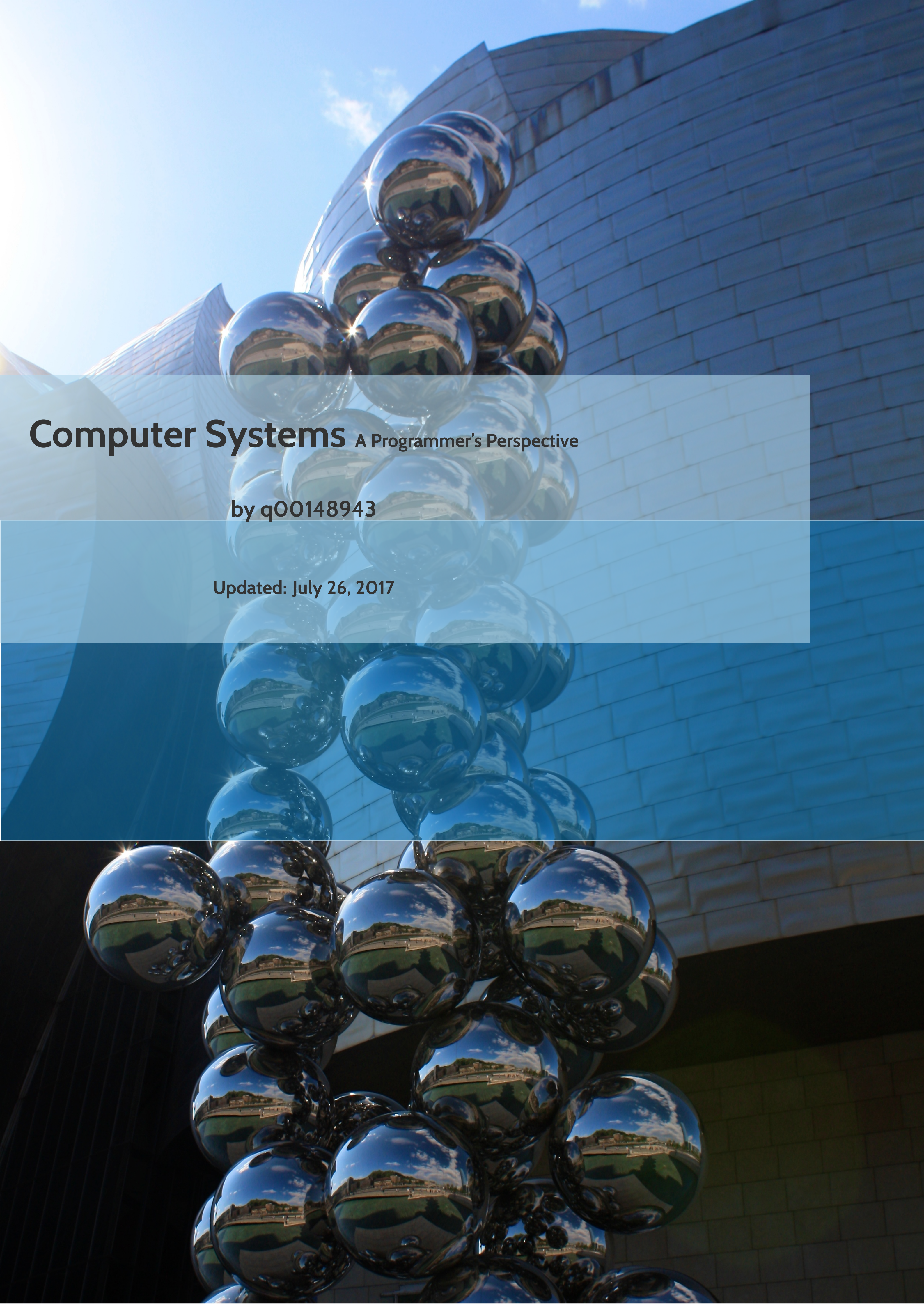

Computer Systems A Programmer's Perspective

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Contents

Program Structure and Execution

1. Numbers

1.1 Number Systems

Definition 1.1.1 – Natural Numbers.

$$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$$

R It is not uncommon for zero to be excluded from the natural numbers. In fact, some exclude zero from the natural numbers and then describe the set of natural numbers that include zero the whole numbers.

$$\mathbb{W} = \{0, 1, 2, 3, \ldots\}$$

For the purposes of these notes, zero will be included within the set of natural numbers.

Definition 1.1.2 – Integers.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Definition 1.1.3 – Positive Integers.

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

Definition 1.1.4 – Rational Numbers.

$$\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$$

Definition 1.1.5 – Proper Fraction. Given $m < n$, then the fraction m/n is called proper.

Definition 1.1.6 – Improper Faction. Given $m > n$, then the fraction m/n is called improper.

1.2 Prime Numbers

Definition 1.2.1 – Greatest Common Divisor. Suppose that m and n are positive integers. The greatest common divisor is the largest divisor (factor) common to both m and n.

Definition 1.2.2 – Relatively Prime. Two integers m and n are relatively prime to each other, $m \perp n$, if they share no common positive integer divisors (factors) except 1.

$$m \perp n \text{ if } \gcd(m,n) = 1.$$

1.2.1 Listing of Prime Numbers 2-997

2	3	5	7	11	13	17	19	23	29	31	37
41	43	47	53	59	61	67	71	73	79	83	89
97	101	103	107	109	113	127	131	137	139	149	151
157	163	167	173	179	181	191	193	197	199	211	223
227	229	233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349	353	359
367	373	379	383	389	397	401	409	419	421	431	433
439	443	449	457	461	463	467	479	487	491	499	503
509	521	523	541	547	557	563	569	571	577	587	593
599	601	607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733	739	743
751	757	761	769	773	787	797	809	811	821	823	827
829	839	853	857	859	863	877	881	883	887	907	911
919	929	937	941	947	953	967	971	977	983	991	997

II Running Programs on a System

2. Simplifying Univariate Polynomials

The definition of a univariate polynomial expression is based on the expanded canonical form of some polynomial expression. It might be that the original expression might not be in the expanded canonical form, so a process called simplifying by expanding will be introduced to manipulate the expression such that it can be written in its expanded canonical form.

This process of simplifying by expanding polynomial expressions will be developed to the extent that it will be used to simplify multivariate polynomials. We will start by simplifying univariate monomial expressions.

2.1 Simplifying Degree 0 Univariate Monomials

Definition 2.1.1 – Indeterminate.

$$x$$

An indeterminate is a symbol that is treated as a variable, but does not stand for anything else but itself and is used as a placeholder.

- it does **not** designate a constant or a parameter
- it is **not** an unknown that could be solved for
- it is **not** a variable designating a function argument

[**wikipedia:indeterminate**]

Definition 2.1.2 – Degree of the Indeterminate.

$$x^k$$

The exponent of an indeterminate power, k is called the degree of the indeterminate.

wikipedia:polynomial

Definition 2.1.3 – Coefficient.

$$Cx^k$$

A coefficient, C is a real number multiplicative factor.

Definition 2.1.4 – Univariate Monomial.

$$C_k x^k$$

A univariate monomial is made up of two factors. The first factor of a monomial, C_k, is the coefficient. The second factor of each monomial, x^k, is an indeterminate raised to a non-negative integer power k.

Degree 0 univariate polynomial expressions are made up of univariate monomials, C_0, called constants. The power identity is an indeterminate raised to a power of 0 has a value of 1. Thus, $x^0 = 1$ and results in the monomial $C_0 \cdot 1$. The canonical form of a product does not show the multiplicative identity factor, so what remains of this monomial product is only the coefficient factor C_0 and from now on will be referred to as a constant.

Degree 0 univariate polynomial expressions are usually a monomial in their canonical form if C_0 is a non-zero real number. The exception is if $C_0 = 0$, the additive identity, then the result is the zero polynomial, which can be considered a degree -1 polynomial.

The expression can be manipulated into its monomial canonical form by simplifying the expression. Simplifying the expression can be defined as evaluating the expression by following order of operations, which is the same as evaluating an arithmetic expression.

Definition 2.1.5 – Univariate Like Terms.

$$C_1 x^k = C_2 x^k$$

Two or more univariate monomials are defined as having like terms if each monomial has the same term, which will be the same indeterminate raised to the same positive integer power.

Sometimes the word term is used to describe monomials (including both the coefficient and the term), which may be confusing when trying to define like terms. For this reason, we will refer to the summands of a polynomial as monomials.

The monomials $5x^1$ and $3x^1$ can be described as having like terms because they share the common term x^1. One could also say that $5x^1$ and $3x^1$ are like terms by definition and consequenlty giving the reader an impression that $5x^1$ and $3x^1$ are terms themselves.

R A degree 1 indeterminate does not display the multiplicative identity in the exponent when its in canonical form.

Example 2.1 – id:20141121-093747.

Express $13x^0$ in canonical form.

S

Solution:

13	PoID(**??**)

■

2.2 Simplifying Degree 1 Univariate Monomials

Example 2.2 – id:20141120-202042.

Express $5x^1$ in canonical form

S

Solution:

$5x$	MId(**??**)

■

Example 2.3 – id:20141121-093439.

Express $7x^1 + 5$ in canonical form.

S

Solution:

$7x + 5$	MId(**??**)

■

If the constant monomial is 0, the additive identity, then the canonical form of a degree 1 univariate polynomial is a degree 1 monomial.

Example 2.4 – id:20141120-203846.

Simplify by expanding $6x + 7x$

S

Solution:
Notice that the indeterminate of each monomial is of degree 1; however, the exponent 1 is not shown. The monomials $6x$ and $7x$ have a like term of x.

$(6 + 7)x$	DPF(**??**)
$13x$	OOA(**??**)

Notice that the sum of two monomials that have like terms can be found by adding the coefficients of the monomials. The distributive property in the factoring direction provides some insight to why we can add the coefficients of monomials that have like terms.

S

Less Steps Solution:

$13x$ OOA(**??**)

Example 2.5 – id:20141027-075159.

Simplify by expanding $7\,\mathrm{cm} + 8\,\mathrm{cm}$

S

Solution:

$$
\begin{aligned}
(7+8)\,\mathrm{cm} &\qquad \text{DPF(\textbf{??})}\\
15\,\mathrm{cm} &\qquad \text{OOA(\textbf{??})}
\end{aligned}
$$

R Remember, if a monomial does not have a coefficient factor, then it's implied that the coefficient factor is 1, the multiplicative identity, and consequently its not explicitly shown.

Example 2.6 – id:20141121-185558.

Simplify by expanding $x + 5x$

S

Solution:
It can be useful when simplifying expressions to make the multiplicative identity (MId) factor explicit.

$$
\begin{aligned}
1x + 5x &\qquad \text{MId(\textbf{??})}\\
(1+5)x &\qquad \text{DPF(\textbf{??})}\\
6x &\qquad \text{OOA(\textbf{??})}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
1x + 5x &\qquad \text{MId(\textbf{??})}\\
6x &\qquad \text{OOA(\textbf{??})}
\end{aligned}
$$

As one becomes more experienced, there is no reason to make the multiplicative identity coefficient explicit.

S

Less Steps Solution:

$6x$	OOA(**??**)

■

Example 2.7 – id:20141121-190857.

Simplify by expanding $8x - 6x$

S

Solution:

$8x + \neg 6x$	DOS(**??**)
$(8 + \neg 6)x$	DPF(**??**)
$2x$	OOA(**??**)

S

Less Steps Solution:

$8x + \neg 6x$	DOS(**??**)
$2x$	OOA(**??**)

■

Example 2.8 – id:20141121-193636.

Simplify by expanding $3x - 5x$

S

Solution:

$3x + \neg 5x$	DOS(**??**)
$(3 + \neg 5)x$	DPF(**??**)
$\neg 2x$	OOA(**??**)
$-2x$	ONeg(**??**)

S

Less Steps Solution:

$$\begin{aligned} 3x + \neg 5x && \text{DOS}(\textbf{??}) \\ \neg 2x && \text{OOA}(\textbf{??}) \\ -2x && \text{ONeg}(\textbf{??}) \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} -2x && \text{OOA}(\textbf{??}) \end{aligned}$$

■

Example 2.9 – id:20141106-150622.

Simplify by expanding $13x - x$

S

Solution:

$$\begin{aligned} 13x - 1x && \text{MId}(\textbf{??}) \\ 13x + \neg 1x && \text{DOS}(\textbf{??}) \\ (13 + \neg 1)x && \text{DPF}(\textbf{??}) \\ 12x && \text{OOA}(\textbf{??}) \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} 13x + \neg x && \text{DOS}(\textbf{??}) \\ 12x && \text{OOA}(\textbf{??}) \end{aligned}$$

■

It is possible for a univariate monomial to have more than two terms in its non-canonical form. The associative property of addition will be used to help simplify these expressions.

Example 2.10 – id:20141121-184652.

Simplify by expanding $3x + 7x + 8x$

S

$(3x + 7x) + 8x$	APA(??)
$(3 + 7)x + 8x$	DPF(??)
$10x + 8x$	OOA(??)
$(10 + 8)x$	DPF(??)
$18x$	OOA(??)

S

Less Steps Solution:

$(3x + 7x) + 8x$	APA(??)
$10x + 8x$	OOA(??)
$18x$	OOA(??)

You might have noticed that this expression could be simplified in one step by adding the coefficient of the three monomials $3x$, $7x$ and $8x$, which have the like term x.

S

Less Steps Solution:

$18x$	OOA(??)

Example 2.11 – id:20141106-152020.

Simplify by expanding $4x - 2x - x$

S

Solution:

$4x - 2x - 1x$	MId(??)
$4x + \neg 2x + \neg 1x$	DOS(??)
$(4 + \neg 2)x + \neg 1x$	DPF(??)
$2x + \neg 1x$	OOA(??)
$(2 + \neg 1)x$	DPF(??)
$1x$	OOA(??)
x	MId(??)

S

Less Steps Solution:

$$\begin{aligned} 4x + \neg 2x + \neg x && \text{DOS(??)} \\ x && \text{OOA(??)} \end{aligned}$$

Example 2.12 – id:20141108-194431.

Simplify by expanding $-3 \cdot 7x - 2x \cdot 4$

S

Solution:

$$\begin{aligned} \neg 3 \cdot 7x - 2x \cdot 4 && \text{ONeg(??)} \\ \neg 3 \cdot 7x + \neg 2x \cdot 4 && \text{DOS(??)} \\ \neg 3 \cdot 7 \cdot x + \neg 2 \cdot x \cdot 4 && \text{JTC(??)} \\ \neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x && \text{CPM(??)} \\ (\neg 3 \cdot 7) \cdot x + (\neg 2 \cdot 4) \cdot x && \text{APM(??)} \\ \neg 21 \cdot x + \neg 8 \cdot x && \text{OOM(??)} \\ \neg 21x + \neg 8x && \text{CTJ(??)} \\ (\neg 21 + \neg 8)x && \text{DPF(??)} \\ \neg 29x && \text{OOA(??)} \\ -29x && \text{ONeg(??)} \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} -3 \cdot 7x + \neg 2x \cdot 4 && \text{DOS(??)} \\ \neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x && \text{CPM(??)} \\ \neg 21x + \neg 8x && \text{OOM(??)} \\ -29x && \text{OOA(??)} \end{aligned}$$

Example 2.13 – id:20141108-194156.

Simplify by expanding $3 \cdot 5x + 3x \cdot 4$

S

Solution:

$$
\begin{aligned}
3 \cdot 5 \cdot x + 3 \cdot x \cdot 4 &\qquad \text{JTC(??)} \\
3 \cdot 5 \cdot x + 3 \cdot 4 \cdot x &\qquad \text{CPM(??)} \\
(3 \cdot 5) \cdot x + (3 \cdot 4) \cdot x &\qquad \text{APM(??)} \\
15 \cdot x + 12 \cdot x &\qquad \text{OOM(??)} \\
15x + 12x &\qquad \text{CTJ(??)} \\
(15 + 12)x &\qquad \text{DPF(??)} \\
27x &\qquad \text{OOA(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
3 \cdot 5 \cdot x + 3 \cdot 4 \cdot x &\qquad \text{CPM(??)} \\
15x + 12x &\qquad \text{OOM(??)} \\
27x &\qquad \text{OOA(??)}
\end{aligned}
$$

Example 2.14 – id:20141108-173613.

Simplify by expanding $8x \cdot 5$

S

Solution:

$$
\begin{aligned}
8 \cdot x \cdot 5 &\qquad \text{JTC(??)} \\
8 \cdot 5 \cdot x &\qquad \text{CPM(??)} \\
(8 \cdot 5) \cdot x &\qquad \text{APM(??)} \\
40 \cdot x &\qquad \text{OOM(??)} \\
40x &\qquad \text{CTJ(??)}
\end{aligned}
$$

2.3 Simplifying Degree 2 Univariate Monomials

Example 2.15 – id:20151018-184931.
Simplify $x \cdot x$

S

Solution:

$1x^1 \cdot 1x^1$	MId(**??**)
$1x^{(1+1)}$	PrCBPo(**??**)
$1x^2$	OOA(**??**)
x^2	MId(**??**)

■

Example 2.16 – id:20141120-202842.

Express by expanding $1x^2$ in canonical form.

S

Solution:

x^2	MId(**??**)

■

Example 2.17 – id:20141106-151138.

Simplify by expanding $4x^2 + 12x^2$

S

Solution:

$(4+12)x^2$	DPF(**??**)
$16x^2$	OOA(**??**)

S

Less Steps Solution:

$16x^2$	OOA(**??**)

■

Example 2.18 – id:20141106-154547.

Simplify by expanding $x^2 - x + x^2 + x$

S

Solution:

$$
\begin{aligned}
1x^2 - 1x + 1x^2 + 1x & \quad \text{MId(??)} \\
1x^2 + \neg 1x + 1x^2 + 1x & \quad \text{DOS(??)} \\
1x^2 + 1x^2 + \neg 1x + 1x & \quad \text{CPA(??)} \\
(1+1)x^2 + (\neg 1 + 1)x & \quad \text{DPF(??)} \\
2x^2 + 0x & \quad \text{OOA(??)} \\
2x^2 & \quad \text{MId(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
x^2 + \neg x + x^2 + x & \quad \text{DOS(??)} \\
x^2 + x^2 + \neg x + x & \quad \text{CPA(??)} \\
2x^2 & \quad \text{OOA(??)}
\end{aligned}
$$

■

Example 2.19 – id:20141108-194709.

Simplify by expanding $-2x4x - x \cdot -x3$

S

Solution:

$$
\begin{aligned}
-2x4x - 1x \cdot 1x3 & \quad \text{MId(??)} \\
\neg 2x4x - 1x \cdot 1x3 & \quad \text{ONeg(??)} \\
\neg 2 \cdot x \cdot 4 \cdot x + \neg 1 \cdot x \cdot 1 \cdot x \cdot 3 & \quad \text{JTC(??)} \\
\neg 2 \cdot 4 \cdot x \cdot x + \neg 1 \cdot \neg 1 \cdot 3 \cdot x \cdot x & \quad \text{CPM(??)} \\
\neg 2 \cdot 4 \cdot x^2 + \neg 1 \cdot \neg 1 \cdot 3 \cdot x^2 & \quad \text{PrCBPo(??)} \\
(\neg 2 \cdot 4) \cdot x^2 + (\neg 1 \cdot \neg 1 \cdot 3)x^2 & \quad \text{APM(??)} \\
\neg 8x^2 + 3x^2 & \quad \text{OOM(??)} \\
(\neg 8 + 3)x^2 & \quad \text{DPF(??)} \\
\neg 5x^2 & \quad \text{OOA(??)} \\
-5x^2 & \quad \text{ONeg(??)}
\end{aligned}
$$

S

Less Steps Solution:

$-2x4x + \neg x \cdot -x3$	DOS(??)
$\neg 2 \cdot 4 \cdot x \cdot x + 3 \cdot \neg x \cdot \neg x$	CPM(??)
$\neg 2 \cdot 4 \cdot x^2 + 3 \cdot x^2$	PrCBPo(??)
$\neg 8x^2 + 3x^2$	OOM(??)
$-5x^2$	OOA(??)

Example 2.20 – id:20141108-191616.

Simplify by expanding $-5x \cdot 4x$

S

Solution:

$\neg 5x \cdot 4x$	ONeg(??)
$\neg 5 \cdot x \cdot 4 \cdot x$	JTC(??)
$\neg 5 \cdot 4 \cdot x \cdot x$	CPM(??)
$\neg 5 \cdot 4 \cdot x^2$	PrCBPo(??)
$(\neg 5 \cdot 4) \cdot x^2$	APM(??)
$\neg 20 \cdot x^2$	OOM(??)
$\neg 20x^2$	CTJ(??)
$-20x^2$	ONeg(??)

S

Less Steps Solution:

$\neg 5 \cdot 4 \cdot x \cdot x$	CPM(??)
$\neg 5 \cdot 4 \cdot x^2$	PrCBPo(??)
$-20x^2$	OOM(??)

2.4 Simplifying Degree 3 Univariate Monomials

2.5 Simplifying Degree 1 Univariate Binomials

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

Degree 1 univariate polynomial expressions can be expressed with at most two different terms and consequently this expression in its canonical form has at most two monomial summands – called a binomial.

Example 2.21 – id:20141109-090809.

Simplify by expanding $5(x + 4)$

S

Solution:

$5(1x + 4)$	MId(**??**)
$5 \cdot 1x + 5 \cdot 5$	DPF(**??**)
$5 \cdot 1 \cdot x + 5 \cdot 5$	JTC(**??**)
$5 \cdot x + 25$	OOM(**??**)
$5x + 25$	CTJ(**??**)

S

Less Steps Solution:

$5x + 20$	DPE(**??**)

Example 2.22 – id:20141109-091015.

Simplify by expanding $5(3x - 9)$

S

Solution:

$5(3x + \neg 9)$	DOS(**??**)
$5 \cdot 3x + 5 \cdot \neg 9$	DPE(**??**)
$5 \cdot 3 \cdot x + 5 \cdot \neg 9$	JTC(**??**)
$15 \cdot x + \neg 45$	OOM(**??**)
$15x + \neg 45$	CTJ(**??**)
$15x - 45$	DOS(**??**)

S

Less Steps Solution:

$5(3x + \neg 9)$	DOS(**??**)
$15x + \neg 40$	DPE(**??**)
$15x - 40$	DOS(**??**)

Example 2.23 – id:20141109-092448.

Simplify by expanding $-(5x + 7)$

Solution:

$$\begin{aligned}
\neg 1(5x + 7) &\quad \text{MId(??)}\\
\neg 1 \cdot 5x + \neg 1 \cdot 7 &\quad \text{DPE(??)}\\
\neg 1 \cdot 5 \cdot x + \neg 1 \cdot 7 &\quad \text{JTC(??)}\\
\neg 5 \cdot x + \neg 7 &\quad \text{OOM(??)}\\
\neg 5x + \neg 7 &\quad \text{CTJ(??)}\\
\neg 5x - 7 &\quad \text{DOS(??)}\\
-5x - 7 &\quad \text{ONeg(??)}
\end{aligned}$$

Less Steps Solution:

$-5x - 7$ DPE(**??**)

Example 2.24 – id:20141109-092651.

Simplify by expanding $-13(7x - 9)$

Solution:

$$\begin{aligned}
\neg 13(7x + \neg 9) &\quad \text{DOS(??)}\\
\neg 13 \cdot 7x + \neg 13 \cdot \neg 9 &\quad \text{DPE(??)}\\
\neg 13 \cdot 7 \cdot x + \neg 13 \cdot \neg 9 &\quad \text{JTC(??)}\\
\neg 91 \cdot x + 117 &\quad \text{OOM(??)}\\
\neg 91x + 117 &\quad \text{CTJ(??)}\\
-91x + 117 &\quad \text{ONeg(??)}
\end{aligned}$$

Less Steps Solution:

$-13(7x + \neg 9)$	DOS(??)
$-91x + 117$	DPE(??)

■

Example 2.25 – id:20141109-092910.

Simplify by expanding $a(x + b)$, where $a, b \in \mathbb{Z}$

S

Solution:

$a(1x + b)$	MId(??)
$a \cdot 1x + a \cdot b$	DPE(??)
$a \cdot 1 \cdot x + a \cdot b$	JTC(??)
$1 \cdot a \cdot x + a \cdot b$	CPM(??)
$1ax + ab$	JTC(??)
$ax + ab$	MId(??)

S

Less Steps Solution:

$ax + ab$	DPE(??)

■

Example 2.26 – id:20141109-093220.

Simplify by expanding $5(x + 2) + 4$

S

Solution:

$5(1x + 2) + 4$	MId(??)
$5 \cdot 1x + 5 \cdot 2 + 4$	DPE(??)
$5 \cdot 1 \cdot x + 5 \cdot 2 + 4$	JTC(??)
$5 \cdot x + 10 + 4$	OOM(??)
$5x + 10 + 4$	CTJ(??)
$5x + 14$	OOA(??)

S

Less Steps Solution:

$$\begin{aligned} 5x + 10 + 4 & \quad \text{DPE(??)} \\ 5x + 14 & \quad \text{OOA(??)} \end{aligned}$$

Example 2.27 – id:20141109-093419.

Simplify by expanding $7x + 5(4x + 8)$

S

Solution:

$$\begin{aligned} 7x + 5 \cdot 4x + 5 \cdot 8 & \quad \text{DPE(??)} \\ 7 \cdot x + 5 \cdot 4 \cdot x + 5 \cdot 8 & \quad \text{JTC(??)} \\ 7 \cdot x + 20 \cdot x + 40 & \quad \text{OOM(??)} \\ 7x + 20x + 40 & \quad \text{CTJ(??)} \\ (7 + 20)x + 40 & \quad \text{DPF(??)} \\ 27x + 40 & \quad \text{OOA(??)} \end{aligned}$$

Example 2.28 – id:20141109-094928.

Simplify by expanding $4(3x + 4) + x + 6$

S

Solution:

$$\begin{aligned} 4(3x + 4) + 1x + 6 & \quad \text{MId(??)} \\ 4 \cdot 3x + 4 \cdot 4 + 1x + 6 & \quad \text{DPE(??)} \\ 4 \cdot 3 \cdot x + 4 \cdot 4 + 1 \cdot x + 6 & \quad \text{JTC(??)} \\ 12 \cdot x + 16 + 1 \cdot x + 6 & \quad \text{OOM(??)} \\ 12x + 16 + 1x + 6 & \quad \text{CTJ(??)} \\ 12 + 1x + 16 + 6 & \quad \text{CPA(??)} \\ (12 + 1)x + 16 + 6 & \quad \text{DPF(??)} \\ 13x + 22 & \quad \text{OOA(??)} \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} 12x + 16 + x + 6 & \quad \text{DPE(??)} \\ 12x + x + 16 + 6 & \quad \text{CPA(??)} \\ 13x + 22 & \quad \text{OOA(??)} \end{aligned}$$

■

Example 2.29 – id:20141109-095151.

Simplify by expanding $5(x - 4) + 3x - 5$

S

Solution:

$$\begin{aligned} 5(1x - 4) + 3x - 5 & \quad \text{MId(??)} \\ 5(1x + \neg 4) + 3x + \neg 5 & \quad \text{DOS(??)} \\ 5 \cdot 1x + 5 \cdot \neg 4 + 3x + \neg 5 & \quad \text{DPE(??)} \\ 5 \cdot 1 \cdot x + 5 \cdot \neg 4 + 3 \cdot x + \neg 5 & \quad \text{JTC(??)} \\ 5 \cdot x + \neg 20 + 3 \cdot x + \neg 5 & \quad \text{OOM(??)} \\ 5x + \neg 20 + 3x + \neg 5 & \quad \text{JTC(??)} \\ 5x + 3x + \neg 20 + \neg 5 & \quad \text{CPA(??)} \\ (5 + 3)x + \neg 20 + \neg 5 & \quad \text{DPF(??)} \\ 8x + \neg 25 & \quad \text{OOA(??)} \\ 8x - 25 & \quad \text{DOS(??)} \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} 5(x + \neg 4) + 3x + \neg 5 & \quad \text{DOS(??)} \\ 5x + \neg 20 + 3x + \neg 5 & \quad \text{DPE(??)} \\ 5x + 3x + \neg 20 + \neg 5 & \quad \text{CPA(??)} \\ 8x + \neg 25 & \quad \text{OOA(??)} \\ 8x - 25 & \quad \text{DOS(??)} \end{aligned}$$

■

Example 2.30 – id:20141109-095536.

Simplify by expanding $8x - 5 - 4(x - 3)$

S

Solution:

$$\begin{aligned}
8x - 5 - 4(1x - 3) &\quad \text{MId(??)}\\
8x + \neg 5 + \neg 4(1x + \neg 3) &\quad \text{DOS(??)}\\
8x + \neg 5 + \neg 4 \cdot 1x + \neg 4 \cdot \neg 3 &\quad \text{DPE(??)}\\
8 \cdot x + \neg 5 + \neg 4 \cdot 1 \cdot x + \neg 4 \cdot \neg 3 &\quad \text{JTC(??)}\\
8 \cdot x + \neg 5 + \neg 4 \cdot x + 12 &\quad \text{OOM(??)}\\
8x + \neg 5 + \neg 4x + 12 &\quad \text{CTJ(??)}\\
8x + \neg 4x + \neg 5 + 12 &\quad \text{CPA(??)}\\
(8 + \neg 4)x + \neg 5 + 12 &\quad \text{DPF(??)}\\
4x + 7 &\quad \text{OOA(??)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
8x + \neg 5 + \neg 4(x + \neg 3) &\quad \text{DOS(??)}\\
8x + \neg 5 + \neg 4x + 12 &\quad \text{DPE(??)}\\
8x + \neg 4x + \neg 5 + 12 &\quad \text{CPA(??)}\\
4x + 7 &\quad \text{OOA(??)}
\end{aligned}$$

■

Example 2.31 – id:20141109-095842.

Simplify by expanding $5(x + 3) + 3(x + 2)$

S

Solution:

$$\begin{aligned}
5 \cdot x + 5 \cdot 3 + 3 \cdot x + 3 \cdot 2 &\quad \text{DPE(??)}\\
5 \cdot x + 15 + 3 \cdot x + 6 &\quad \text{OOM(??)}\\
5x + 15 + 3x + 6 &\quad \text{CTJ(??)}\\
5x + 3x + 15 + 6 &\quad \text{CPA(??)}\\
(3 + 5)x + 15 + 6 &\quad \text{DPF(??)}\\
8x + 21 &\quad \text{OOA(??)}
\end{aligned}$$

S

Less Steps Solution:

$$5x + 15 + 3x + 6 \qquad \text{DPE(??)}$$
$$5x + 3x + 15 + 6 \qquad \text{CPA(??)}$$
$$8x + 21 \qquad \text{OOA(??)}$$

■

2.6 Simplifying Degree 2 Univariate Binomials

Example 2.32 – id:20141106-152339.

Simplify by expanding $3x^2 + 2x + 5x^2 + 4x$

S

Solution:

$$3x^2 + 5x^2 + 2x + 4x \qquad \text{CPA(??)}$$
$$(3+5)x^2 + (2+4)x \qquad \text{DPF(??)}$$
$$8x^2 + 6x \qquad \text{OOA(??)}$$

If needed we could continue and express it in the simplified factored form using the distributive property

$$(4x + 3)2x \qquad \text{DPF(??)}$$

S

Less Steps Solution:

$$3x^2 + 5x^2 + 2x + 4x \qquad \text{CPA(??)}$$
$$8x^2 + 6x \qquad \text{OOA(??)}$$

■

Example 2.33 – id:20141107-121834.

Simplify by expanding $\left(\sqrt{9 - x^2}\right)^2$

S

Solution:

$$\left(\sqrt{9-1x^2}\right)^2 \qquad \text{MId(??)}$$
$$\left(\sqrt{9+\neg 1x^2}\right)^2 \qquad \text{DOS(??)}$$
$$\left[\left(9+\neg x^2\right)^{\frac{1}{2}}\right]^2 \qquad \text{RTPo(??)}$$
$$9+\neg 1x^2 \qquad \text{PoPo(??)}$$
$$\neg 1x^2+9 \qquad \text{CPA(??)}$$
$$\neg x^2+9 \qquad \text{MId(??)}$$
$$-x^2+9 \qquad \text{ONeg(??)}$$

S

Less Steps Solution:

$$9-x^2 \qquad \text{PoPo(??)}$$

It might be easier to view this using a variable substitution for the radicand, $9-x^2$*. Let* $k=9+\neg 1x^2$.

$$\left(\sqrt{k}\right)^2 \qquad \text{MId(??)}$$
$$\left(\sqrt{k}\right)^2 \qquad \text{DOS(??)}$$
$$\left[(k)^{\frac{1}{2}}\right]^2 \qquad \text{RTPo(??)}$$
$$k \qquad \text{PoPo(??)}$$
$$9+\neg 1x^2 \qquad \text{CPA(??)}$$
$$\neg 1x^2+9 \qquad \text{CPA(??)}$$
$$\neg x^2+9 \qquad \text{MId(??)}$$
$$-x^2+9 \qquad \text{ONeg(??)}$$

D

Dependencies:example **??**-20141105-144223

Example 2.34 – id:20141209-145211.

Simplify by expanding $2x(2x+4)+x^2\cdot 2\cdot 1+0$

S

Solution:

$$
\begin{aligned}
2x \cdot 2x + 2x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0 & \quad \text{DPE(??)} \\
2 \cdot x \cdot 2 \cdot x + 2 \cdot x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0 & \quad \text{JTC(??)} \\
2 \cdot 2 \cdot x \cdot x + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0 & \quad \text{CPM(??)} \\
2 \cdot 2 \cdot x^2 + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0 & \quad \text{PrCBPo(??)} \\
4 \cdot x^2 + 8 \cdot x + 2 \cdot x^2 + 0 & \quad \text{OOM(??)} \\
4x^2 + 8x + 2x^2 + 0 & \quad \text{CTJ(??)} \\
4x^2 + 2x^2 + 8x & \quad \text{APA(??)} \\
(4+2)x^2 + 8x & \quad \text{DPF(??)} \\
6x^2 + 8x & \quad \text{OOA(??)}
\end{aligned}
$$

D

Dependencies:example **??**-20141209-144203

2.7 Simplifying Degree 2 Univariate Trinomials

Example 2.35 – id:20141109-133008.

Simplify by expanding $(x+5)(x-8)$

S

Solution:

$$
\begin{aligned}
(1x+5)(1x-8) & \quad \text{MId(??)} \\
(1x+5)(1x+\neg 8) & \quad \text{DOS(??)} \\
1x(1x+\neg 8) + 5(1x+\neg 8) & \quad \text{DPE(??)} \\
1x \cdot 1x + 1x \cdot \neg 8 + 5 \cdot 1x + 5 \cdot \neg 8 & \quad \text{DPE(??)} \\
1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot \neg 8 + 5 \cdot 1 \cdot x + 5 \cdot \neg 8 & \quad \text{JTC(??)} \\
1 \cdot 1 \cdot x \cdot x + \neg 8 \cdot 1 \cdot x + 1 \cdot 5 \cdot x + \neg 8 \cdot 5 & \quad \text{CPM(??)} \\
1 \cdot 1 \cdot x^2 + \neg 8 \cdot 1 \cdot x + 1 \cdot 5 \cdot x + \neg 8 \cdot 5 & \quad \text{PrCBPo(??)} \\
1 \cdot x^2 + \neg 8 \cdot x + 5 \cdot x + \neg 40 & \quad \text{OOM(??)} \\
1x^2 + \neg 8x + 5x + \neg 40 & \quad \text{CTJ(??)} \\
1x^2 + \neg 3x + \neg 40 & \quad \text{OOA(??)} \\
1x^2 - 3x - 40 & \quad \text{DOS(??)} \\
x^2 - 3x - 40 & \quad \text{MId(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$\begin{aligned}
(x+5)(x+\neg 8) &\qquad \text{DOS(??)}\\
x(x+\neg 8)+5(x+\neg 8) &\qquad \text{DPE(??)}\\
x^2+\neg 8x+5x+\neg 40 &\qquad \text{DPE(??)}\\
x^2-3x-40 &\qquad \text{OOA(??)}
\end{aligned}$$

■

Example 2.36 – id:20141109-133316.

Simplify by expanding $(x+a)(x+b)$, where $a, b \in \mathbb{Z}$

S

Solution:

$$\begin{aligned}
(1x+a)(1x+b) &\qquad \text{MId(??)}\\
1x(1x+b)+a(1x+b) &\qquad \text{DPE(??)}\\
1x\cdot 1x+1x\cdot b+a\cdot 1x+a\cdot b &\qquad \text{DPE(??)}\\
1\cdot x\cdot 1\cdot x+1\cdot x\cdot b+a\cdot 1\cdot x+a\cdot b &\qquad \text{JTC(??)}\\
1\cdot 1\cdot x\cdot x+1\cdot b\cdot x+1\cdot a\cdot x+a\cdot b &\qquad \text{CPM(??)}\\
1\cdot 1\cdot x^2+1\cdot b\cdot x+1\cdot a\cdot x+a\cdot b &\qquad \text{PrCBPo(??)}\\
1\cdot x^2+1\cdot b\cdot x+1\cdot a\cdot x+a\cdot b &\qquad \text{OOM(??)}\\
1x^2+1bx+1ax+ab &\qquad \text{CTJ(??)}\\
1x^2+(1b+1a)x+ab &\qquad \text{DPF(??)}\\
x^2+(b+a)x+ab &\qquad \text{MId(??)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
x(x+b)+a(x+b) &\qquad \text{DPE(??)}\\
x^2+(b+a)x+ab &\qquad \text{DPE(??)}
\end{aligned}$$

■

Example 2.37 – id:20141109-140659.

Simplify by expanding $(2x+3)(5x+13)$

S

Solution:

$$
\begin{aligned}
2x(5x+13)+3(5x+13) && \text{DPE(??)}\\
2x\cdot 5x+2x\cdot 13+3\cdot 5x+3\cdot 13 && \text{DPE(??)}\\
2\cdot x\cdot 5\cdot x+2\cdot x\cdot 13+3\cdot 5\cdot x+3\cdot 13 && \text{JTC(??)}\\
2\cdot 5\cdot x\cdot x+2\cdot 13\cdot x+5\cdot 3\cdot x+3\cdot 13 && \text{CPM(??)}\\
2\cdot 5\cdot x^2+2\cdot 13\cdot x+5\cdot 3\cdot x+3\cdot 13 && \text{PrCBPo(??)}\\
10\cdot x^2+26\cdot x+16\cdot x+39 && \text{OOM(??)}\\
10x^2+26x+15x+39 && \text{CTJ(??)}\\
10x^2+41x+39 && \text{OOA(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
2x(5x+13)+3(5x+13) && \text{DPE(??)}\\
10x^2+26x+15x+39 && \text{DPE(??)}\\
10x^2+41x+39 && \text{OOA(??)}
\end{aligned}
$$

■

Example 2.38 – id:20141109-141019.

Simplify by expanding $(-3x-5)(7x+8)$

S

Solution:

$$
\begin{aligned}
(\neg 3x-5)(7x+8) && \text{ONeg(??)}\\
(\neg 3x+\neg 5)(7x+8) && \text{DOS(??)}\\
\neg 3x(7x+8)+\neg 5(7x+8) && \text{DPE(??)}\\
\neg 3x\cdot 7x+\neg 3x\cdot 8+\neg 5\cdot 7x+\neg 5\cdot 8 && \text{DPE(??)}\\
\neg 3\cdot x\cdot 7\cdot x+\neg 3\cdot x\cdot 8+\neg 5\cdot 7\cdot x+\neg 5\cdot 8 && \text{JTC(??)}\\
\neg 3\cdot 7\cdot x\cdot x+\neg 3\cdot 8\cdot x+\neg 5\cdot 7\cdot x+\neg 5\cdot 8 && \text{CPM(??)}\\
\neg 3\cdot 7\cdot x^2+\neg 3\cdot 8\cdot x+\neg 5\cdot 7\cdot x+\neg 5\cdot 8 && \text{PrCBPo(??)}\\
\neg 21\cdot x^2+\neg 24\cdot x+\neg 35\cdot x+\neg 40 && \text{OOM(??)}\\
\neg 21x^2+\neg 24x+\neg 35x+\neg 40 && \text{CTJ(??)}\\
\neg 21x^2+\neg 59x+\neg 40 && \text{OOA(??)}\\
\neg 21x^2-59x-40 && \text{DOS(??)}\\
-21x^2-59x-40 && \text{ONeg(??)}
\end{aligned}
$$

S

Less Steps Solution:

$(-3x + \neg 5)(7x + 8)$	DOS(**??**)
$\neg 3x(7x + 8) + \neg 5(7x + 8)$	DPE(**??**)
$\neg 21x^2 + \neg 24x + \neg 35x + \neg 40$	CTJ(**??**)
$-21x^2 - 59x - 40$	OOA(**??**)

■

Example 2.39 – id:20141109-141347.

Simplify by expanding $(ax + b)(cx + d)$, where $a, b, c, d \in \mathbb{Z}$

S

Solution:

$ax(cx + d) + b(cx + d)$	DPE(**??**)
$ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d$	DPE(**??**)
$a \cdot x \cdot c \cdot x + a \cdot x \cdot d + b \cdot c \cdot x + b \cdot d$	JTC(**??**)
$a \cdot c \cdot x \cdot x + a \cdot d \cdot x + b \cdot c \cdot x + b \cdot d$	CPM(**??**)
$a \cdot c \cdot x^2 + a \cdot d \cdot x + b \cdot c \cdot x + b \cdot d$	PrCBPo(**??**)
$acx^2 + adx + bcx + bd$	CTJ(**??**)
$acx^2 + (ad + bc)x + bd$	DPF(**??**)

S

Less Steps Solution:

$ax(cx + d) + b(cx + d)$	DPE(**??**)
$acx^2 + (ad + bc)x + bd$	DPE(**??**)

■

Example 2.40 – id:20141105-161225.

Simplify by expanding $\left(2 - \frac{x}{2}\right)^2$.

S

Solution:

$$
\begin{aligned}
&\left(2 - \frac{1x}{2}\right)^2 && \text{MId(??)}\\
&\left(2 + \neg\frac{1x}{2}\right)^2 && \text{DOS(??)}\\
&\left(2 + \neg\frac{1x}{2}\right)\left(2 + \neg\frac{1x}{2}\right) && \text{PoTF(??)}\\
&2\left(2 + \neg\frac{1x}{2}\right) + \neg\frac{1x}{2}\left(2 + \neg\frac{1x}{2}\right) && \text{DPE(??)}\\
&2 \cdot 2 + 2 \cdot \neg\frac{1x}{2} + \neg\frac{1x}{2} \cdot 2 + \neg\frac{1x}{2} \cdot \neg\frac{1x}{2} && \text{DPE(??)}\\
&2 \cdot 2 + 2 \cdot \neg\frac{1}{2} \cdot x + \neg\frac{1}{2} \cdot x \cdot 2 + \neg\frac{1}{2} \cdot x \cdot \neg\frac{1}{2} \cdot x && \text{JTC(??)}\\
&2 \cdot 2 + 2 \cdot \neg\frac{1}{2} \cdot x + \neg\frac{1}{2} \cdot 2 \cdot x + \neg\frac{1}{2} \cdot \neg\frac{1}{2} \cdot x \cdot x && \text{CPM(??)}\\
&2 \cdot 2 + 2 \cdot \neg\frac{1}{2} \cdot x + \neg\frac{1}{2} \cdot 2 \cdot x + \neg\frac{1}{2} \cdot \neg\frac{1}{2} \cdot x^2 && \text{PrCBPo(??)}\\
&4 + \neg 1 \cdot x + \neg 1 \cdot x + \frac{1}{4} \cdot x^2 && \text{OOM(??)}\\
&4 + \neg 1x + \neg 1x + \frac{1}{4}x^2 && \text{CTJ(??)}\\
&\frac{1}{4}x^2 + \neg 1x + \neg 1x + 4 && \text{CPA(??)}\\
&\frac{1}{4}x^2 + \neg 2x + 4 && \text{OOA(??)}\\
&\frac{1}{4}x^2 - 2x + 4 && \text{DOS(??)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
&7x + 20x + 40 && \text{DPE(??)}\\
&27x + 40 && \text{OOA(??)}
\end{aligned}
$$

2.8 Simplifying Degree n Univariate Polynomials

Definition 2.8.1 – Univariate Polynomial Expression.

$$\sum_{k=0}^{n} C_k x^n = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0} \tag{2.1}$$

A univariate polynomial in an indeterminate x is an expression made up of one or more summands of the form $C_k x^k$, which are called monomials. The first factor of each monomial, C_k, is a numerical factor called the coefficient where $C_k \in$. The second factor of each monomial, x^k, is an indeterminate raised to a non-negative integer power

i.

wikipedia:polynomial

Definition 2.8.2 – Degree of the Univaritate Polynomial.

$$C_n x^n + C_{n-1}x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

The degree of the univariate polynomial is determined by the monomial with the largest degree of the indeterminate.

III Interaction and Communication Between Programs

3. Functions

Property 3.0.1 – Function Value Argument Substitution (FVAS).

$$f(a) \tag{3.1a}$$

Example 3.1 – id:20151012-201647.
Find $f(2)$ given $f(x) = 2x^3 - 3x^2 - 12x + 1$

S

Solution:

$f(x) = 2x^3 + \neg 3x^2 + \neg 12x + 1$	DOS(**??**)
$f(2) = 2\,[2]^3 + \neg 3\,[2]^2 + \neg 12\,[2] + 1$	SPE(**??**)
$f(2) = 2(8) + \neg 3(4) + \neg 12(2) + 1$	OOE(**??**)
$f(2) = 16 + \neg 12 + \neg 24 + 1$	OOM(**??**)
$f(2) = -19$	OOA(**??**)

D

Dependencies:example **??**-20151012-190708

Example 3.2 – id:20151012-203549.
Find $f(-1)$ given $f(x) = 2x^3 - 3x^2 - 12x + 1$

S

Solution:

$$
\begin{aligned}
f(x) &= 2x^3 + \neg 3x^2 + \neg 12x + 1 && \text{DOS(??)} \\
f(\neg 1) &= 2\left[\neg \mathbf{1}\right]^3 + \neg 3\left[\neg \mathbf{1}\right]^2 + \neg 12\left[\neg \mathbf{1}\right] + 1 && \text{SPE(??)} \\
f(\neg 1) &= 2(\neg 1) + \neg 3(1) + \neg 12(\neg 1) + 1 && \text{OOE(??)} \\
f(\neg 1) &= \neg 2 + \neg 3 + 12 + 1 && \text{OOM(??)} \\
f(\neg 1) &= 8 && \text{OOA(??)}
\end{aligned}
$$

D

Dependencies:example **??**-20151012-190708

3.1 Inverse Functions

3.2 Inverses

Property 3.2.1 – Cosine Inverse (ArcCos).

$$\cos^{-1}(\cos\theta) = \theta \tag{3.2a}$$

Property 3.2.2 – Sine Inverse (ArcSin).

$$\sin^{-1}(\sin\theta) = \theta \tag{3.3a}$$

Property 3.2.3 – Tangent Inverse (ArcTan).

$$\tan^{-1}(\tan\theta) = \theta \tag{3.4a}$$

Property 3.2.4 – Exponential Inverse (EI).

$$\log_a(a^x) = x \tag{3.5a}$$

Property 3.2.5 – Logarithmic Inverse (LI).

$$a^{\log_a x} = x \tag{3.6a}$$

Property 3.2.6 – Power Inverse (PoI).

$$(b^m)^{\frac{1}{m}} = b \tag{3.7a}$$

Bibliography