Combinatorial Max-Flow via Weighted Push-Relabel on Shortcut Graphs

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Joint work with Aaron Bernstein, Joakim Blikstad, Thatchaphol Saranurak, Ta-Wei Tu

Max-Flow

Input: directed (uncap.) graph G=(V,E) and vertices s,t

Find max # of edge-disjoint s-t paths <=> Find s-t flow of maximum value s.t. each edge has congestion <= 1

History and Our Results

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Continuous optimization: Daitch-Spielman, CKMST, Madry, Sherman (approx)

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- augmenting paths easier to understand and implement

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This work: n^2 polylog(n) and simpler
- exploits rich combinatorial structure
- even implemented by Blikstad-Tu (2025)

Roadmap

combinatorial structure of max-flow: this talk

- short flow paths --> fast algorithm
- combinatorial preconditioning
- expanders and expander hierarchy

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weighted push-relabel: generalization of the push-relabel algorithm for max-flow (skip)

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Call this a short flow. Can be computed by blocking flow or standard push-relabel

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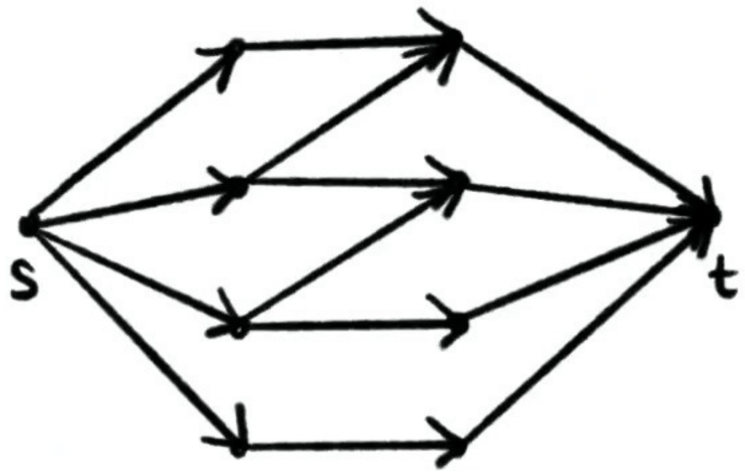

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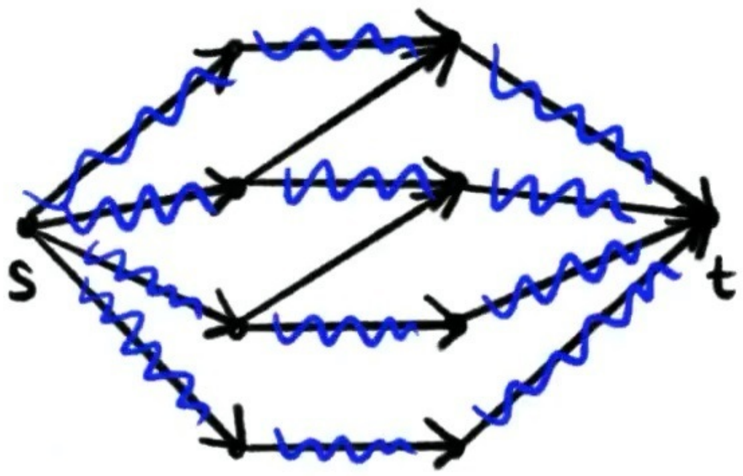

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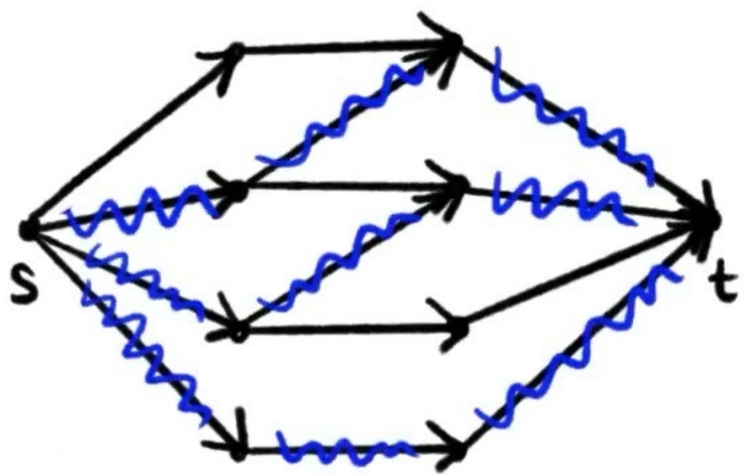

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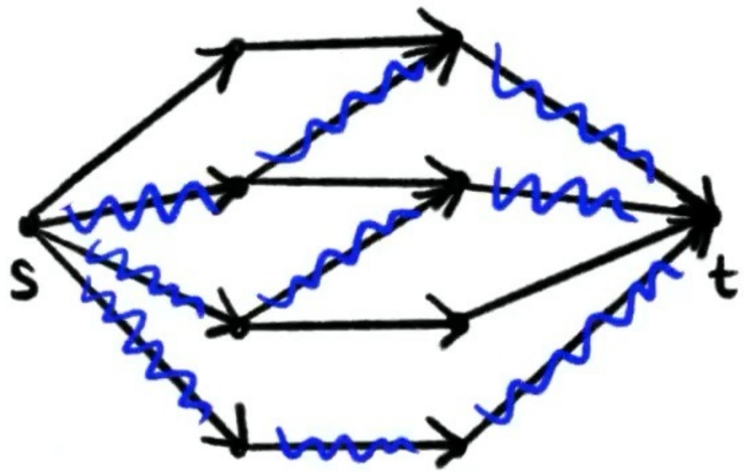

Short flow ~= greedy maximal matching which is 1/2-approx.

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So average path length <= h on the residual graph =><=

Combinatorial Preconditioning

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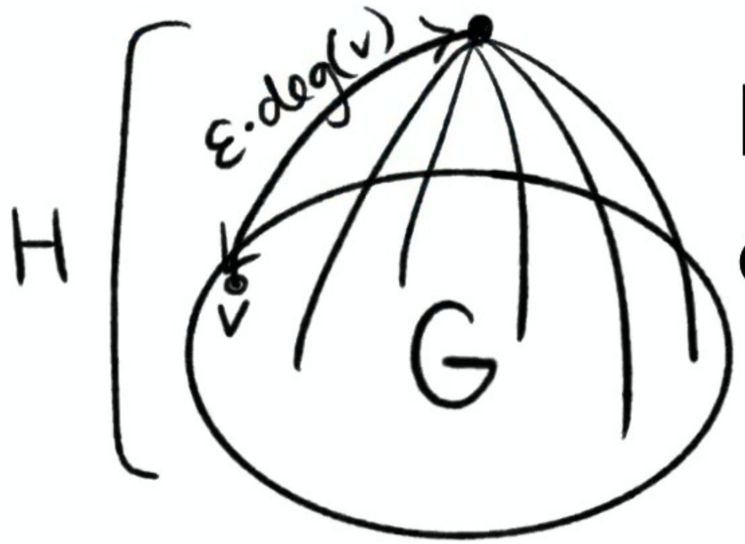

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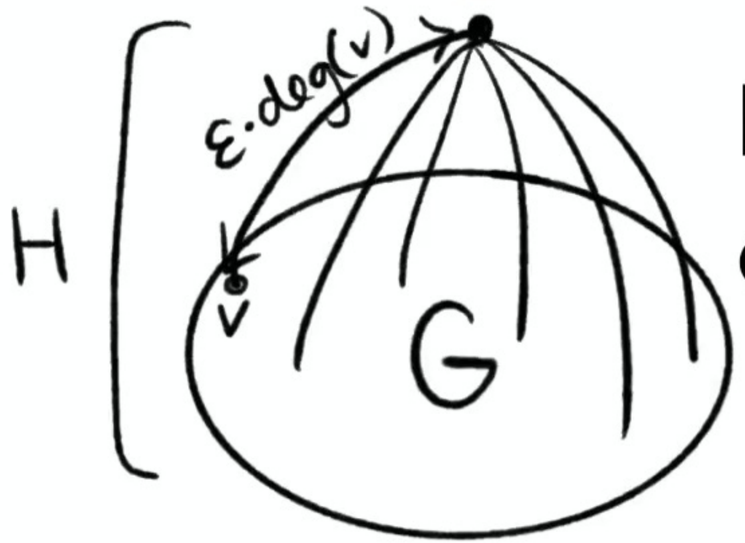

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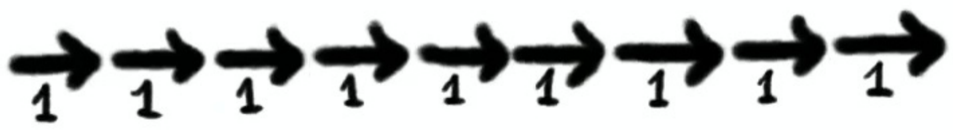

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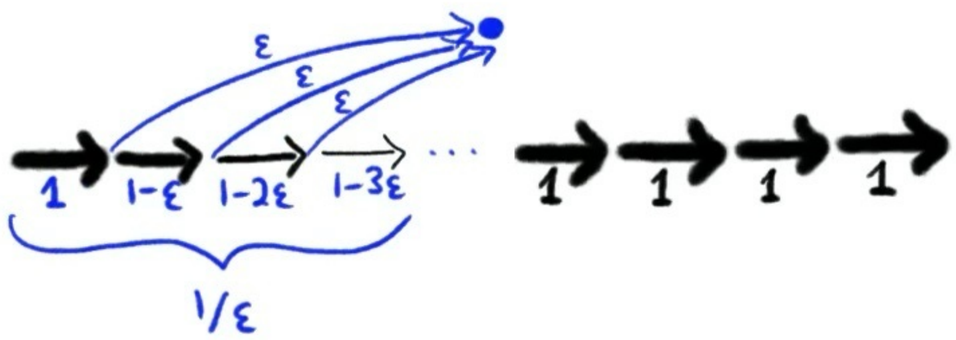

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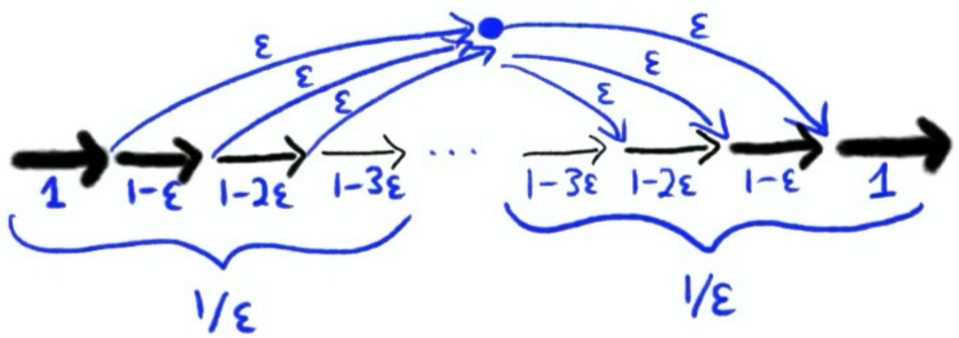

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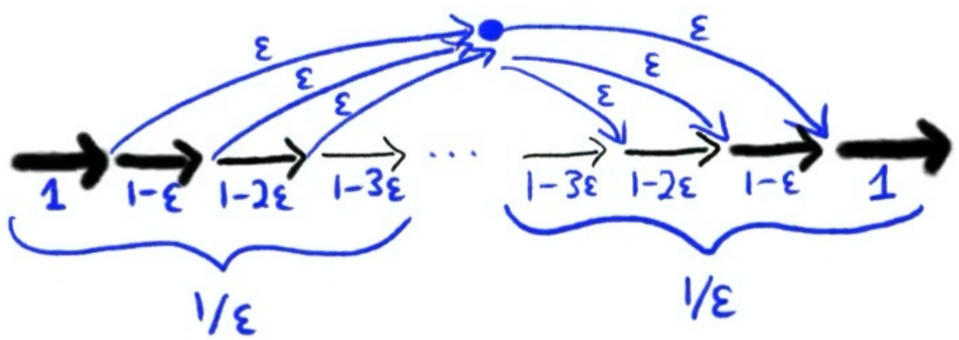

Each vertex participates in <= deg(v) paths, each leaking ε => ε*deg(v) total

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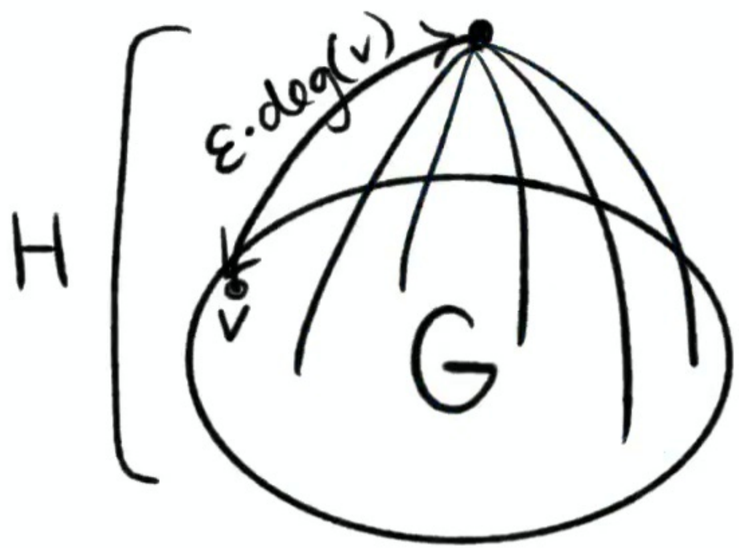

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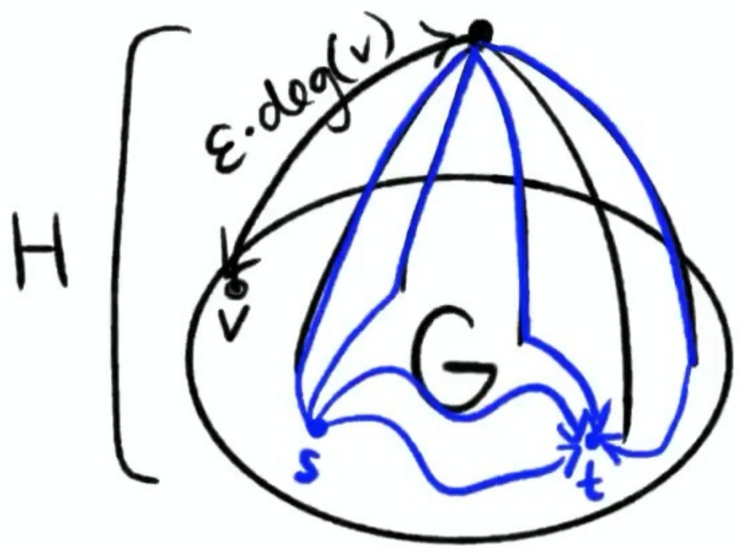

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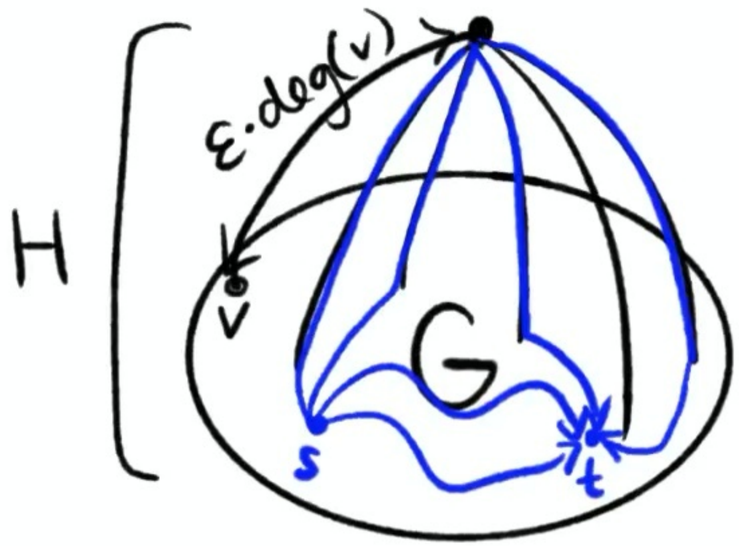

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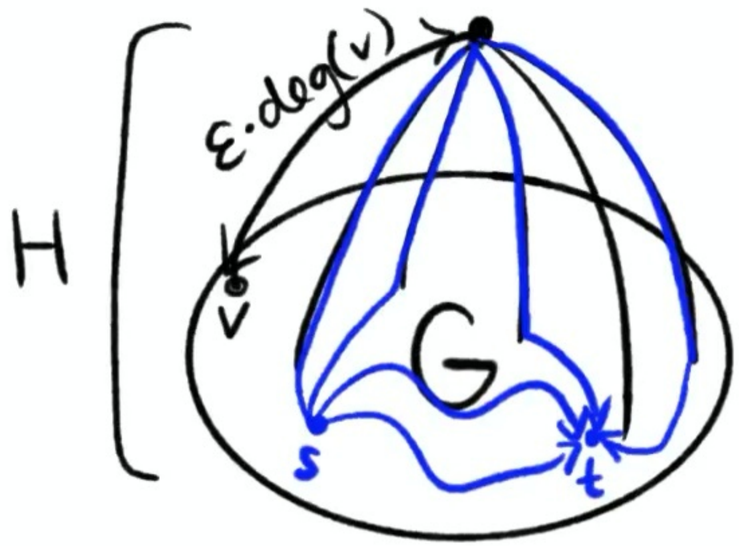

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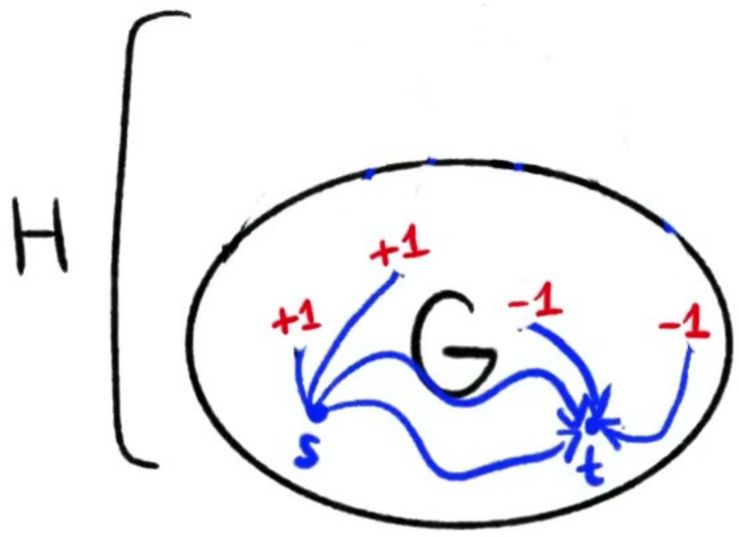

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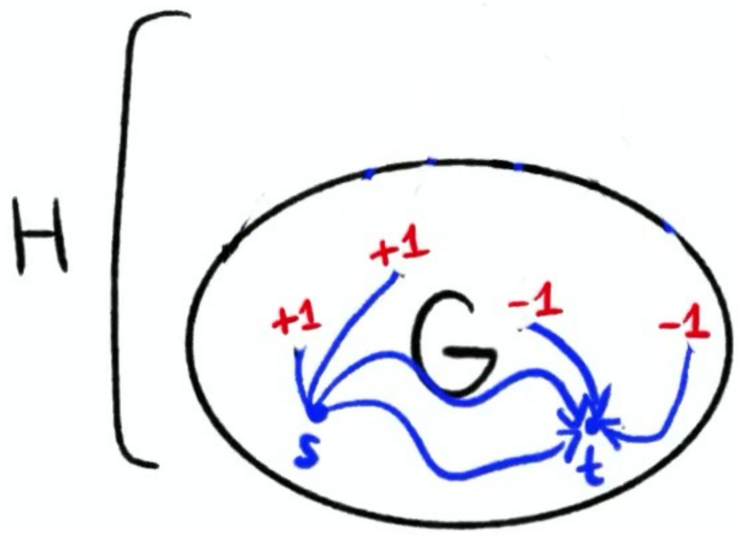

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Induces vertex demands.

Each vertex demand <= $\varepsilon * \deg(v)$ in absolute value.

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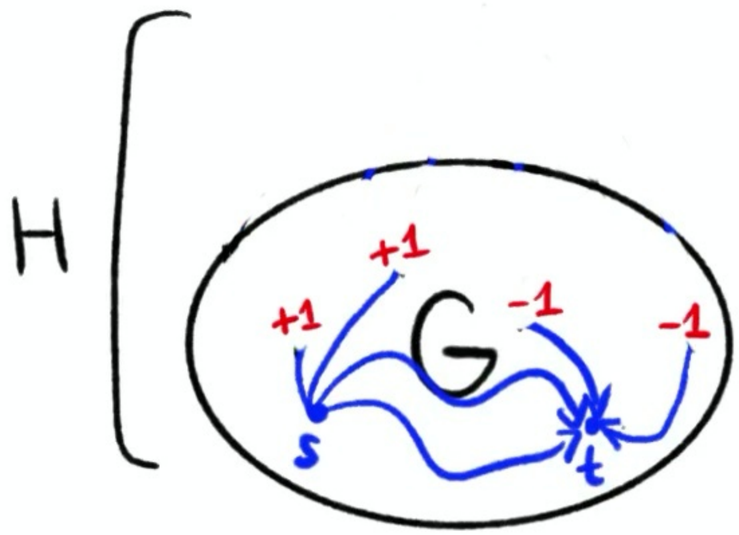
Flow definition of expander: G is a ϕ-expander if for any vertex demand with absolute value <= deg(v) at each vertex v, can be routed with congestion <= 1/ϕ



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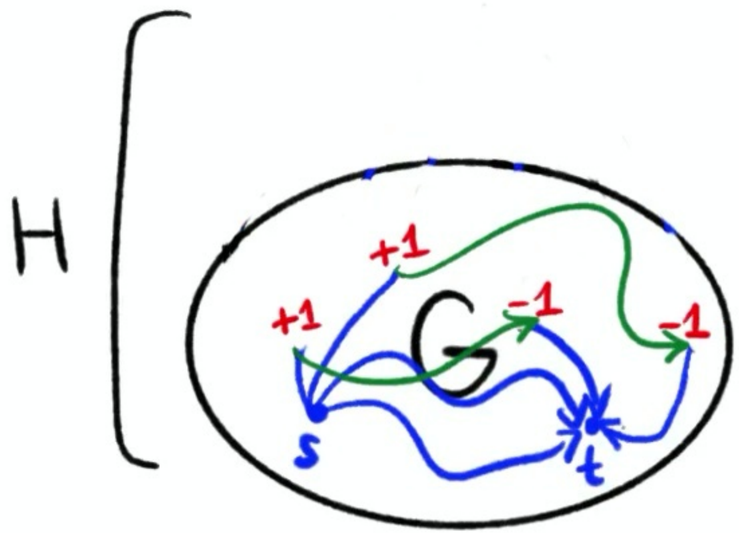

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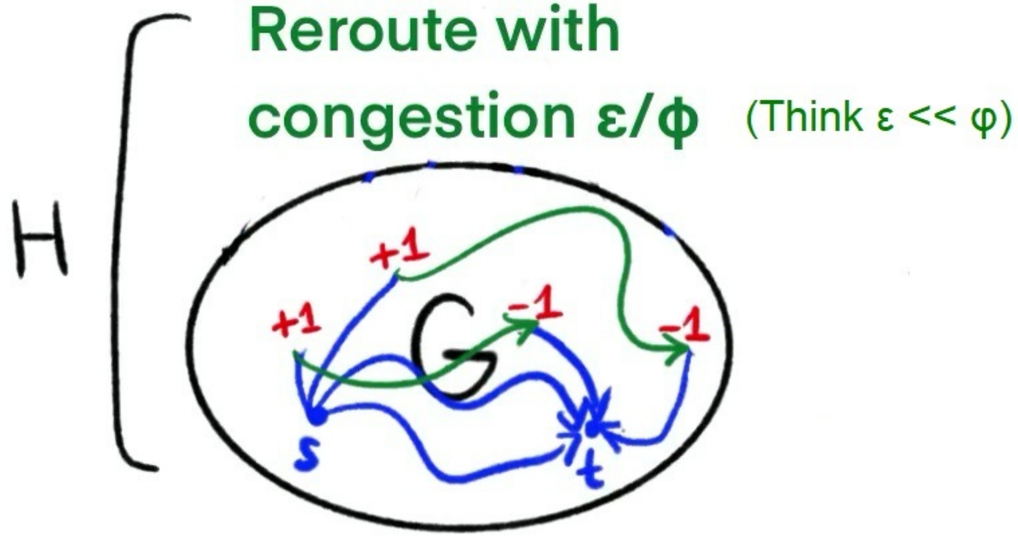

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In general, G may not be an expander.

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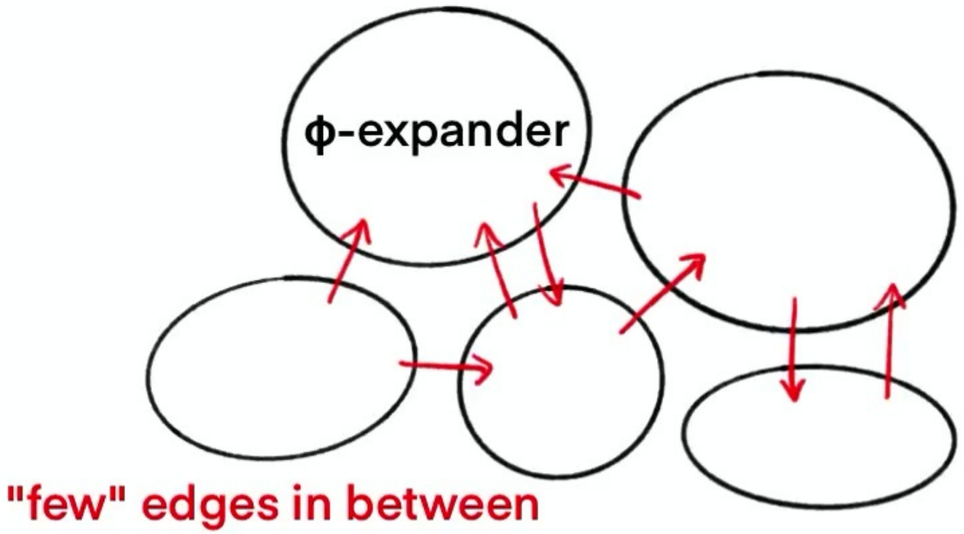

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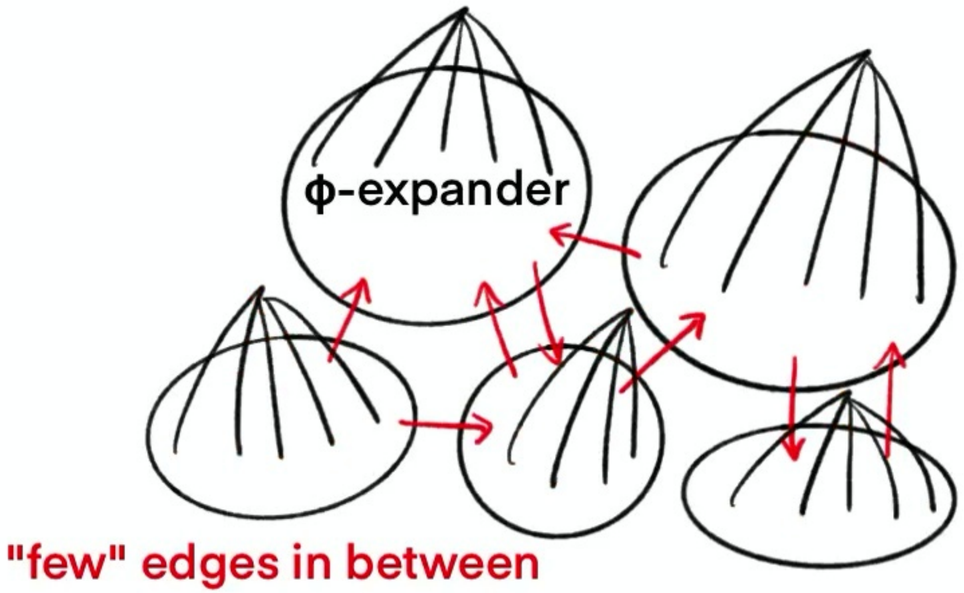

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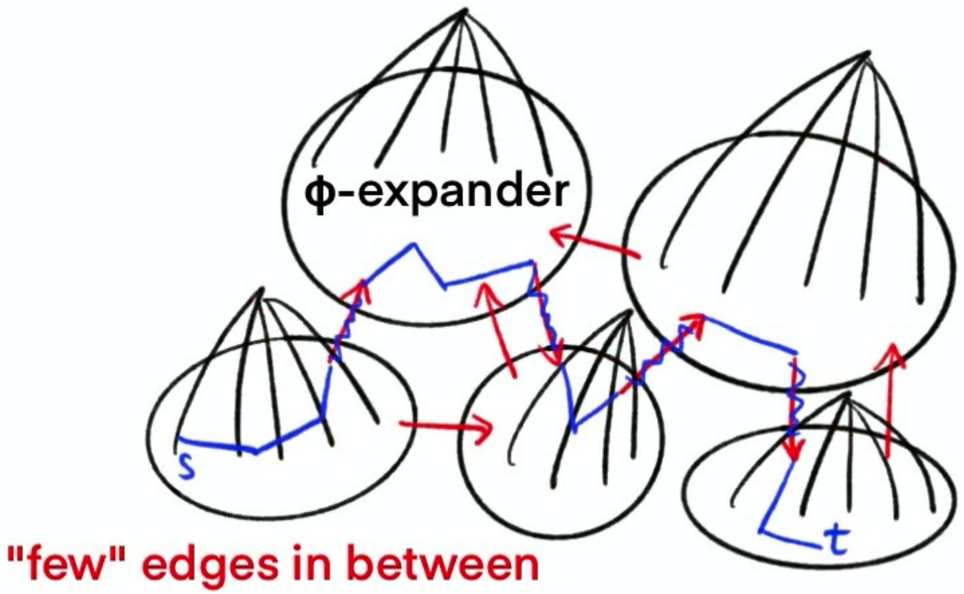

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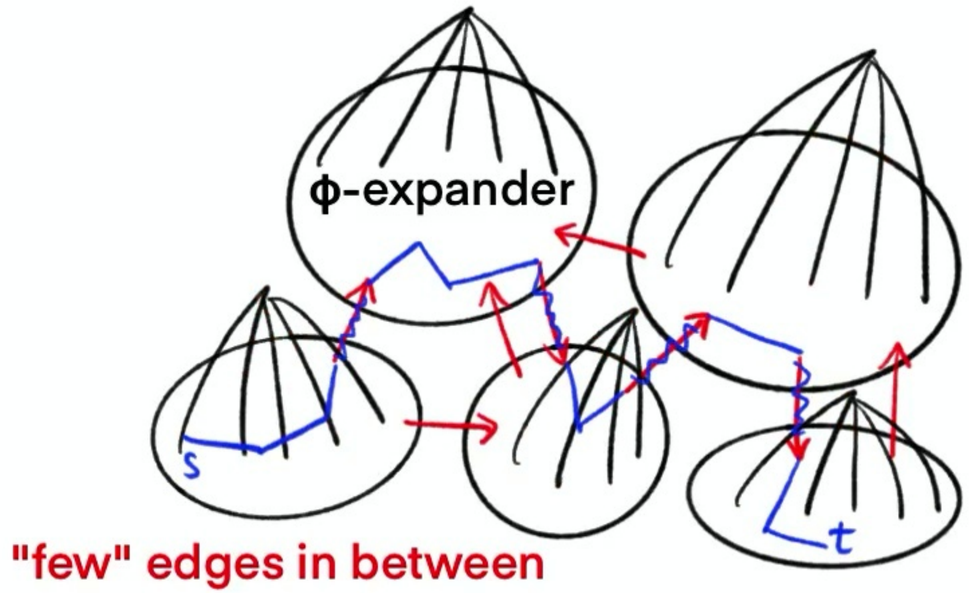

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What if it uses too many edges in between?

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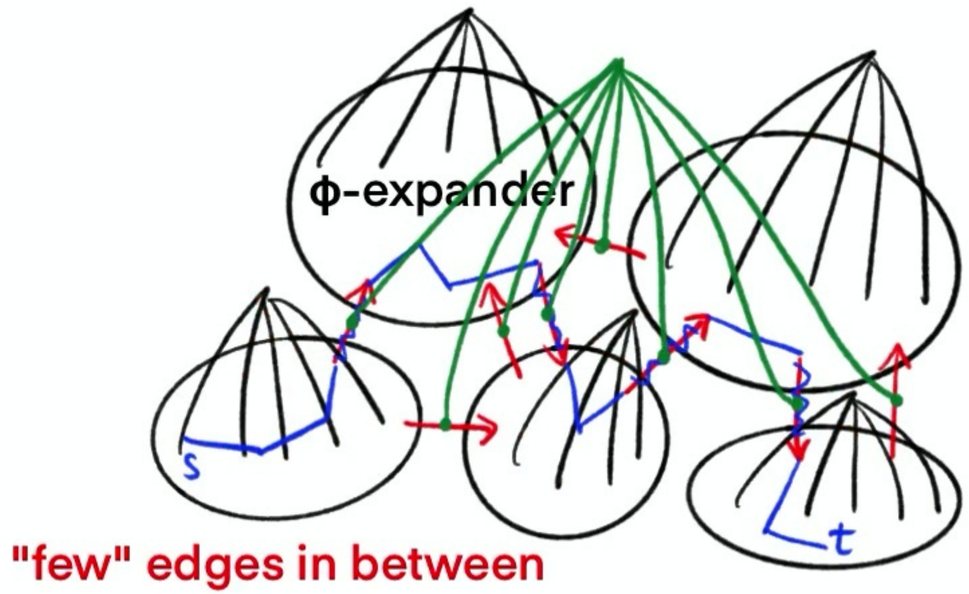

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Add another star. If these edges "expand", then done. Otherwise, next-level decomposition.

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DAG instances are quite general, includes bipartite matching.

Missing Details

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Constant-approximation is sufficient

- re-run algorithm on residual graph to halve the remaining flow
- repeat O(log n) times for 1/poly(n) error

Conclusion

Combinatorial structure for max-flow: short paths, star preconditioning, expanders and DAGs

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Open: combinatorial beyond n^2? (And Goldberg-Rao's $m^{1.5}$)
- already hard on DAGs!