CS 206	Data Structures	Spring 2017
	Project 1	
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Matrix Overview

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an *$m \times n$ matrix*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The i-th row of A is the $1 \times n$ matrix $[a_{i1}, a_{i2}, ..., a_{in}]$. The j-th column of A is the $m \times 1$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

The (i,j)-th element or entry of A is the element a_{ij}, that is, the number in the i-th row and j-th column of A. A convenient shorthand notation for expressing the matrix A is to write $A = [a_{ij}]$, which indicates that A is the matrix with its (i,j)-th element equal to a_{ij}.

Matrix Arithmetic

The basic operations of matrix arithmetic are as follows.

Addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The *sum* of A and B, denoted by A + B, is the $m \times n$ matrix that has

$a_{ij} + b_{ij}$ as its (i,j)-th element. In other words, $A + B = [a_{ij} + b_{ij}]$. The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, since the sum of the matrices is defined only when both matrices have the same number of rows and the same number of columns.

Example)

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Multiplication

Let A be an $m \times k$ matrix and B be a $k \times n$ matrix. The product of A and B, denoted by AB, is the $m \times n$ matrix with its (i,j)-th entry equal to the sum of the products of the corresponding elements from the i-th row of A and the j-th column of B. In other words, if $AB = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ .. + a_{ik}b_{kj}$$

Example)

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

Identity Matrix

A matrix with the same number of rows as columns is called a *square matrix*. The *identity matrix* of order n is the n × n square matrix $\mathbf{I}_n$ $[d_{ij}]$, where $d_{ij} = 1$ if i = j and $d_{ij} = 0$ if i ≠ j. Hence

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when A is an m × n matrix, we have

$$A\mathbf{I}_n = \mathbf{I}_m A = A$$

Transpose

Let $A = [a_{ij}]$ be an m × n matrix. The *transpose* of A, denoted by A^t, is the n x m matrix obtained by interchanging the rows and columns of A. In other words, if $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for i = 1, 2, ... , n and j = 1, 2, ... , m.

Example)

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \text{then} \quad A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Symmetric Matrix

A square matrix A is called *symmetric* if $A = A^t$. Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all $1 \le i \le n$ and $1 \le j \le n$.

Example)

$$\text{The matrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is symmetric.}$$

Inverse

If A and B are n × n matrices with $AB = BA = \mathbf{I}_n$, then B is called the *inverse* of A and written A^{-1}.

Part I.

Specify a Matrix ADT that consists of the following operations:

- Create a matrix of size m x n. This operation should read in each entry of the matrix from a file in which the first line contains the first row of the matrix, the second line contains the second row of the matrix and so on.
- Create an identity matrix of size n.
- Add two matrices.
- Multiply two matrices.
- Transpose a give matric.
- Check a matrix is symmetric.
- Check whether two matrices are the same or not

The specification of each operation should clearly state under what condition the operation returns a valid result and under what situation the operation will report that the operation cannot be performed normally.

Part II.

(1) Implement the specification that you described in Problem 1. State clearly your implementation decisions.

(2) For each operation you implement, what is its time complexity in terms of big-Oh?

Part III.

Problem 1.

Write a program that performs the following calculations using the Matrix ADT you developed in Part I:

(1)

$$\begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & -2 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$$

(2)

$$\begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$$

Problem 2.

Show that

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

Problem 3.

(1) Add a function *power* that computes A^n to the program in Problem 1, given a square matrix A and a positive number n. (Note that $A^n = A...A$ where A occurs n times in " A...A".) \

(2) What is A^{10} where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

?