CS 204	Discrete Mathematics	Fall 2016
	Homework 5(96 pts)	
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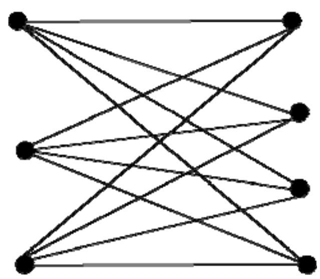
1. (6 pts) Consider the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \le 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2. \end{cases}$$

(a) Compute H(n) for n = 1, 2, ..., 10.
(b) Using the pattern from part (a), guess what H(100) is.

2. (8 pts) The complete bipartite graph $K_{m,n}$ is the simple undirected graph with m+n vertices split into two sets V1 and V2 (|V1| = m, |V2| = n) such that vertices x,y share an edge if and only if $x \in V1$ and $y \in V2$. For example $K_{3,4}$ is the following graph.



(a) Find a recurrence relation for the number of edges in $K_{3,n}$.
(b) Find a recurrence relation for the number of edges in $K_{n,n}$.

3. (10 pts) Consider the following recurrence relation:

$$C(n) = \begin{cases} 0 & \text{if } n = 0 \\ n+3\cdot C(n-1) & \text{if } n > 0. \end{cases}$$

Prove by induction that $C(n) = \dfrac{3^{n+1} - 2n - 3}{4}$ for all $n \ge 0$.

4. (15 pts) Consider the following recurrence relation:

$$G(n) = 1 \qquad \text{if } n = 0$$
$$= G(n-1) + 2n - 1 \qquad \text{if } n > 0.$$

(a) Calculate G(0), G(1), G(2), G(3), G(4), and G(5).

(b) Use sequence of differences to guess at a closed-form solution for G(n).

(c) Prove that your guess is correct.

5. (5 pts) The following recursive definition defines a set $\mathbb{Z}$ of ordered pairs.

B. (2, 4) is in $\mathbb{Z}$.

R1. If (x,y) is in $\mathbb{Z}$ with $x < 10$ and $y < 10$, then (x+1,y+1) is in $\mathbb{Z}$.

R2. If (x,y) is in $\mathbb{Z}$ with $x > 1$ and $y < 10$, then (x-1, y+1) is in $\mathbb{Z}$.

Plot these ordered pairs in the xy-plane.

6. (6 pts) Let S be a set of sets with the following recursive definition.

B. $\emptyset \in S$.

R. If $X \subseteq S$, then $X \in S$.

(a) List three different elements of S.

(b) Explain why S has infinitely many elements.

7. (20 pts) Recall the definition of a line map discussed in class.

(a) Prove by induction that a line map with n distinct lines has at least n+1 regions.

(b) Prove by induction that a line map with n distinct lines has at most 2^n regions.

(c) Part (a) gives a lower bound on the number of regions in a line map. For example, a line map with five lines must have at least six regions. Give an example of a line map that achieves this lower bound, that is, draw a line map with five lines and six regions.

(d) Part (b) says that a line map with three lines can have at most eight regions. Can you draw a line map with three lines that achieves this upper bound? Do so, or explain why you can't.

8. (10 pts) Consider the following recurrence relation:

$$H(n) = \begin{cases} 0 & \text{if } n \le 0 \\ 1 & \text{if } n = 1 \text{ or } n = 2 \\ H(n-1) + H(n-2) - H(n-3) & \text{if } n > 2. \end{cases}$$

Prove that $H(2n) = H(2n - 1) = n$ for all $n \ge 1$.

9. (8 pts) Consider the following recurrence relation:

$$L(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ L(n-1) + L(n-2) & \text{if } n > 2. \end{cases}$$

Let α and β be the constants that are used to compute the Fibonacci numbers as discussed in class. Prove that $L(n) = \alpha^n + \beta^n$ for all $n \in \mathbb{N}$. Use strong induction.

10. (8 pts) Define a Q-sequence recursively as follows.

B. $\langle x, 4-x \rangle$ is a Q-sequence (of length 2) for any real number x.
R. If $\langle x_1, x_2, ..., x_{j-1}, x_j \rangle$ and $\langle y_1, y_2, ..., y_{k-1}, y_k \rangle$ are Q-sequences, so is

$$\langle x_1 - 1, x_2, ..., x_{j-1}, x_j, y_1, y_2, ..., y_{k-1}, y_k - 3 \rangle$$

(, of which the length is j+k).

Use structural induction to prove that the sum of the numbers in any Q-sequence is 4.