Planar Bisection Notes

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1 Outline

The problem is: given planar graph G with face weights, and balance parameter $b \le 1/2$, find the minimum cost cycle enclosing at least bW weight on each side, where W is the sum of all face weights.

Suppose OPT is the smallest b-balanced cut. We guess a parameter $\lambda \approx OPT/W$. The main goal is to construct a **low-weight spanner**: want a spanner $H \subseteq G$ of total weight $\text{poly}(\epsilon^{-1}\log n) \cdot OPT$ such that there exists a $(b-\epsilon)$-balanced cut with weight at most $(1+\epsilon)OPT$ in the spanner.

2 Step 1: find collection of laminar cycles

The first step is to find a collection $\mathcal{C}$ of cycles with the following two properties:

1. The total cost of all cycles is $\text{poly}(\epsilon^{-1}\log n) \cdot OPT$
2. If a cycle C is inside a region R, then it contains the (unique) hole H of R with more than half the weight, and moreover, $c(C)/w(C \setminus H) > \epsilon^{-1}\lambda$.

The construction roughly goes as follows: greedily pick cycles C in each region R satisfying $c(C)/w(C \cap R) \le \epsilon^{-1}\lambda$. Also, periodically remove heavy nesting of cycles, so that cycle weight drops by half every other cycle in each chain, which means the height of the decomposition is $O(\log W)$.

(1) follows from the fact that there are $O(\log W)$ levels, and each level consists of cycles with disjoint interior which can be charged to their faces, giving total $\le \epsilon^{-1}\lambda \cdot W = \epsilon^{-1}OPT$ cost of cycles per level.

(2) follows from the fact that if a cycle with $c(C)/w(C \setminus H) \le \epsilon^{-1}\lambda$ existed, then we would have greedily added it to our collection. And if it didn't contain the hole H, then it should have been added before whatever holes in R it contains, and then NOT deleted since its weight is necessarily less than half of R (since it's disjoint from H).

3 Step 2: prize-collecting

Consider a region R (with holes). First, contract each hole into a single vertex v with **volume** $\phi(v)$ equal to ϵ^{-1} **times** the cost of the cycle/hole. (All non-hole vertices u get $\phi(u) = 0$.) We then run prize-collecting on the holes, computing a set of edges Z connecting some holes that satisfies:

1. The total cost of Z is at most twice the sum of the vertex volumes: $c(Z) \le 2\sum \phi(v)$
2. For any subgraph $H \subseteq G$, there is a set U of vertices such that
   (a) $\sum_{v \in U} \phi(v) \le c(H)$ (i.e., we can pay to exempt some vertices from condition (b))
   (b) If vertices $u, v \notin U$ are connected in H, then they're connected in Z

We add the computed edges Z to our spanner. Since $c(Z) \le 2\sum \phi(v)$ and $\sum \phi(v)$ is simply the sum of holes (which can, again, be charged to the weight inside the holes), we have $c(Z) \le O(\epsilon^{-2}\log W)\, OPT$.

4 Spanner

For each region without a hole, compute a boundary spanner of cost at most $O(\epsilon^{-4})$ times the cycle cost. For each region with holes, for each connected component of Z, compute a boundary spanner where the "boundary" is an Euler tour of the holes together with the component in Z (where edges of Z are traveled twice, once in each direction). The boundary spanner cost can be charged to existing edges, so the total cost of adding the boundary spanners is at most $O(\epsilon^{-4})$ times the cost so far, or $O(\epsilon^{-4}) \cdot O(\epsilon^{-2} \log W)\, OPT = O(\epsilon^{-6} \log W)\, OPT$.

5 Transforming OPT

Recall that our goal is to transform OPT so that it only uses edges on the spanner. Fix a cycle O in OPT. For each region R with a hole that O intersects, set $H := O \cap R$, and use property (2) of prize-collection to obtain a set U such that $\sum_{v \in U} \phi(v) \le c(H)$. Now "cut up" O along the boundary of each hole in U (if it completely or partially overlaps with O, that is). We can charge the extra cutting to the boundaries of the holes in U, which gets charged to $O \cap R$ at an $\epsilon : 1$ ratio (since $\phi(v)$ was defined with extra factor ϵ^{-1}). Actually, it's $2\epsilon : 1$ ratio since we have to cut on both sides of the hole. Since $O \cap R$ is disjoint over all O and R, the total extra cutting is at most $2\epsilon\, OPT$.

Define OPT' to be the new transformed OPT ($OPT' \le (1 + 2\epsilon)OPT$), and fix a cycle O in OPT'. We now consider the holes in each region *not* in U. First, if a region R has no holes, then replacing the path $O \cap R$ with its path in the boundary spanner changes the weight by at most $(\epsilon/\lambda)\, c(O \cap R)$ (otherwise, we would've added the cycle formed by the two paths into $\mathcal{C}$). Summing over all $(\epsilon/\lambda)\, c(O \cap R)$ (recall that $\lambda \approx OPT/W$) we get a weight difference of $O(\epsilon W)$.

Next, if R has holes and O intersects any boundary of R (that is, the outer boundary of R or an inner hole), then replace by the path in boundary spanner of the relevant connected component in Z. Through the *cyclic double cover* trick, we can make sure the new and old paths do not enclose the largest hole in R. So the new path must also change the weight by at most $(\epsilon/\lambda)\, c(O \cap R)$ (otherwise, we would've added the cycle formed by the two paths into $\mathcal{C}$). Again, the weight difference is $O(\epsilon W)$.

The last case is when O is completely contained within a region R. Then, O must contain the largest hole in R (otherwise, we would've added the cycle formed by the two paths into $\mathcal{C}$). This is handled with an ad-hoc argument, by replacing O with a "canonical" cycle in R containing the largest hole.

In total, we obtain a spanner of total size $O(\epsilon^{-6} \log W)\, OPT$ such that OPT can be modified with extra cost at most $O(\epsilon)OPT$ and weight error $O(\epsilon W)$.