Algorithms: New Techniques for Old Problems

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  - “What if the input graph is an expander?”

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  - “Continuous formulation of a discrete problem?”

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- Iterative methods [CKMST'11, Sherman'13]
  - "Continuous formulation of a discrete problem?"
- I work on preconditioning and iterative methods for graphs

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- Precondition by **expander decomposition**

Expanders

- Given unweighted, undirected graph G, its **conductance** is

$$\Phi(G) = \min_{S \subseteq V} \frac{|E(S, V \setminus S)|}{\min\{\mathbf{vol}(S), \mathbf{vol}(V \setminus S)\}}$$

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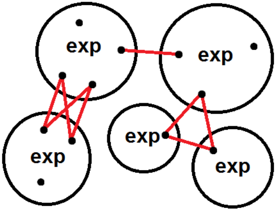
- Examples of ϕ-expanders ($\phi \geq \frac{1}{\text{polylog}(n)}$): hypercube, random graphs, peer-to-peer networks

Expander decomposition

- Given graph $G = (V, E)$, partition V into $V_1, \ldots, V_k$ s.t.
  - Each induced graph $G[V_i]$ is a ϕ-expander for $\phi \geq 1/n^{o(1)}$
  - At most half of edges go between expanders



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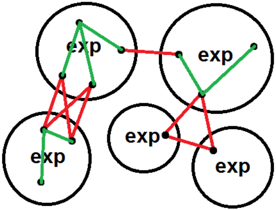

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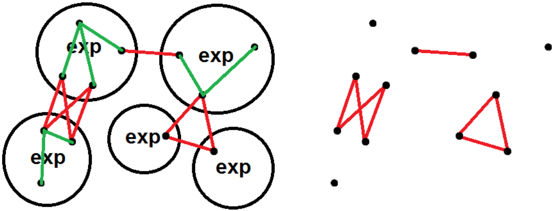

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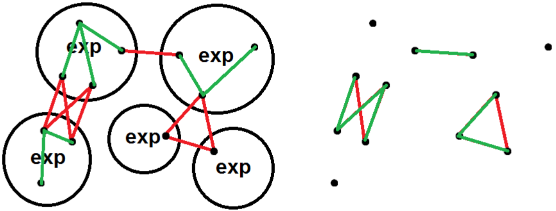

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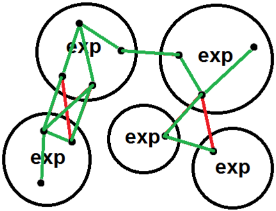

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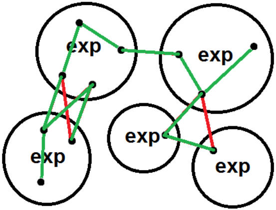

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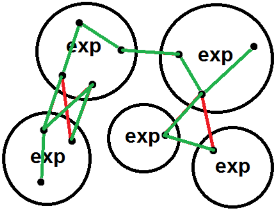

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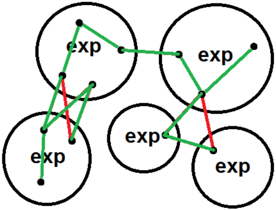

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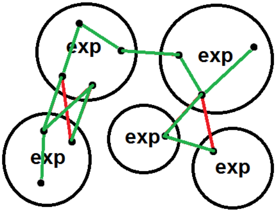

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- [CG**L**NPS'20]: deterministic exp. decomp. in $m^{1+o(1)}$ time
  - Derandomize cut-matching game [KRV'07]

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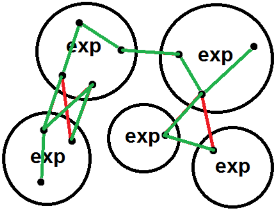
      Dynamic version: framework of [NSW'17]



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  - Derandomize graph sparsification by random sampling [BK'96]

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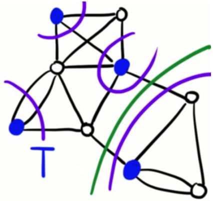

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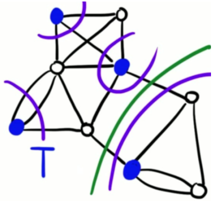

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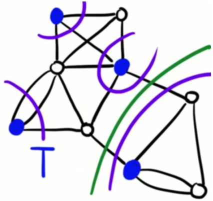

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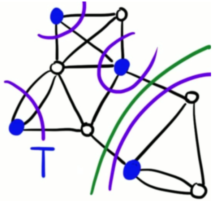

- Randomized: sample each vertex into T w.p. $1/k$
- Derandomization: overhead of $k^{O(1)} \log n$
- Theorem: if G is a ϕ-expander, then deterministic min-cut in $\phi^{-O(1)}\text{polylog}(n)$ many s–t max-flows

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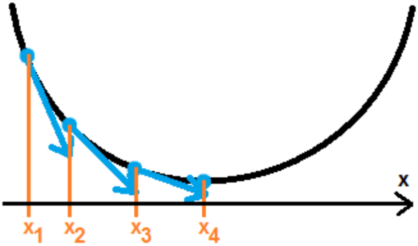

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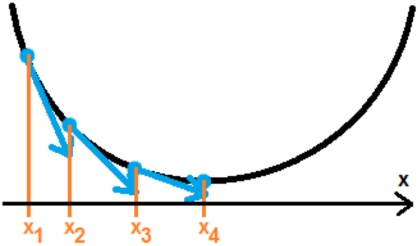

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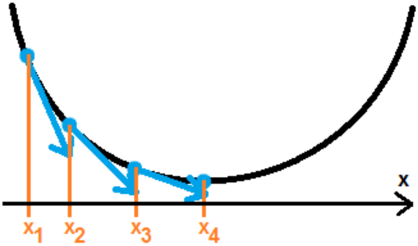

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- multiplicative weight update / gradient descent in online algorithms, convex programming, machine learning

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