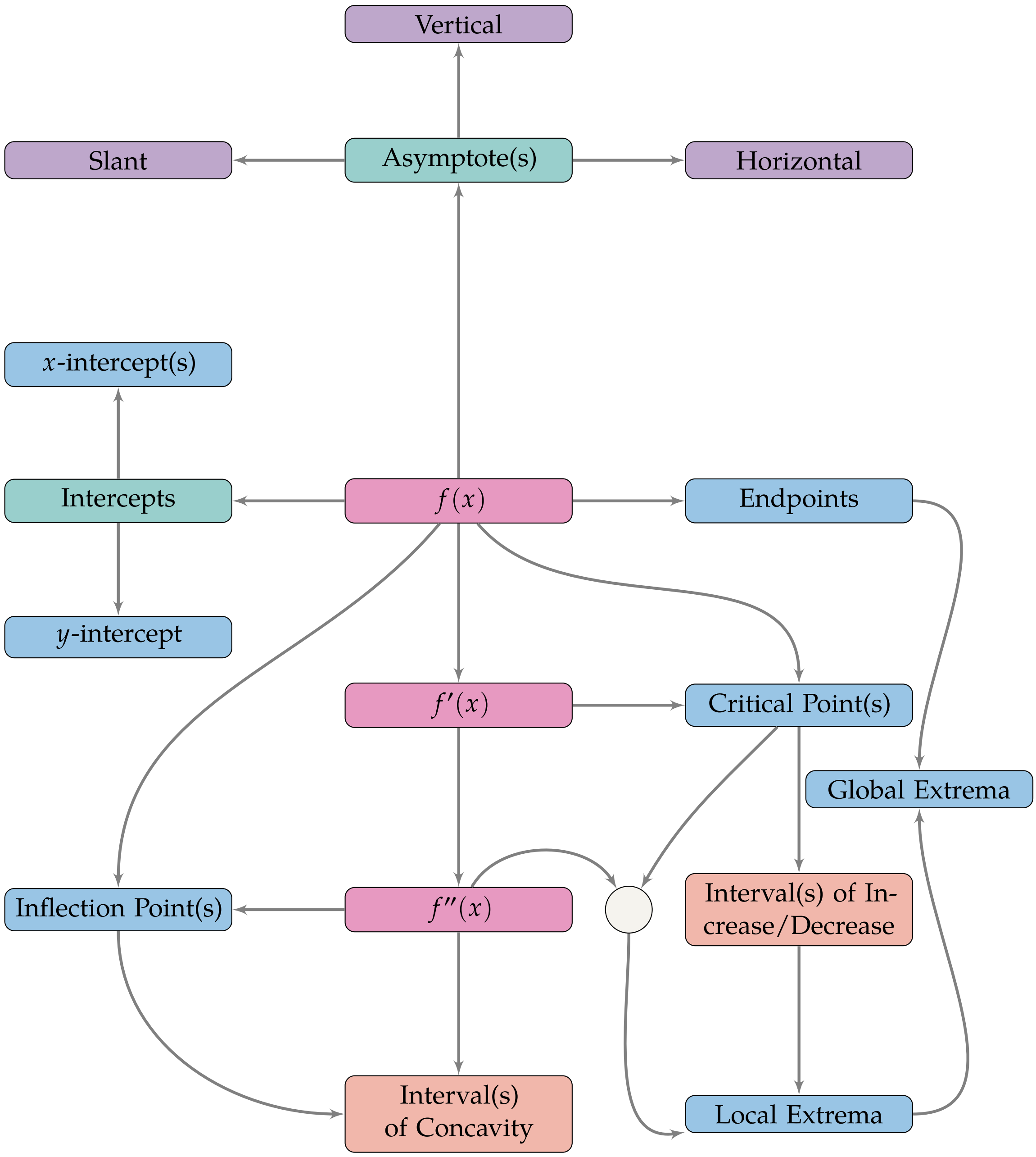



Figure 20.1: Overview of Curve Sketching

# Bibliography

## Books

## Website

## Articles