最速降线 (Brachistochrone Problem)

详情参考 Jupyter Notebook

孙琪

Euler-Lagrange 方程：

假设满足边界条件 $y(a)=A$ 和 $y(b)=B$ 的函数使得泛函

$$J[y]=\int_a^b L(x, y(x), y'(x))\,dx$$

取到极值，则 $y(x)$ 满足

$$\frac{\partial L}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\frac{\partial L}{\partial y'}(x, y(x), y'(x)) = 0$$

Beltrami Identity

若 $\frac{\partial L}{\partial x}=0$，则成立有 $L - y'(x)\frac{\partial L}{\partial y'} = C$。

证明：首先考虑一般情况 $L(x, y(x), y'(x))$，由链式法则知：

$$\frac{dL}{dx} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}y'(x) + \frac{\partial L}{\partial y'}y''(x)$$

$$\xrightarrow{\text{移项}} \quad \frac{\partial L}{\partial y}y'(x) = \frac{dL}{dx} - \frac{\partial L}{\partial y'}y''(x) - \frac{\partial L}{\partial x}$$

在 Euler-Lagrange 方程中乘以 $y'(x)$ 得：

$$\frac{\partial L}{\partial y}y'(x) - y'(x)\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

由此可得：

$$\frac{dL}{dx} - \frac{\partial L}{\partial y'}y''(x) - \frac{\partial L}{\partial x} - y'(x)\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

$$-\frac{\partial L}{\partial x} + \frac{d}{dx}\left(L - y'(x)\frac{\partial L}{\partial y'}\right) = 0$$

特别的，当 $\frac{\partial L}{\partial x}=0$ 时有：

$$\frac{d}{dx}\left(L - y'(x)\frac{\partial L}{\partial y'}\right) = 0 \Rightarrow L - y'(x)\frac{\partial L}{\partial y'} = C. \qquad \square$$

最速降线

对于最速降线问题，$L(y, y'(x)) = \left[\frac{1+(y'(x))^2}{2g\,y(x)}\right]^{\frac{1}{2}}$，计算得

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial y'} = \frac{1}{2}\left[\frac{1+(y'(x))^2}{2g\,y(x)}\right]^{-\frac{1}{2}}\frac{2y'(x)}{2g\,y(x)} = y'(x)\left[1+(y'(x))^2\right]^{-\frac{1}{2}}(2g\,y(x))^{-\frac{1}{2}}$$

由 Beltrami 等式得：

$$\left[1+(y'(x))^2\right]^{\frac{1}{2}}(2g\,y(x))^{-\frac{1}{2}} - (y'(x))^2\left[1+(y'(x))^2\right]^{-\frac{1}{2}}(2g\,y(x))^{-\frac{1}{2}} = C$$

$$\left[1+(y'(x))^2\right]^{\frac{1}{2}}(2g\,y(x))^{-\frac{1}{2}}\left[1-\frac{(y'(x))^2}{1+(y'(x))^2}\right] = C$$

$$\left[1+(y'(x))^2\right]^{-\frac{1}{2}}(2g\,y(x))^{-\frac{1}{2}} = C$$

$$\left[1+(y'(x))^2\right]y(x) = \frac{1}{2gC^2} := k^2 \quad (\text{由于 } g, C^2 \text{ 均大于}0)$$

换言之，有

$$1+\left(\frac{dy}{dx}\right)^2 = \frac{k^2}{y} \qquad 或 \qquad \frac{dx}{dy} = \sqrt{\frac{y}{k^2-y}}$$

做变量替换 $\frac{dx}{dy} = \tan\varphi$，即有

$$\frac{y}{k^2-y} = \frac{\sin^2\varphi}{\cos^2\varphi}, \quad y = k^2\sin^2\varphi = \frac{k^2}{2}(1-\cos 2\varphi)$$

由 $\frac{dy}{d\varphi} = 2k^2\sin\varphi\cos\varphi$ 可以得出

$$\frac{dx}{d\varphi} = \sqrt{\frac{y}{k^2-y}}\frac{dy}{d\varphi} = \frac{k\sin\varphi}{k\cos\varphi}2k^2\sin\varphi\cos\varphi = 2k^2\sin^2\varphi = k^2(1-\cos 2\varphi)$$

$$x = \int k^2(1-\cos 2\varphi)\,d\varphi = \frac{k^2}{2}(2\varphi - \sin 2\varphi)$$

记 $\frac{k^2}{2} := R$，$2\varphi = \theta$，则可得 $x(\theta) = R(\theta - \sin\theta)$，$y(\theta) = R(1-\cos\theta)$。 $\square$

算例： 假设起始两点分别为 $(0, 0)$ 和 (x^*, y^*)，则有

$$\begin{cases} x^* = x(\theta^*) = R(\theta^* - \sin\theta^*) \\ y^* = y(\theta^*) = R(1 - \cos\theta^*) \end{cases}$$

可知：

① θ^* 是非线性方程 $\frac{y^*}{x^*} - \frac{1-\cos\theta^*}{\theta^* - \sin\theta^*} = 0$ 的解 (Newton法)

② $R = \frac{y^*}{1-\cos\theta^*}$.