Congestion-Approximators from the Bottom Up

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Joint with Satish Rao (Berkeley),
Di Wang (Google)

Max-Flow Problem

Given undirected (uncapacitated) graph

$G=(V,E)$ and vector b in $\mathbb{R}^V$ with $\sum b_v=0$

want net flow b_v at each vertex v

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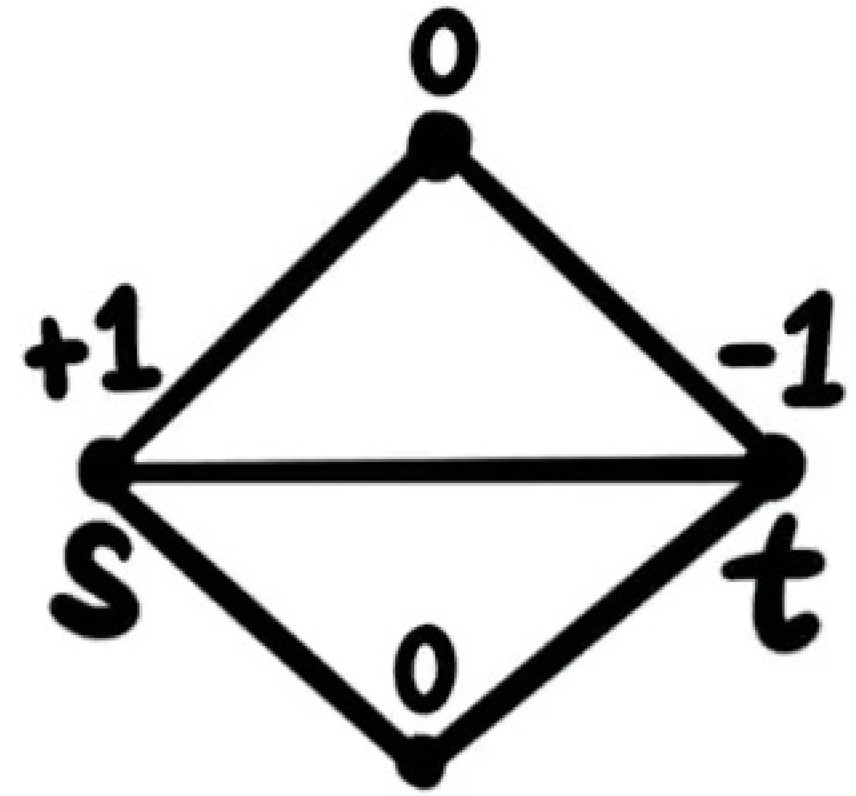

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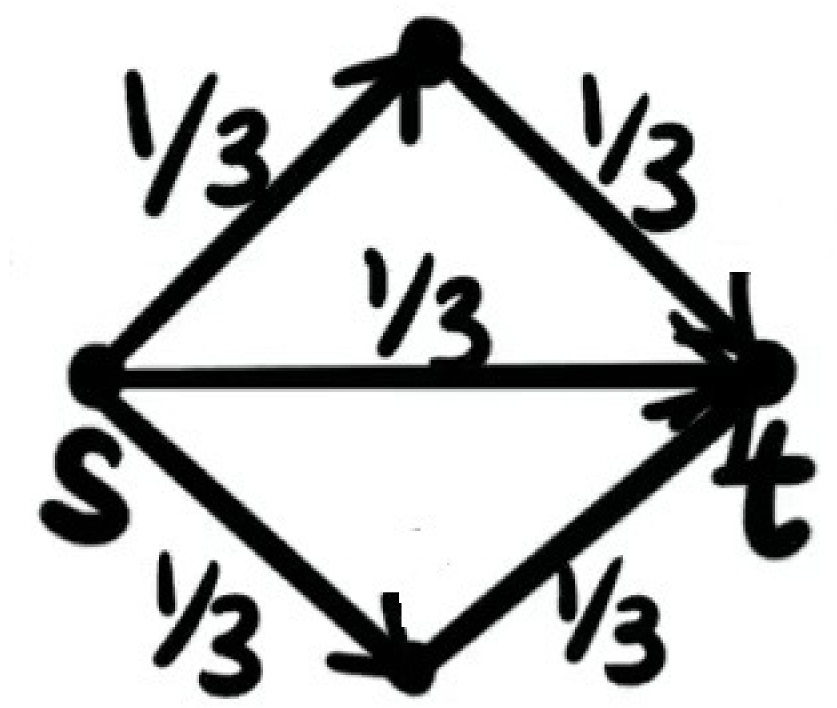

$b = \mathbb{1}_s - \mathbb{1}_t$

congestion = 1/3

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Flow/Cut Duality

Given a subset of vertices S, consider

$$\frac{\partial S}{|b(S)|} \quad \begin{array}{l} \leftarrow \text{size of boundary} \\ \leftarrow \left|\sum_{v \in S} b_v\right| \end{array}$$

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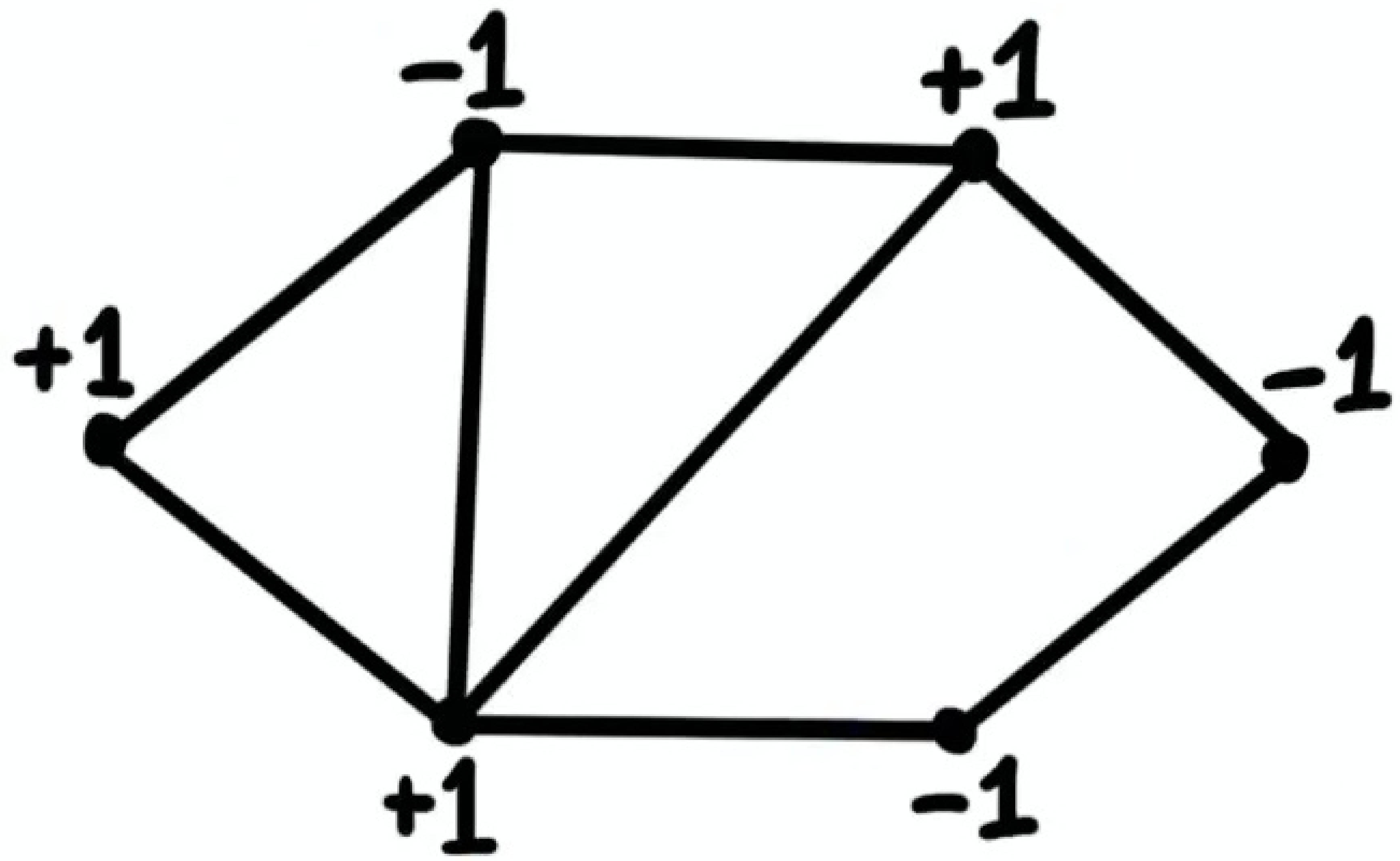

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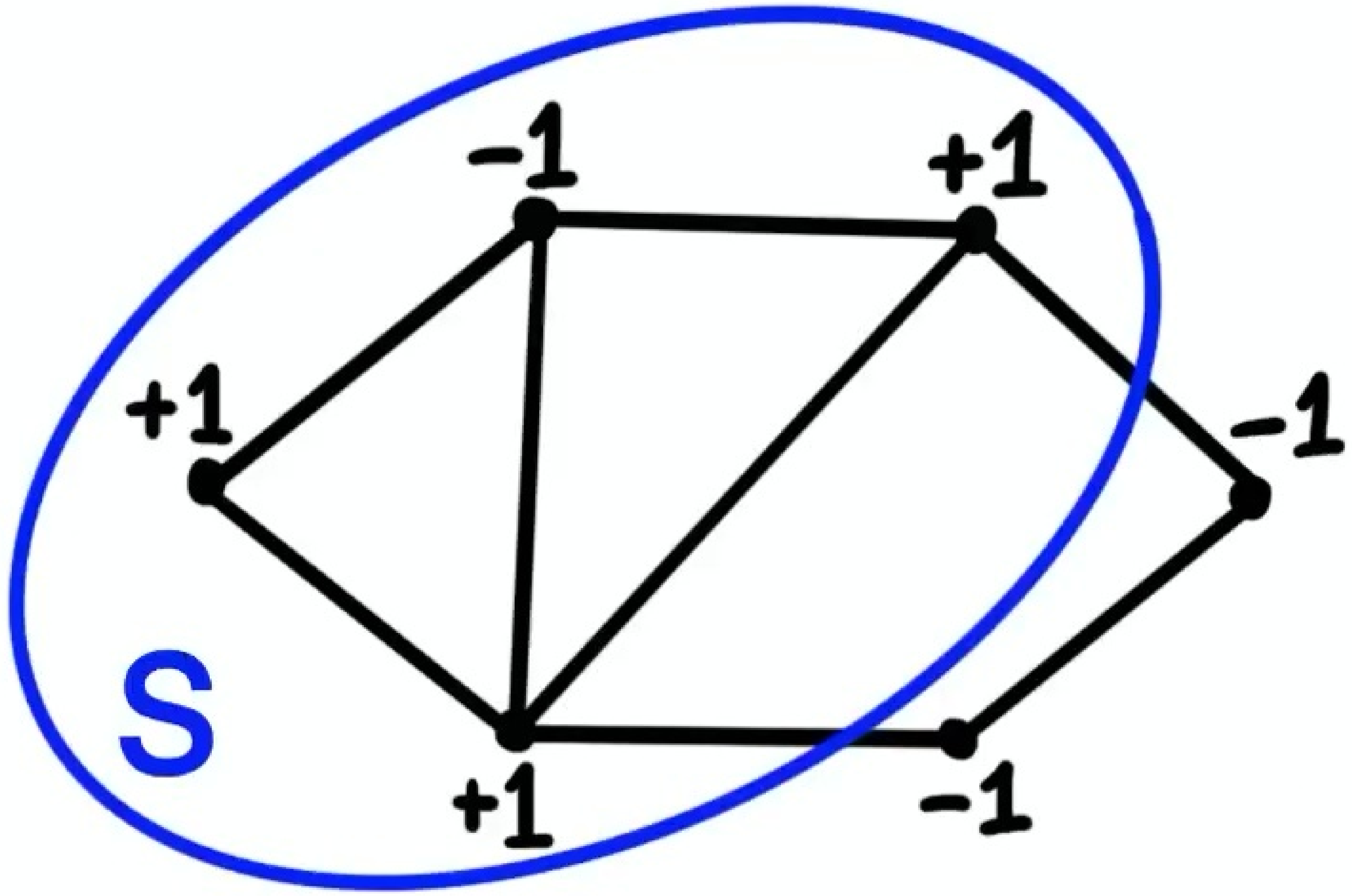

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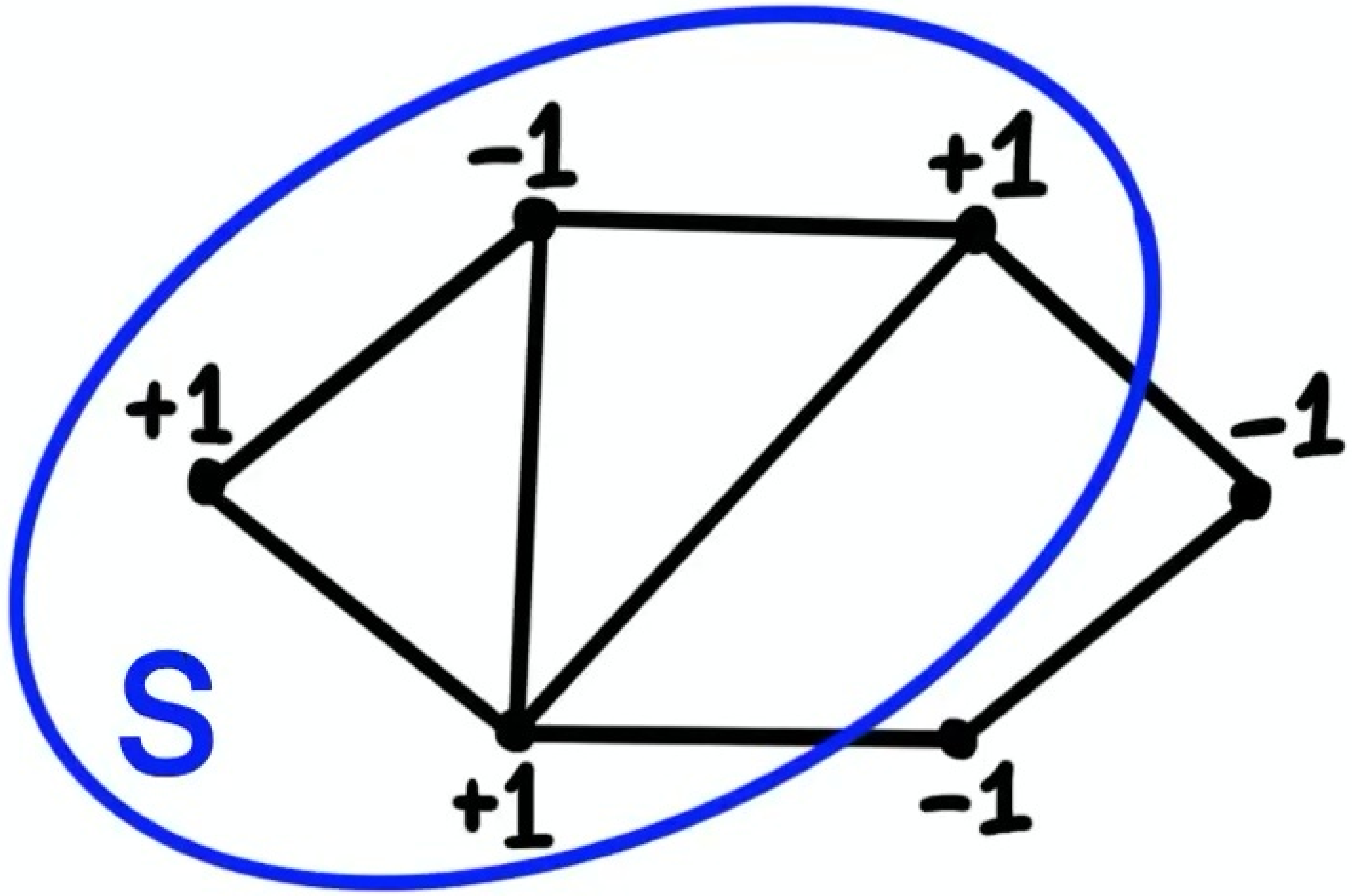

$$\frac{\partial S}{|b(S)|} \quad \begin{matrix} \leftarrow \text{size of boundary} \\ \leftarrow \left|\sum_{v \in S} b_v\right| \end{matrix}$$

$\partial S = 2$

$|b(S)| = 2$

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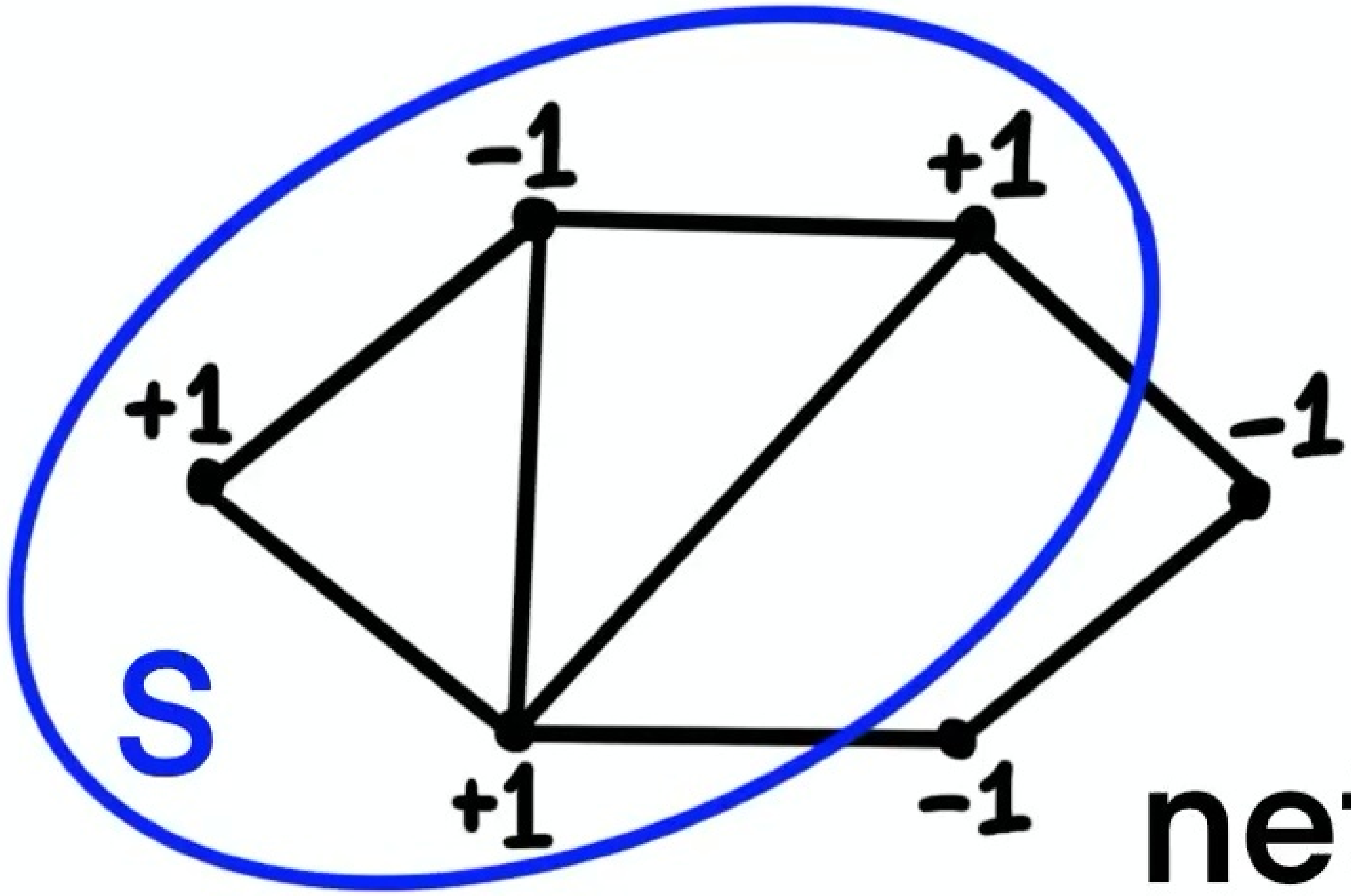

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net $|b(S)|=2$ flow must go across $\partial S=2$ edges

$\rightarrow$ any flow has congestion >= 1

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family $\mathcal{C}$ of subsets S s.t.

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(b(V)=0)

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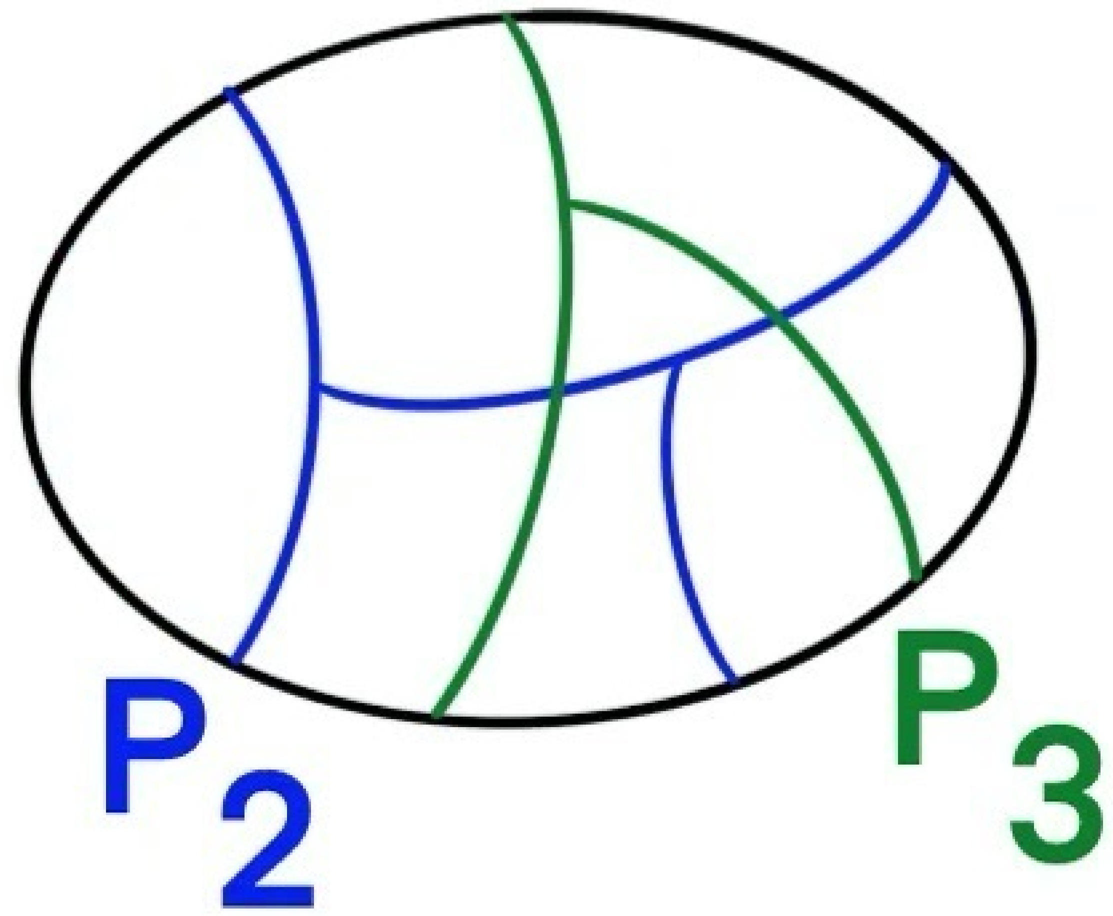
This work: polylog-quality congestion approximator without recursive max-flow

Our Congestion Approximator Construction

Theorem: Consider partitions $P_1, P_2, \ldots P_L$ of V s.t.



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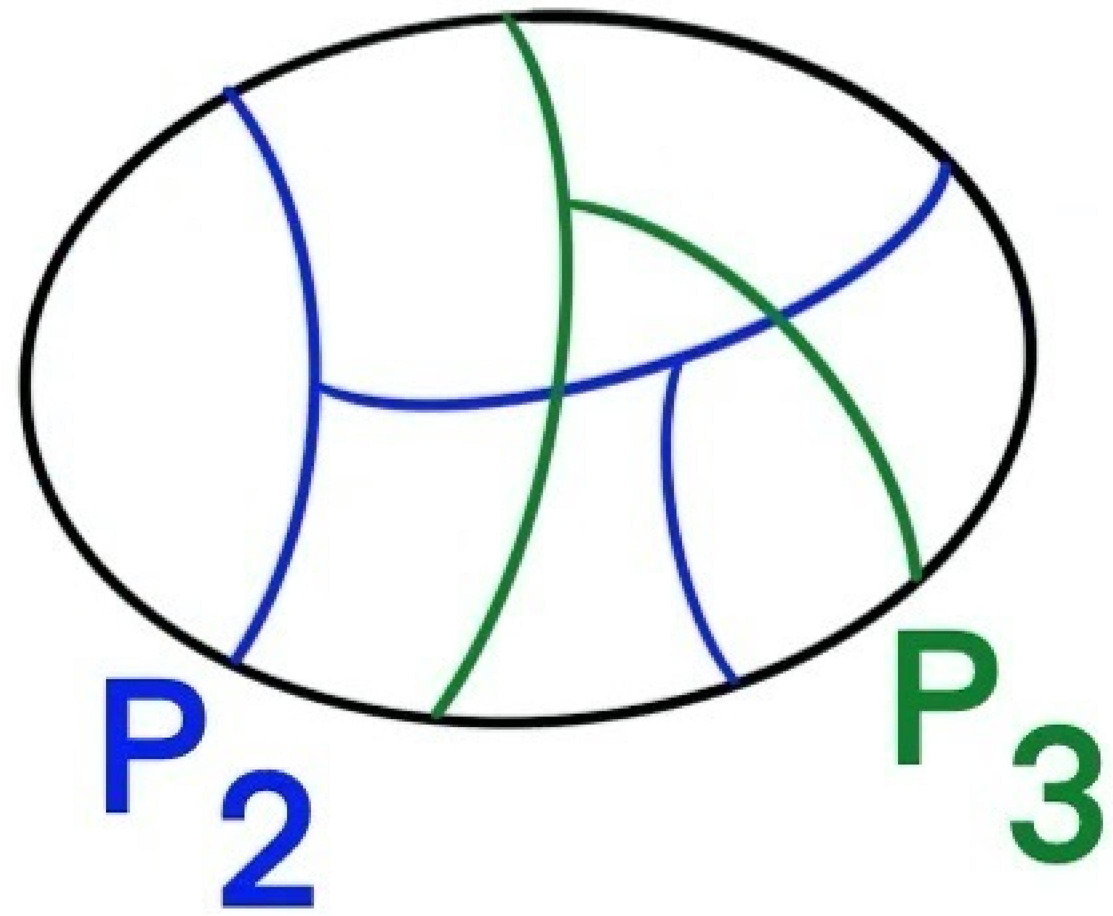

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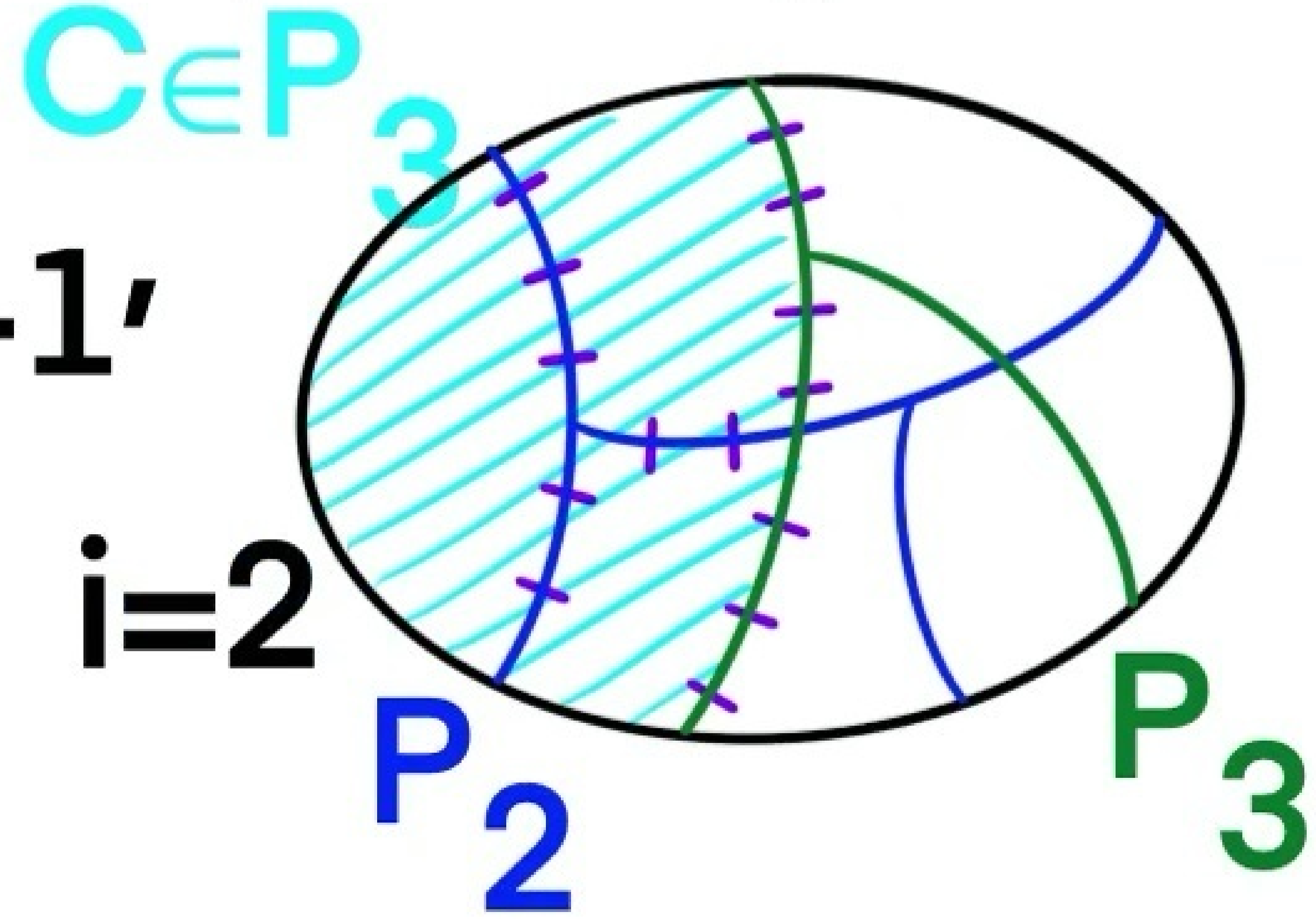

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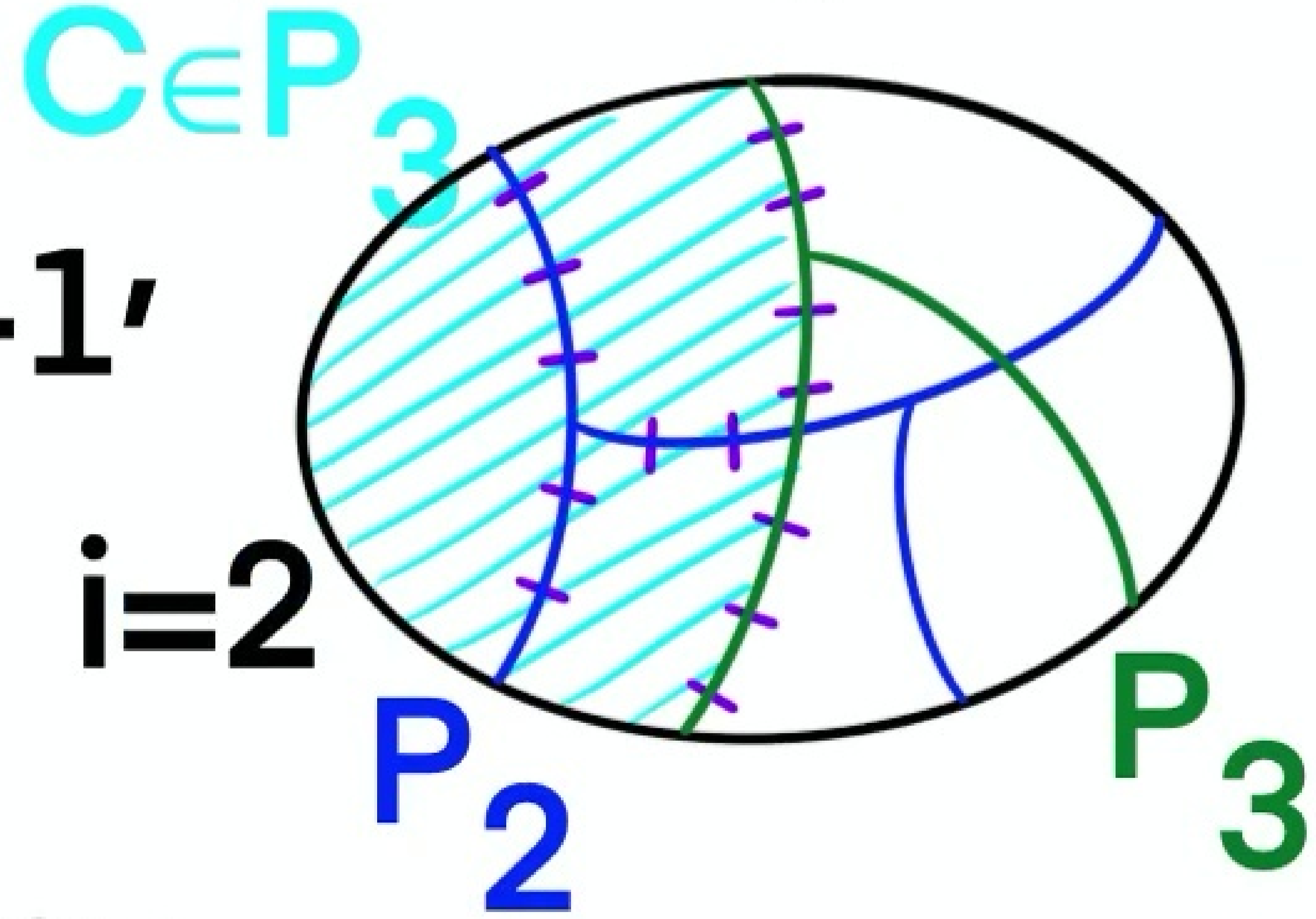

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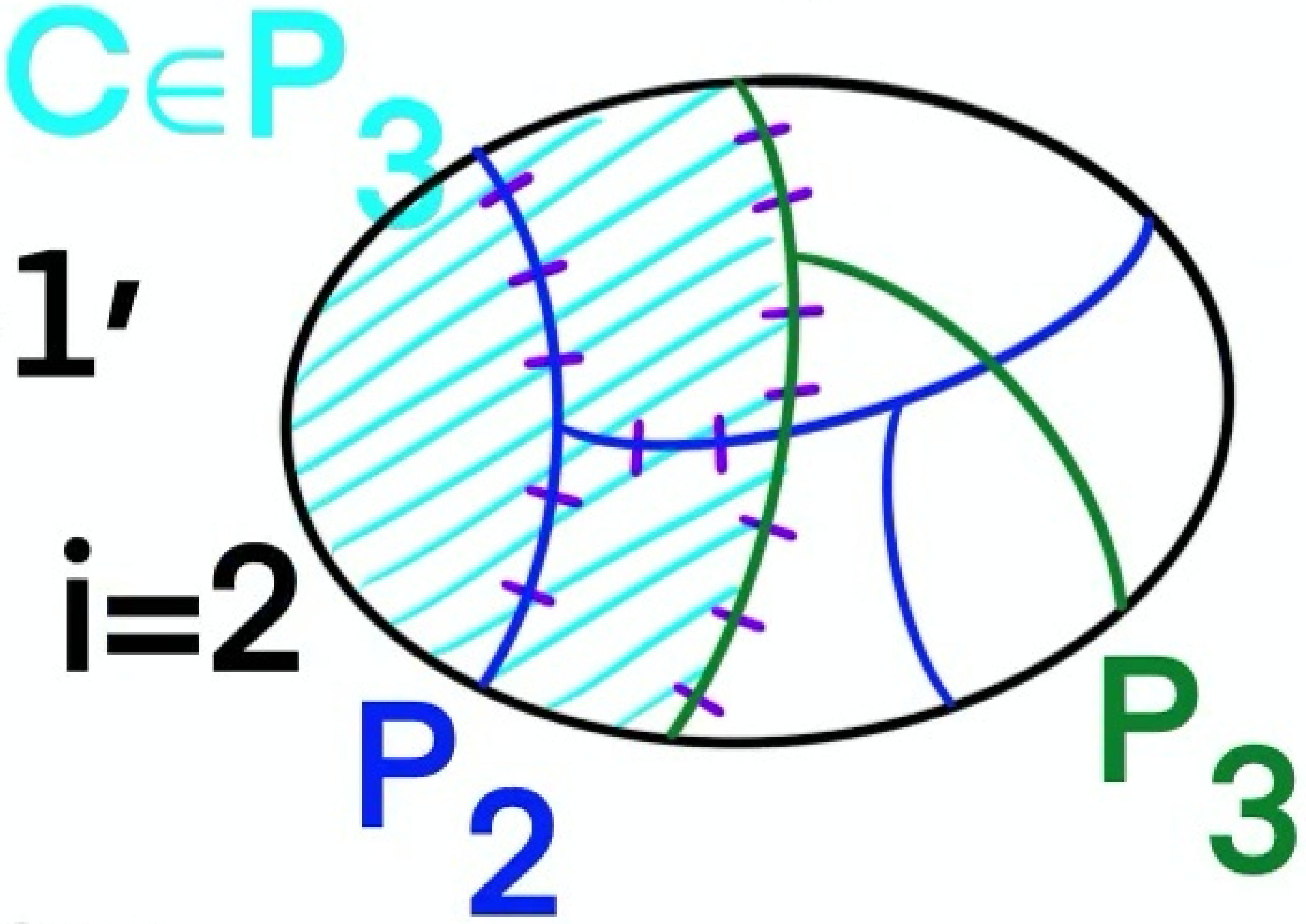

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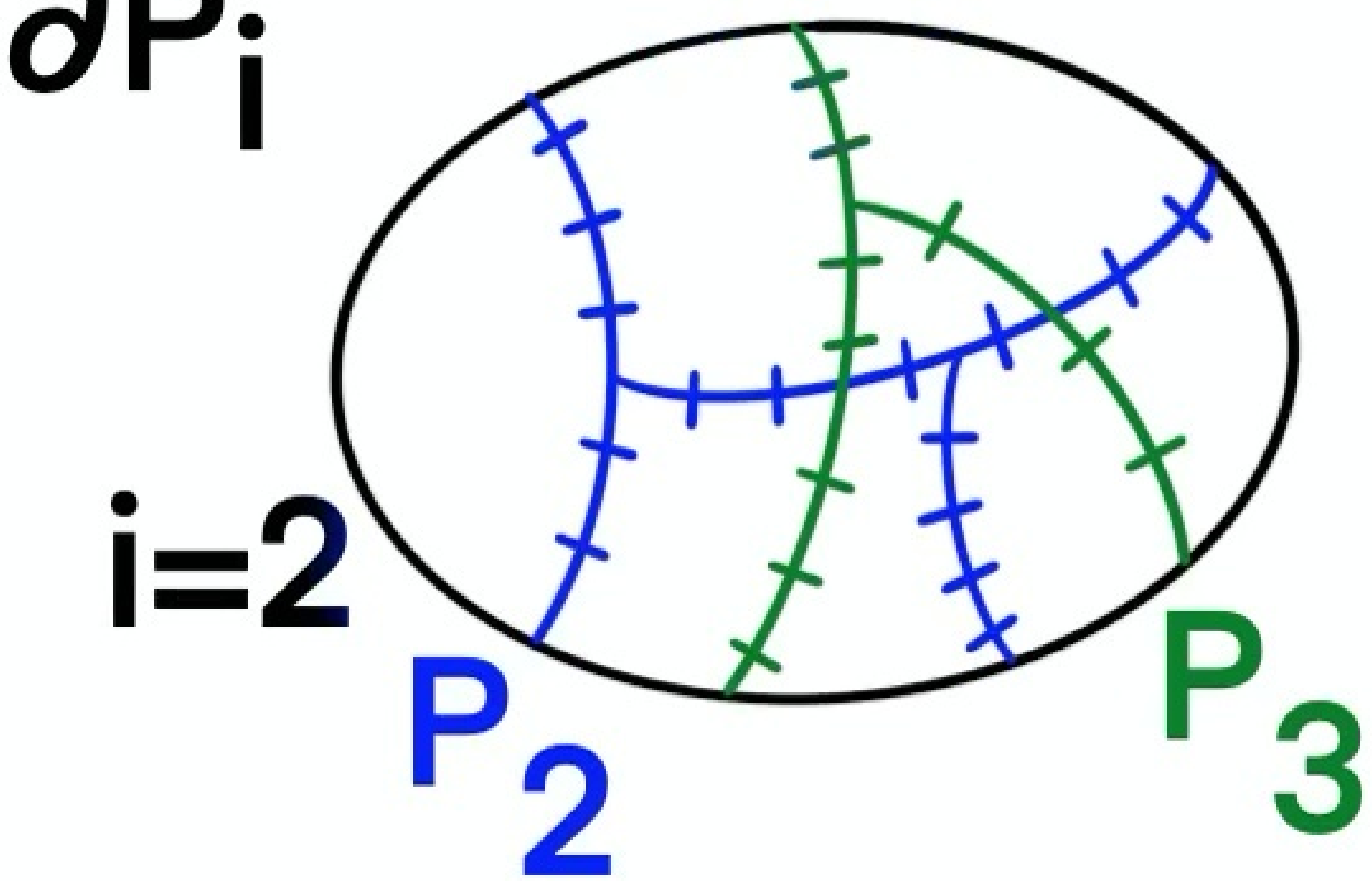

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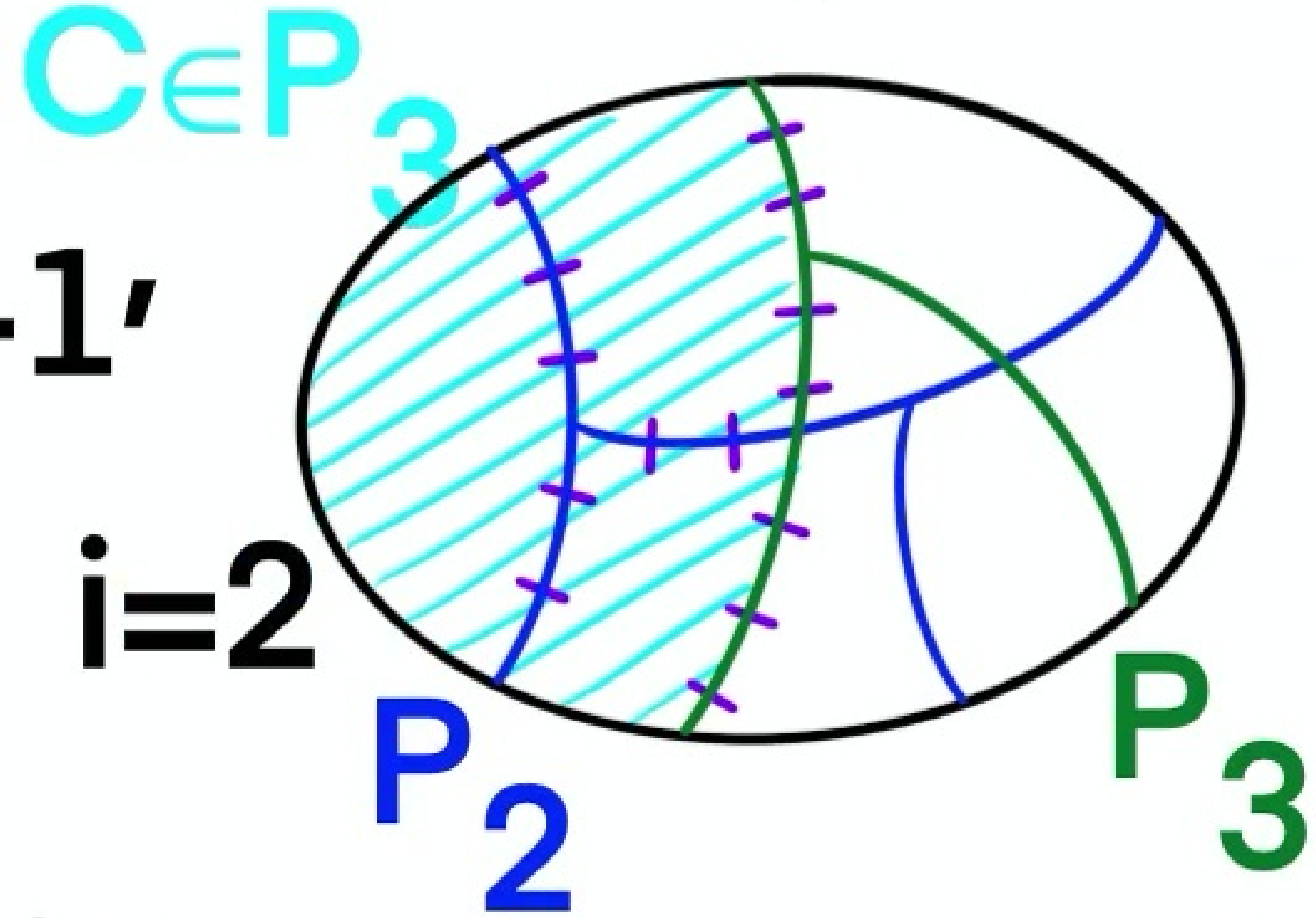

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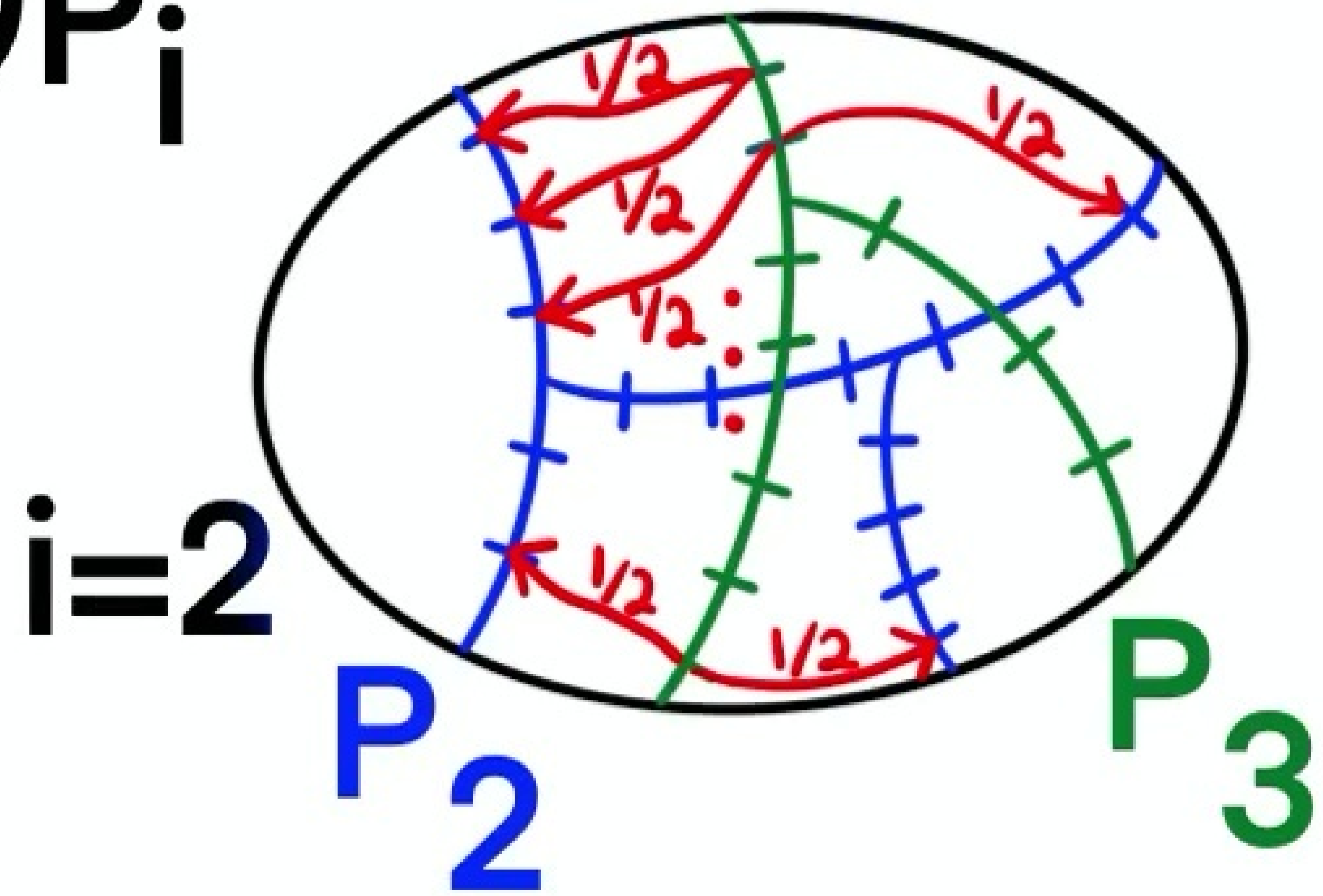

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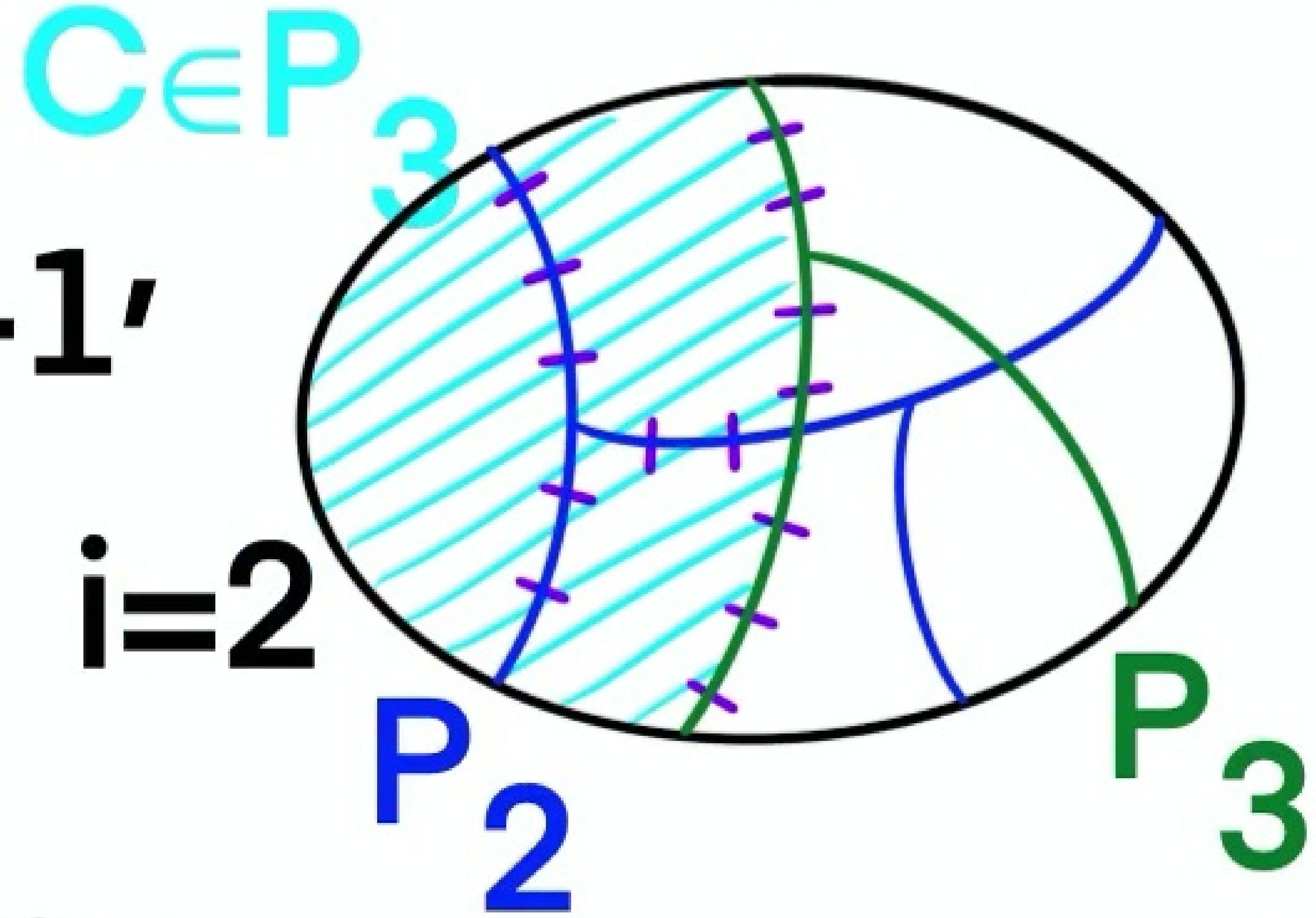

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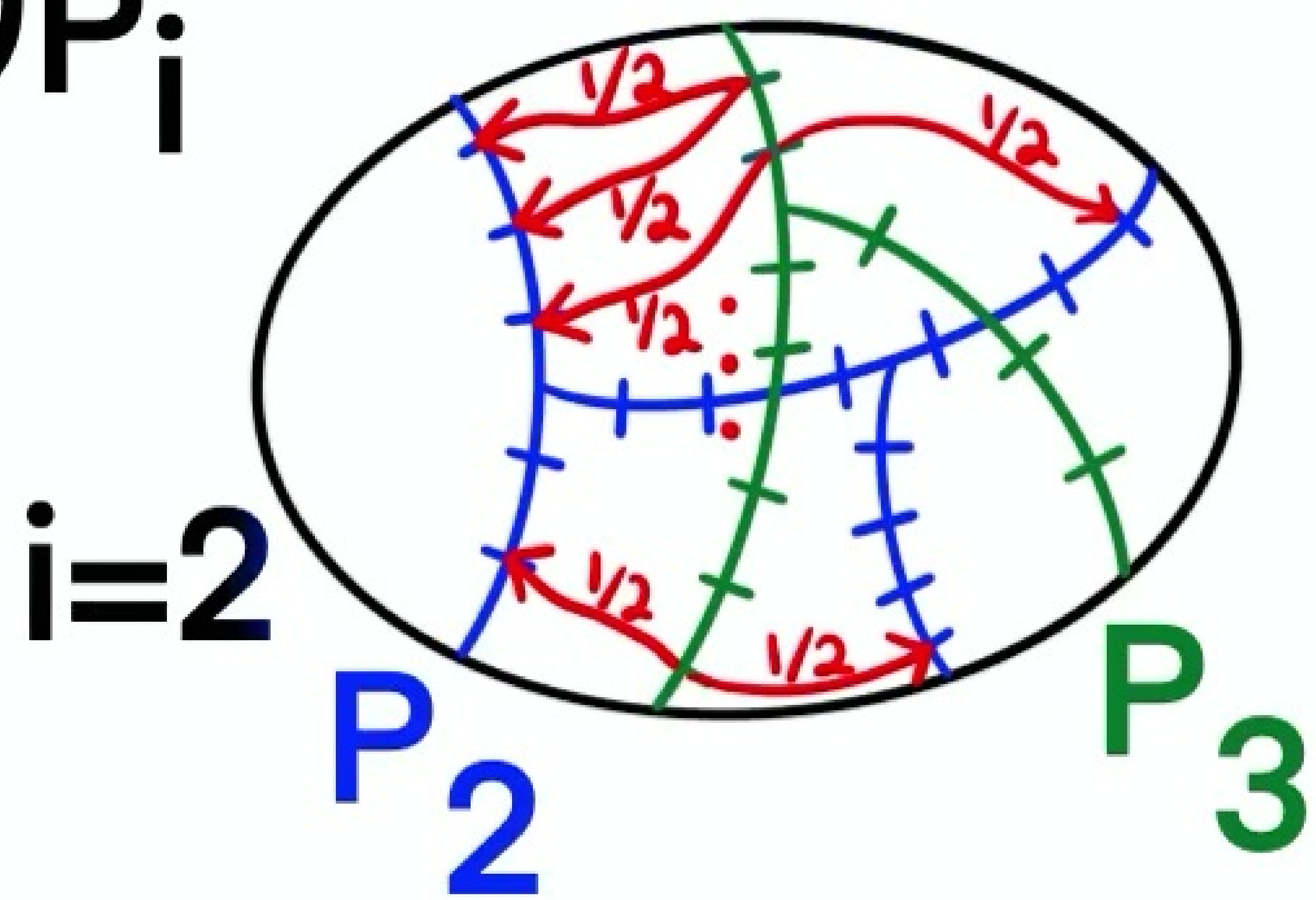

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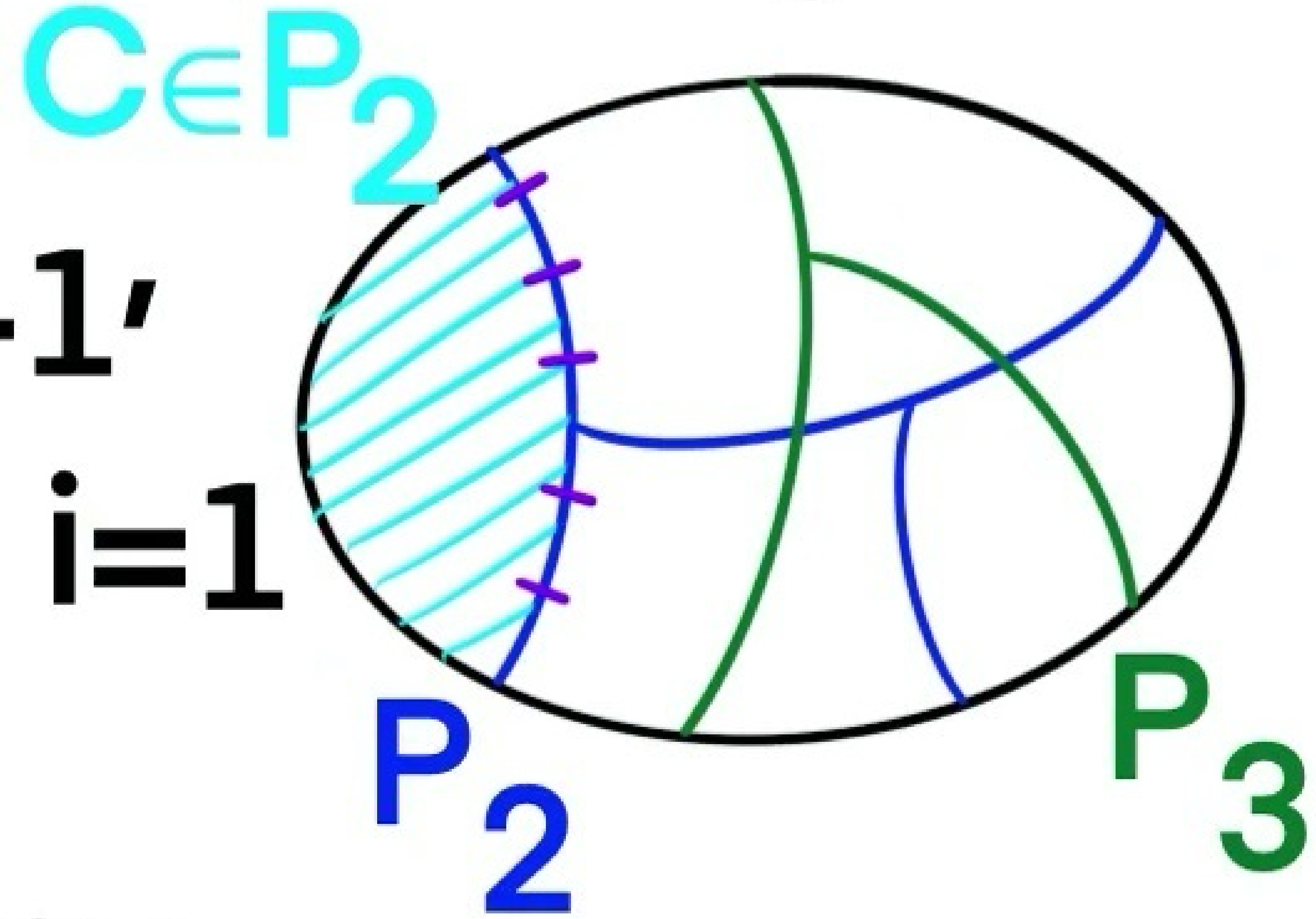

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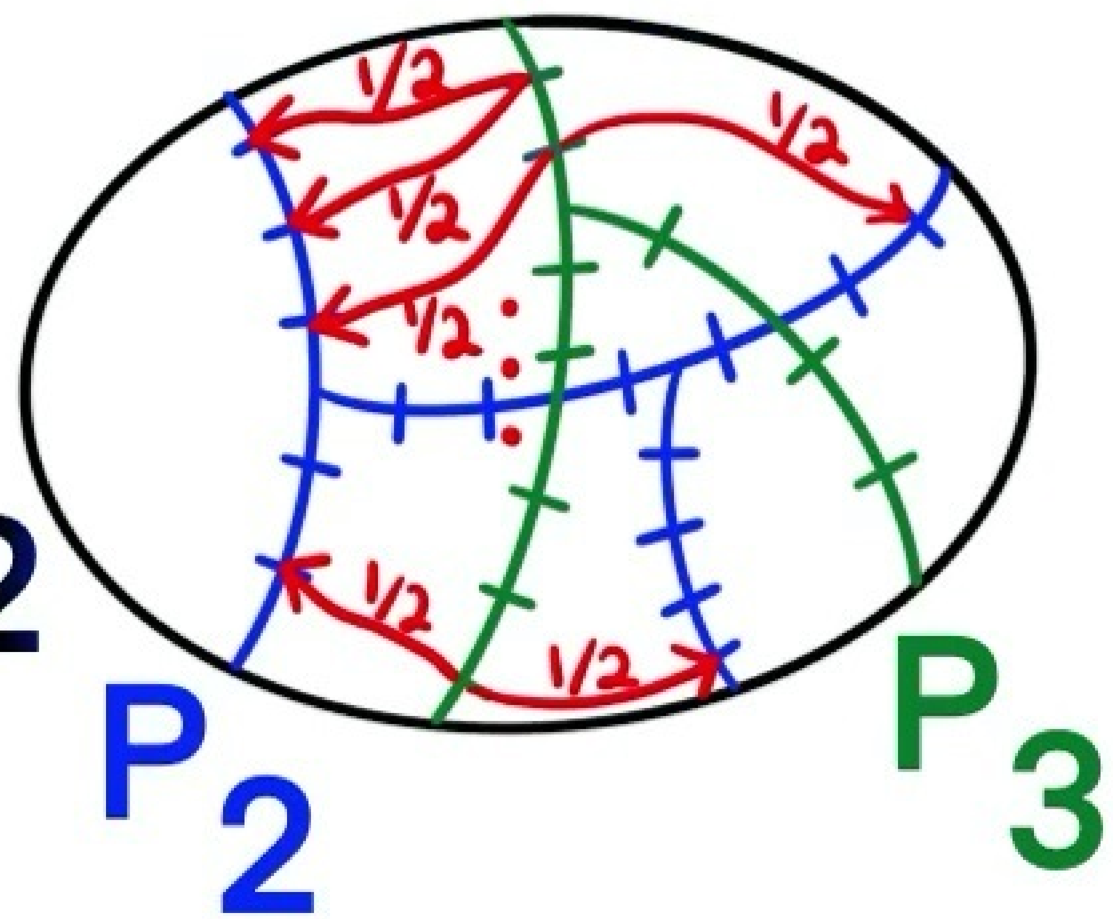

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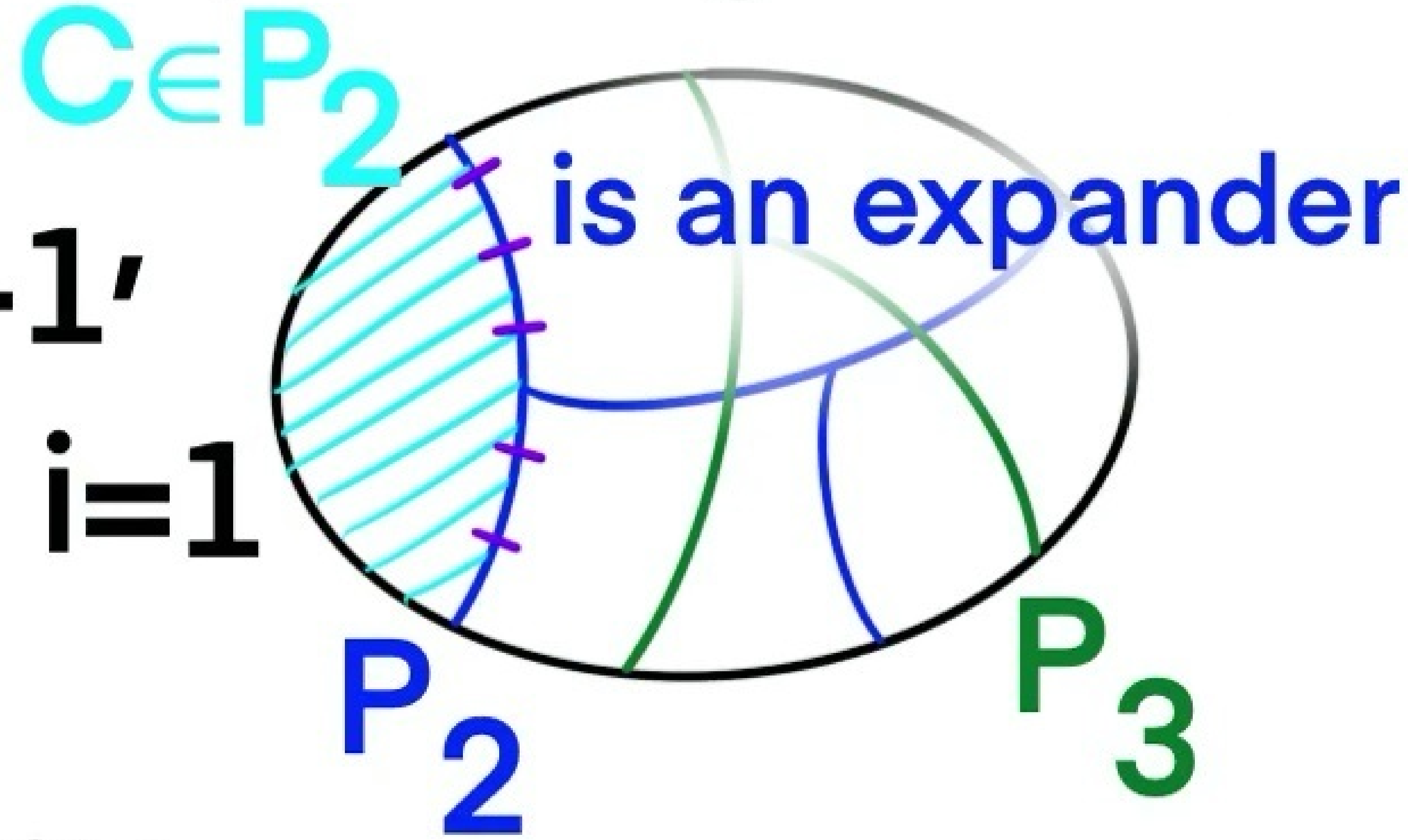

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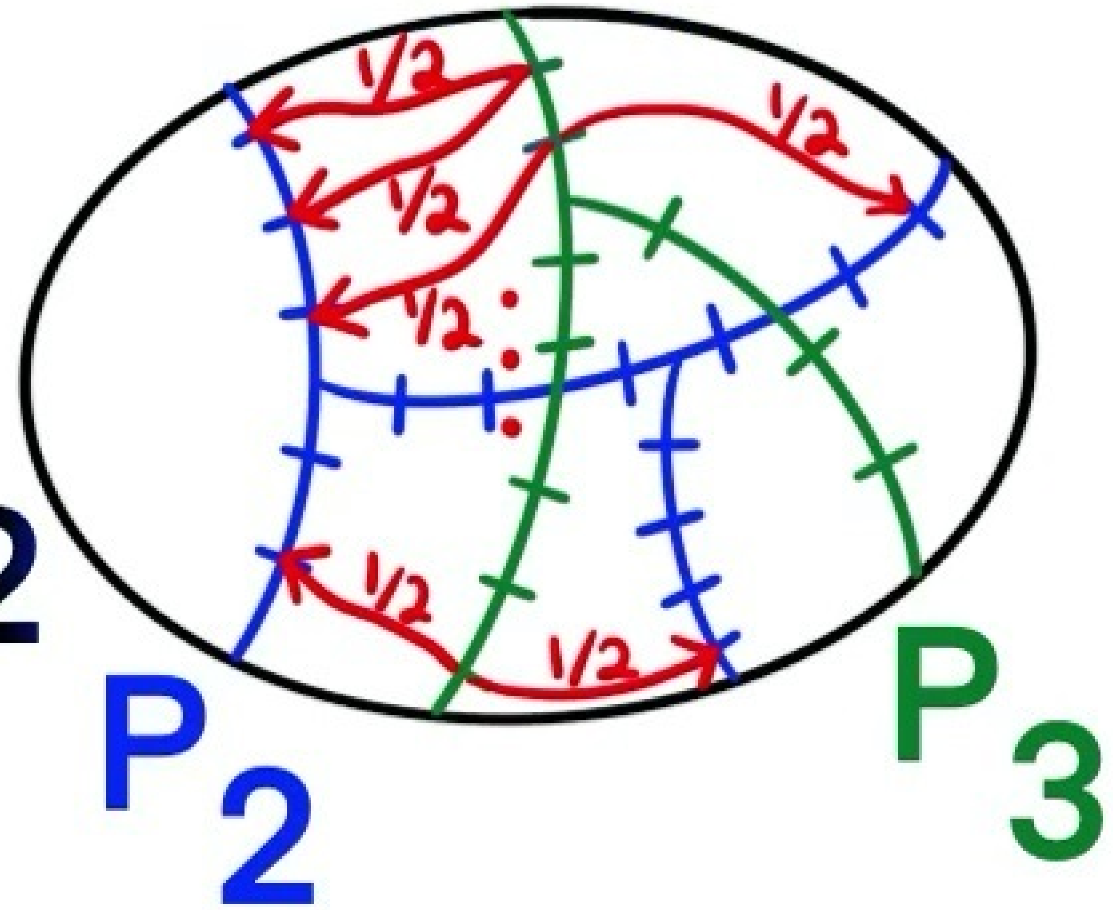

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For each i, let $R_{>=i}$ be the common refinement of partitions P_i, P_{i+1}, ..., P_L

Then $\mathcal{C} = \cup_i R_{>=i}$ is a congestion approximator with quality polylog.

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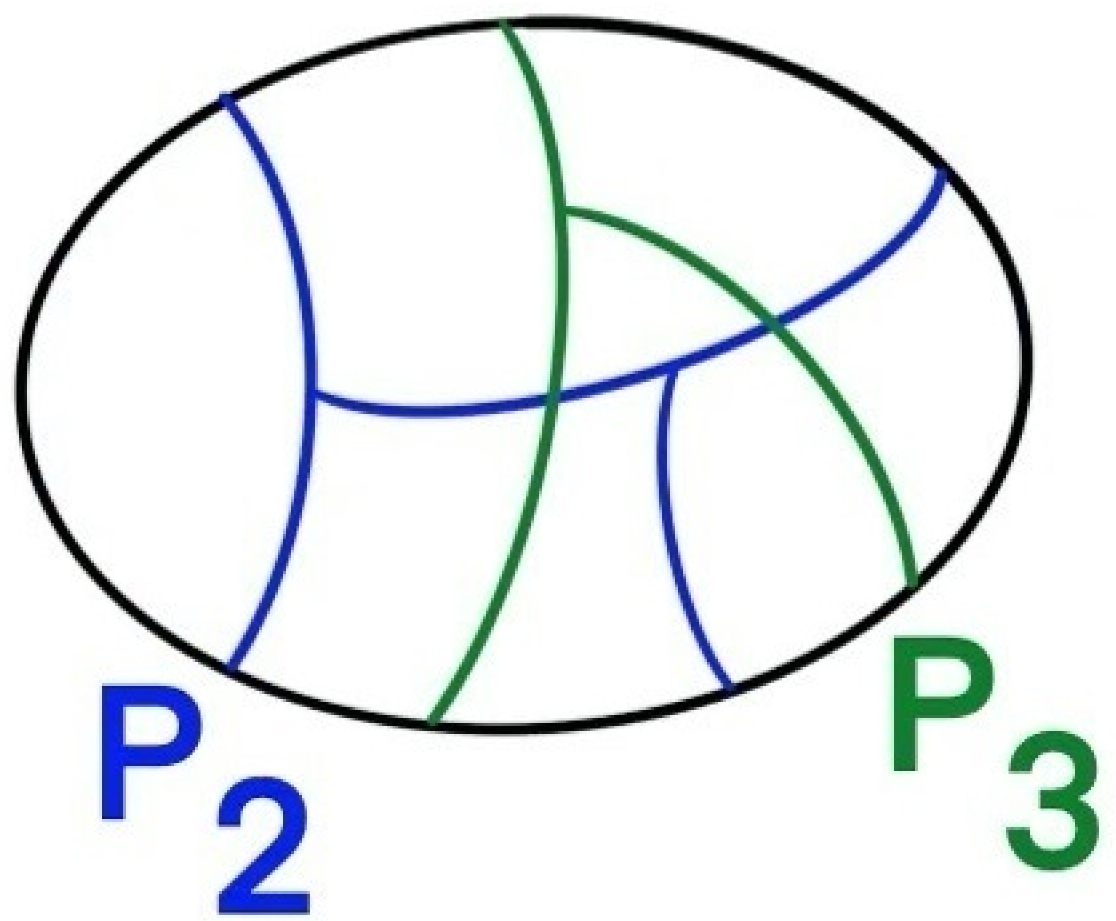

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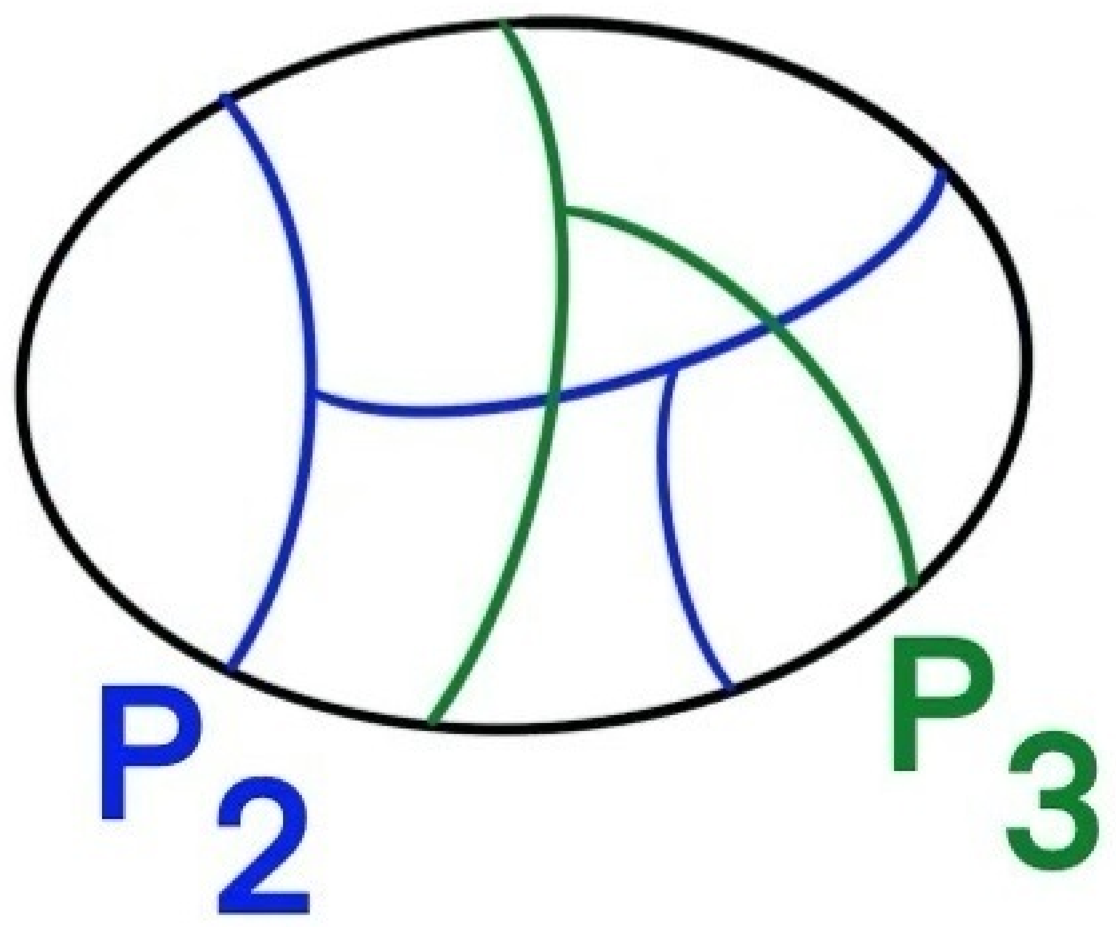

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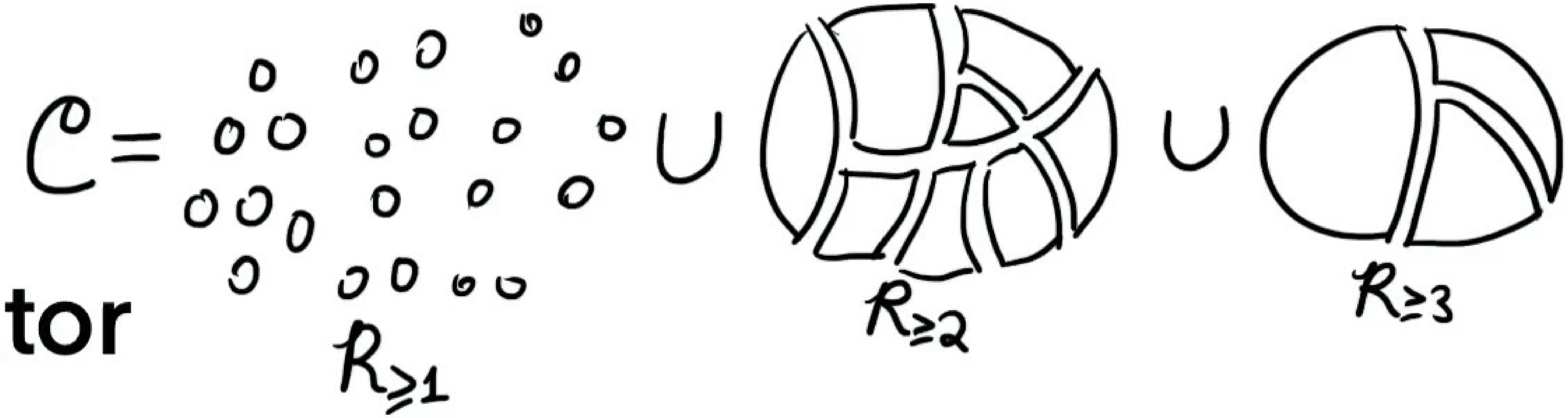

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Max-flow calls required, but use structure of $P_1, \ldots, P_{i-1}$ to build "pseudo"-congestion approximator sufficient for the specialized max-flow calls

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Number of polylog factors still high (unspecified).
Open question: reduce number of polylog factors?