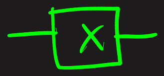
Topics

- Review matrices
- Hermitian / unitary
- eigenvalues / eigenvectors
- single qubit gates / states
- python and Qiskit



Review matrices:

M × N dimensional object

$$\begin{array}{cc} & \longleftarrow N \longrightarrow \\ \begin{array}{c}\uparrow\\ M\\ \downarrow\end{array} & \begin{pmatrix} m_{00} & m_{01} & m_{02} & \cdots \\ m_{10} & & & \\ m_{20} & & & \\ \vdots & & & \end{pmatrix}\end{array}$$

Vector is just $\overset{\text{column vector}}{\begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \end{pmatrix}}$, $\overset{\text{row vector}}{\begin{pmatrix} V_0 & V_1 & V_2 & \cdots \end{pmatrix}}$, $\vec{V} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

Specialized notation for vectors: Dirac Bra-ket Notation

$|\varphi_1\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\langle \varphi_2| = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ $\overset{\text{ket}}{|\psi\rangle}$ = column vector , $\overset{\text{bra}}{\langle\psi|}$ = row vector

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \overset{1}{\cancel{1\cdot 1}} + \overset{2}{\cancel{1\cdot 2}} + \overset{3}{\cancel{1\cdot 3}} = 6$$

$$\langle \varphi_2 | \varphi_1 \rangle = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6$$

$$(|\psi\rangle)^{*T} = (|\psi\rangle)^{\dagger}$$ ← Hermitian conjugate

$4^* = 4$, $(4+i)^* = 4-i$, $|\psi_1\rangle = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $|\psi_1\rangle^* = \begin{pmatrix}1^*\\2^*\\3^*\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} = |\psi_1\rangle$

$$|\psi_3\rangle = \begin{pmatrix}1+i\\2-3i\\3+4i\end{pmatrix},\quad |\psi_3\rangle^* = \begin{pmatrix}1-i\\2+3i\\3-4i\end{pmatrix}$$

$$|\psi_3\rangle^\dagger = \begin{pmatrix}1-i\\2+3i\\3-4i\end{pmatrix}^T = \begin{pmatrix}1-i & 2+3i & 3-4i\end{pmatrix}$$

$$(|\psi_3\rangle^\dagger)^\dagger = |\psi_3\rangle$$

Scalar x matrix:

$$2\ ,\ M = \begin{pmatrix}1&2\\3&4\end{pmatrix} \Rightarrow 2\cdot M = \begin{pmatrix}2\cdot 1 & 2\cdot 2\\ 2\cdot 3 & 2\cdot 4\end{pmatrix} = \begin{pmatrix}2&4\\6&8\end{pmatrix}$$

matrix x matrix:

$$M = \begin{pmatrix}1&2\\3&4\end{pmatrix},\ N = \begin{pmatrix}2&1\\1&2\end{pmatrix},\quad MN = \begin{pmatrix}\boxed{1\ \ 2}\\ \boxed{3\ \ 4}\end{pmatrix}\begin{pmatrix}2&1\\1&2\end{pmatrix} = \begin{pmatrix}\boxed{1\cdot 2+2\cdot 1} & \boxed{1\cdot 1+2\cdot 2}\\ \boxed{3\cdot 2+4\cdot 1} & \boxed{3\cdot 1+4\cdot 2}\end{pmatrix} = \begin{pmatrix}4&5\\10&11\end{pmatrix}$$

$$NM = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}5 & 8\\ 7 & 10\end{pmatrix}$$

$$MN \neq NM \,, \quad \underbrace{[M, N]}_{\text{Commutator}} = MN - NM = 0$$

Vector × matrix :

$$v = \begin{pmatrix}1\\ 0\end{pmatrix} , \quad M = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$vM = \xcancel{\begin{pmatrix}1\\ 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}$$

$$Mv = \begin{pmatrix}\boxed{0 \quad 1}\\ 1 \quad 0\end{pmatrix}\begin{pmatrix}\boxed{\begin{matrix}1\\ 0\end{matrix}}\end{pmatrix} = \begin{pmatrix}\boxed{0}\\ 1\end{pmatrix}$$

$$w = \begin{pmatrix}2\\ 1\end{pmatrix} , \quad N = \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$$

$$Nw = \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix} = \begin{pmatrix}1 \cdot 2 + 2 \cdot 1\\ 1 \cdot 2 + 3 \cdot 1\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

Hermitian / Unitary Matrices

Both square matrices $(M \times M)$

Hermitian: $A^\dagger = A$, Ex: Pauli Matrices $(\sigma_x, \sigma_y, \sigma_z, \mathbb{1})$

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$\sigma_y = Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad Y^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = Y$$

$$\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{1}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

Unitary: $A^\dagger A = \mathbb{1} \Rightarrow A^\dagger = A^{-1}$, $A^{-1} A = \mathbb{1}$

$\mathbb{1} M = M$, $1 \times 4 = 4$

$$X \cdot X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

$$Y \cdot Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

$$i = \sqrt{-1}\,, \quad i \cdot i = \sqrt{-1}^{\,2} = -1$$

$$Z \cdot Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

Turn a Hermitian matrix into a unitary matrix:

$$H = H^\dagger$$

$$U = e^{iH}, \quad U^\dagger = \left(e^{iH}\right)^\dagger = e^{-iH^\dagger} = e^{-iH}$$

$$UU^\dagger = e^{iH} e^{-iH} = e^{iH-iH} = e^{0} = \mathbb{1}$$

Hermitian matrices describe measurements on a system

Spectral Theorem: says Hermitian matrices have real eigenvalues.

Unitary matrices describe evolution of a system.

Spectral Theorem: Says unitary matrices have eigenvalues with unit absolute value $|a| = 1$.

proper, character

Eigenvectors / Eigenvalues :

column vector

Let A is $N \times N$ matrix, $|\psi\rangle$ is an eigenvector of A with eigenvalue λ

if $|\psi\rangle \neq 0$ and $\underline{A|\psi\rangle = \lambda|\psi\rangle}$.

To solve the eigensystem write

$$\boxed{(A - \mathbb{1}\lambda)|\psi\rangle = 0}$$

$\cancel{|\psi\rangle = 0}$ or $(A - \mathbb{1}\lambda) = 0$

if $\underbrace{\cancel{(A-\mathbb{1}\lambda)^{-1}}\,\cancel{(A-\mathbb{1}\lambda)}}_{\mathbb{1}}|\psi\rangle = \cancel{(A-\mathbb{1}\lambda)^{-1}\cdot 0} \rightarrow 0$

$\Rightarrow |\psi\rangle = 0$

Only exist solutions with non zero $|\varphi\rangle$ if $(A-\mathbb{1}\lambda)$ cannot be inverted.

If $\det(\text{matrix})=0$ then cannot be inverted.

$$\det(A-\lambda\mathbb{1})=0$$ Characteristic Equation

Example for 2x2:

$$A=\begin{pmatrix}1&2\\1&0\end{pmatrix},\quad \det\left(\begin{pmatrix}1&2\\1&0\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right)=0$$

$$\Rightarrow \det\left(\begin{pmatrix}1-\lambda&2\\1&-\lambda\end{pmatrix}\right)=(1-\lambda)(-\lambda)-2=\lambda^2-\lambda-2$$

$$=(\lambda-2)(\lambda+1)=0$$

$$\Rightarrow \boxed{\lambda=2,-1}$$

$$\det\left(\begin{pmatrix}A&B\\C&D\end{pmatrix}\right)$$

$$=AD-BC$$

$$\boxed{\lambda = 2}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{(A - \mathbb{1}\cdot\lambda)}_{}|\varphi\rangle = 0$$

$$\left(\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \overbrace{\begin{pmatrix} 1-\lambda & 2-0 \\ 1-0 & 0-\lambda \end{pmatrix}}^{} \overset{\lambda=2}{=} \boxed{\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}}$$

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -\varphi_1 + 2\varphi_2 \\ \varphi_1 - 2\varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{-\varphi_1 + 2\varphi_2 = 0}, \text{ and } \varphi_1 - 2\varphi_2 = 0$$

$$\Rightarrow 2\varphi_2 = \varphi_1$$

$$\boxed{|\varphi\rangle = \text{const.} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$$A|\varphi\rangle = \lambda|\varphi\rangle \Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+2 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 \\ 2\cdot 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \checkmark$$

$\underline{\lambda = -1}$

$$(A - \lambda \mathbb{1})|\varphi\rangle = 0$$

$$\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2\varphi_1 + 2\varphi_2 \\ \varphi_1 + \varphi_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad 2\varphi_1 + 2\varphi_2 = 0 \quad \text{and} \quad \varphi_1 + \varphi_2 = 0$$

$$\Rightarrow \varphi_1 = -\varphi_2 \quad \Rightarrow \boxed{|\varphi\rangle = \text{const.} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ with eigenvalues $\lambda = 2, -1$ and eigenvectors $|\varphi\rangle = \text{const.} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$|0\rangle \, , \, |1\rangle$$

$$X|0\rangle = |1\rangle \, , \, X|1\rangle = |0\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \, , \quad |\alpha|^2 + |\beta|^2 = 1$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad X|0\rangle = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = |1\rangle$$

Hadamard

$$-\boxed{H}- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,$$

Phase

$$-\boxed{S}- = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$\pi/8$

$$, \quad -\boxed{T}- = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{1}\,\underset{\substack{\nwarrow \\ \text{eigenvector}}}{|0\rangle}$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1|1\rangle$$

$$X|\psi\rangle = \lambda|\psi\rangle \quad , \quad |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$X|\psi\rangle = 1|\psi\rangle$$

Topics:

- Review: Hermitian/unitary matrices, eigenvalues/vectors, single qubit gates/states
- Bloch sphere
- multi-qubit states/gates
- writing multi-qubit gates/states with Qiskit

Review

Hermitian: M is a matrix, $M^{*T} = \overset{\text{Hermitian conjugate}}{M^\dagger} = M$

Unitary: M is a matrix, $M^\dagger M = \mathbb{1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$

Eigenvalues/vectors: Let A be an N × N matrix

$\underbrace{A|\psi\rangle = \lambda|\psi\rangle}$, λ is an eigenvalue

$|\psi\rangle$ is an eigenvector.

$$\underbrace{\det(A - \lambda\mathbb{1}) = 0}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \det(A - \lambda\mathbb{1}) = \det\begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - 2$$

$$= \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2, -1$$

$\lambda = 2$

$\ldots \quad |\psi\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

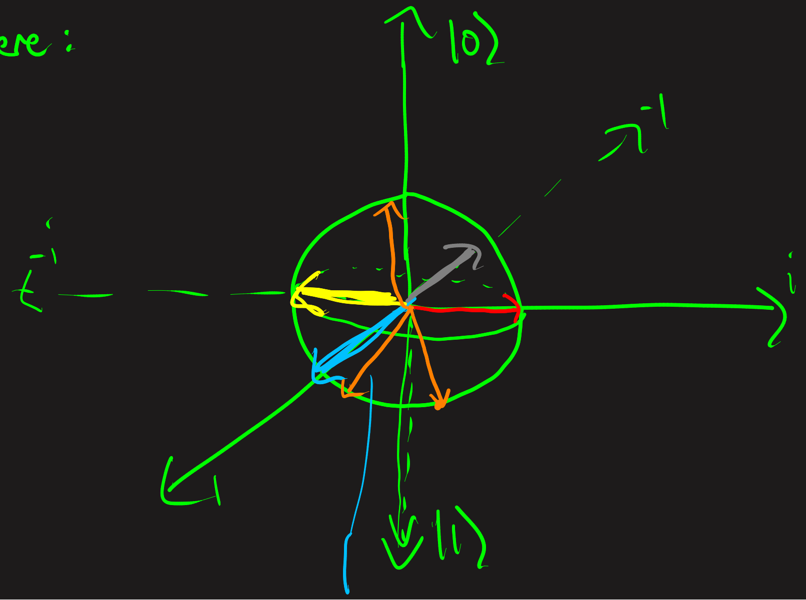
Pauli matrices: (Single qubit gates)

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X, \quad XX^\dagger = \mathbb{1}, \quad Y^\dagger = Y \ldots$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \ldots$$

Bloch Sphere:



$$|\psi\rangle = \left(\frac{1}{\sqrt{5}} |0\rangle + i\sqrt{\frac{4}{5}} |1\rangle \right)$$

$$|\psi\rangle = \left(|0\rangle + i|1\rangle \right) \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = \left(|0\rangle - |1\rangle \right) \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = \left(|0\rangle - i|1\rangle \right) \frac{1}{\sqrt{2}}$$

$\boxed{Ex}$ $|\psi_1\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle \overset{\downarrow}{+} \frac{1}{\sqrt{2}}|1\rangle\right)$,

To measure something

$$|\psi_1\rangle = e^{i\phi}|\psi_1\rangle$$

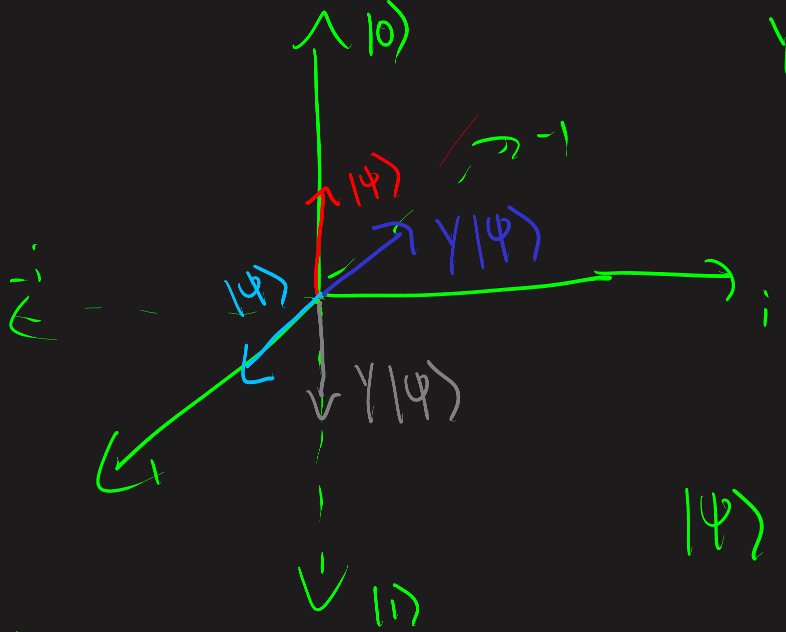
$$\underset{\uparrow\ (|\psi_1\rangle)^+}{\langle\psi_1|}\mathbb{1}|\psi_1\rangle = \langle\psi_1|\psi_1\rangle =$$

$$\langle\psi_1| = e^{-i\phi}(|\psi_1\rangle)^+$$

$$\cancel{e^{i\phi}e^{-i\phi}} \to 1 \ \langle\psi_1|\psi_1\rangle$$

$|\psi = |0\rangle$

Qubit states when we apply gates



$$Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$Y|0\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\i\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \underbrace{i|1\rangle}$$

local phase = i

$$|\psi\rangle = \underbrace{e^{i\phi}}_{\text{global phase}}|\psi\rangle \quad , \quad |\psi\rangle = \left(\overbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}}|1\rangle\right)\underbrace{e^{i\phi}}$$

$$X|0\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$$

$$Y|\psi\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{i}{\sqrt{2}}\left(\overset{|1\rangle}{\begin{pmatrix}0\\1\end{pmatrix}} - \overset{|0\rangle}{\begin{pmatrix}1\\0\end{pmatrix}}\right)$$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

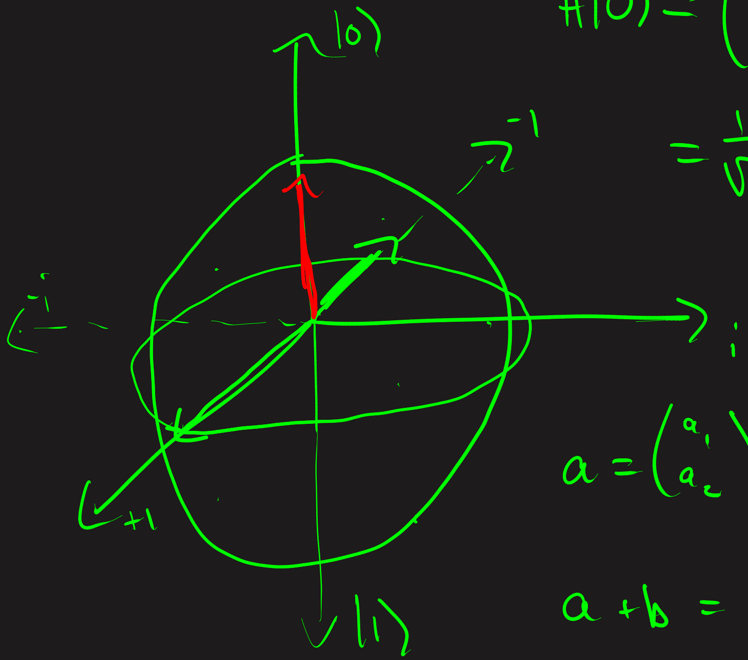
How does H act on $|0\rangle$ and $|1\rangle$?

- what are the Bloch sphere pictures?

$H|0\rangle$, $H|1\rangle$

$$H|1\rangle = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\Bigg(\underset{|0\rangle}{\begin{pmatrix}1\\0\end{pmatrix}} - \underset{|1\rangle}{\begin{pmatrix}0\\1\end{pmatrix}}\Bigg)$$



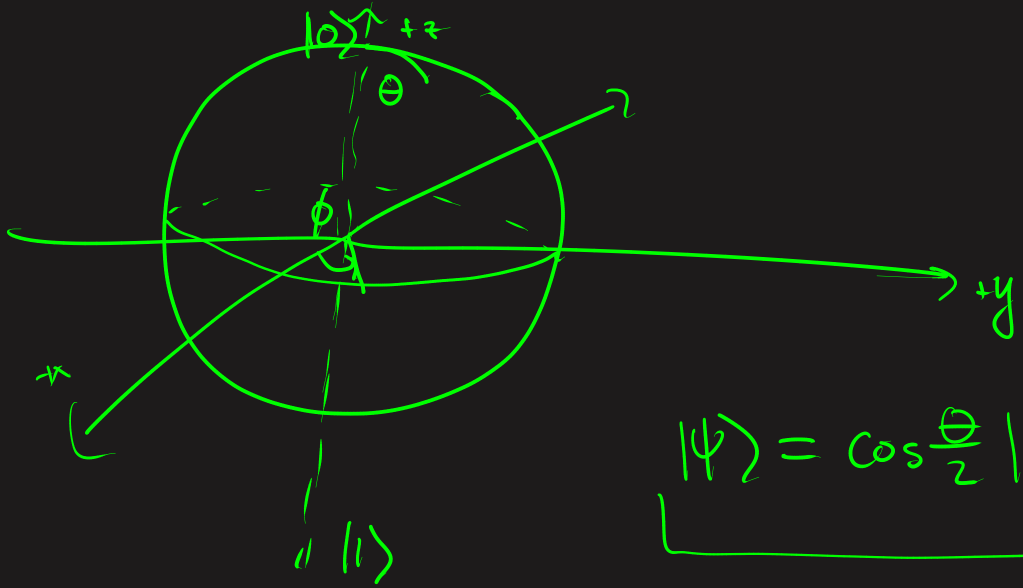
$$H|0\rangle = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\Bigg(\underset{|0\rangle}{\begin{pmatrix}1\\0\end{pmatrix}} + \underset{|1\rangle}{\begin{pmatrix}0\\1\end{pmatrix}}\Bigg)$$

$a = \begin{pmatrix}a_1\\a_2\end{pmatrix}$, $b = \begin{pmatrix}b_1\\b_2\end{pmatrix}$

$a + b = \begin{pmatrix}a_1 + b_1\\ a_2 + b_2\end{pmatrix}$

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}$$



$$|\psi\rangle = \underbrace{\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle}_{\text{most general single qubit state}}$$

$$\theta = 0 \Rightarrow |\psi\rangle = |0\rangle$$

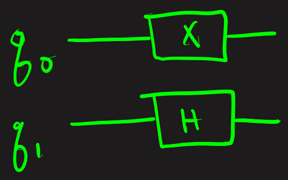
$$\theta = \pi \Rightarrow |\psi\rangle = |1\rangle$$

Or equiv. $|\psi\rangle = e^{i\lambda}\left(|\psi\rangle\right)$

Most general single qubit gate:

$$U(\underbrace{\theta, \phi}_{\text{coords on sphere}}, \underbrace{\lambda}_{\text{global phase}}) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}$$

$$U(\pi, \tfrac{-\pi}{2}, \tfrac{\pi}{2}) = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(\pi, 0, 0) = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For N qubits, need to have a state space that is 2^N-dimensional.

- gates have to $2^N \times 2^N$ unitary matrices.

$$N=2, \quad \text{4-dimensions} \quad 4\updownarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |\psi\rangle, \quad \tilde{U} = {}^{4}\!\downarrow \overset{\longrightarrow 4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

Two qubits: $\underset{a|0\rangle + b|1\rangle}{|\psi_1\rangle}$, $\underset{c|0\rangle + d|1\rangle}{|\psi_2\rangle}$

$|\psi_{tot}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, Basis states for 2-qubit are represented by

$$\underset{\text{1st qubit}\quad\text{2nd qubit}}{|00\rangle} ,\ |01\rangle ,\ |10\rangle ,\ |11\rangle$$

$$|\psi_{tot}\rangle = C_{00}\underline{|00\rangle} + C_{01}\underline{|01\rangle} + C_{10}\underline{|10\rangle} + C_{11}\underline{|11\rangle} ,\quad |C_{00}|^2 + |C_{01}|^2 + |C_{10}|^2 + |C_{11}|^2 =$$

$$V = \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} ,\quad W = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix} ,\quad \underset{2\text{-d}}{V} \overset{\text{tensor product}}{\otimes} \underset{2\text{-d}}{W} = \begin{pmatrix} V_0 \cdot W \\ V_1 \cdot W \end{pmatrix} = \underbrace{\begin{pmatrix} V_0 W_0 \\ V_0 W_1 \\ V_1 W_0 \\ V_1 W_1 \end{pmatrix}}_{4\text{-dimensional}}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$2^N \times 2^N\text{-dim gates} \overset{N=4}{\Longrightarrow} 4 \times 4\text{-dim gates (unitary)}$$

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}, \quad N = \begin{pmatrix} n_{00} & n_{01} \\ n_{10} & n_{11} \end{pmatrix}$$

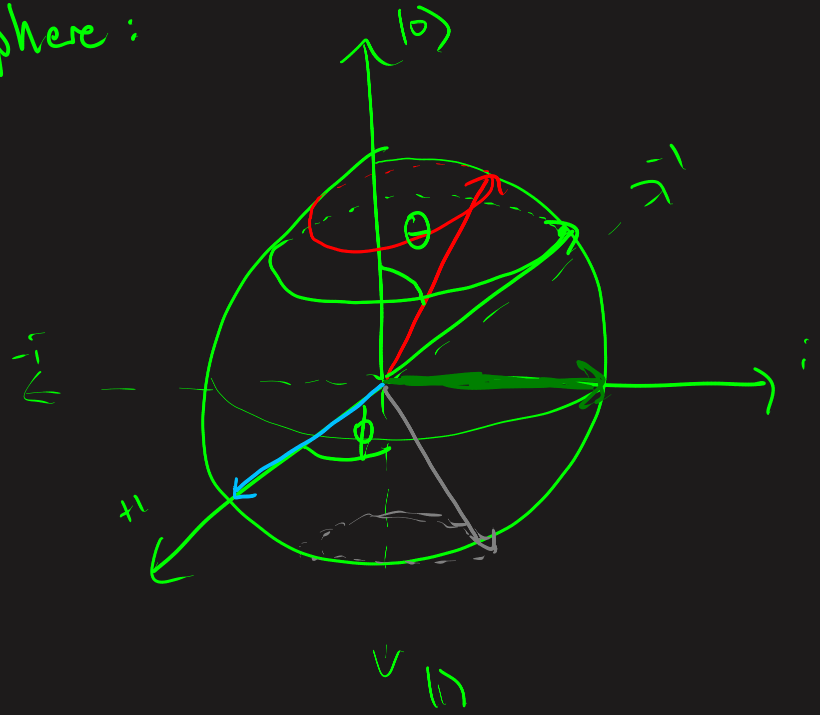
$$M \otimes N = \begin{pmatrix} m_{00} N & m_{01} N \\ m_{10} N & m_{11} N \end{pmatrix} = 4\Big\downarrow \overset{\xrightarrow{4}}{\begin{pmatrix} m_{00}n_{00} & m_{00}n_{01} & m_{01}n_{00} & m_{01}n_{01} \\ m_{00}n_{10} & m_{00}n_{11} & m_{01}n_{10} & m_{01}n_{11} \\ m_{10}n_{00} & m_{10}n_{01} & m_{11}n_{00} & m_{11}n_{01} \\ m_{10}n_{10} & m_{10}n_{11} & m_{11}n_{10} & m_{11}n_{11} \end{pmatrix}}$$

Not all two-qubit gates can be written as tensor-product.

Topics:

- Review: Bloch sphere, U-gate, Qiskit
- Multi-qubit gates/states, using Qiskit
- Qiskit exercise

Bloch Sphere:



$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$a > b$

$a < b$

$a = b$, $|\psi\rangle = a|0\rangle + a|1\rangle$

$= a|0\rangle + ia|1\rangle$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler's ID

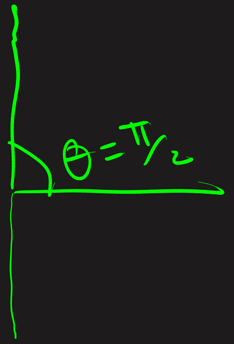
U-gate (most general single-qubit gate)

$$U(\theta, \phi, \lambda) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i(\phi+\lambda)}\cos(\theta/2) \end{pmatrix}$$

$\theta = \pi$

$$\begin{pmatrix} 0 & -e^{i\lambda} \\ e^{i\phi} & 0 \end{pmatrix}$$

$U|0\rangle$ = any other state



$\cos\theta = \frac{x}{z}$

$\cos(\frac{\pi}{2}) = 0$

$\cos(\frac{3\pi}{2}) = 0$

$\cos(0) = 1$

$\cos(\pi) = -1$

$\sin\theta = \frac{y}{z}$

$\Rightarrow \sin 0 = 0$

$\sin(\pi) = 0$

$\sin(\frac{\pi}{2}) = 1$

$\sin(\frac{3\pi}{2}) = -1$

Most common single-qubit gates:

Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{Phase} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}$$

* All unitary. $UU^{\dagger} = \mathbb{1}$, $U^{\dagger} = U^{-1}$

$\underline{\theta = \pi} \Rightarrow \quad U = \begin{pmatrix} 0 & -e^{i\lambda} \\ e^{i\phi} & 0 \end{pmatrix}$ $\qquad e^{ix} = \cos x + i\sin x$

$-e^{i\lambda} = -\cos\lambda - i\sin\lambda = 1 \quad , \quad \underline{\lambda = \pi}$ ✓

$e^{i\phi} = \underbrace{\cos\phi}_{+1} + \cancel{i\sin\phi} = 1 \quad , \quad \phi = 0$

$\Rightarrow U(\theta=\pi, \lambda=\pi, \phi=0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$

What Z from $U(\theta, \phi, \lambda)$.

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad U\Big|_{\theta=0} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\phi+\lambda)} \end{pmatrix}$

$e^{i(\phi+\lambda)} = \cancel{\cos(\overbrace{\phi+\lambda}^{\pi \to -1})} + i\cancel{\sin(\overbrace{\phi+\lambda}^{\pi \to 0})} \quad , \quad \phi+\lambda = -1 \Rightarrow U\Big|_{\theta=0,\ \phi+\lambda=1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Homework: Show how U reduces to two other gates (other than X, Z).

Multi-qubit gates/states:

single qubit

$$|\psi_1\rangle = a|0\rangle + b|1\rangle, \quad |\psi_2\rangle = c|0\rangle + d|1\rangle$$

$$|\psi_{tot}\rangle = |\psi_1\rangle \underset{\substack{\uparrow \\ \text{tensor product}}}{\otimes} |\psi_2\rangle$$

$$\text{1-qubit: } |0\rangle, |1\rangle \ : \ |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \ |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\text{2-qubit: } \underbrace{|0\rangle}_{2d} \otimes \underbrace{|0\rangle}_{2d}, \ |0\rangle \otimes |1\rangle, \ |1\rangle \otimes |0\rangle, \ |1\rangle \otimes |1\rangle$$

$$V = \begin{pmatrix}v_0\\v_1\end{pmatrix}, \ W = \begin{pmatrix}w_0\\w_1\end{pmatrix} \Rightarrow V \otimes W = \begin{pmatrix}v_0 W\\ v_1 W\end{pmatrix} = \begin{pmatrix}v_0 w_0\\ v_0 w_1\\ v_1 w_0\\ v_1 w_1\end{pmatrix}$$

Mean same thing

$$|0\rangle \otimes |0\rangle = |00\rangle, \quad \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 \cdot \begin{pmatrix}1\\0\end{pmatrix}\\ 0 \cdot \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = |01\rangle, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = |10\rangle, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|1\rangle \otimes |1\rangle = |11\rangle, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\overbrace{\langle 00|10\rangle}^{\text{'Dot' product}} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right),$$

Can I write this as tensor product of two states? No!

Bell pair (entangled)

$$|\psi_2\rangle = \frac{1}{\sqrt{4}} \left(|00\rangle + |10\rangle + |01\rangle + |11\rangle \right)$$

$$(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = \underbrace{|00\rangle + |01\rangle + |10\rangle + |11\rangle}_{}$$

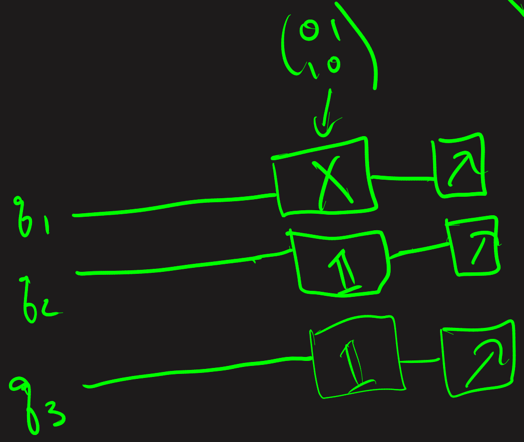
multi-qubit gates:

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}, \quad N = \begin{pmatrix} n_{00} & n_{01} \\ n_{10} & n_{11} \end{pmatrix}$$

unitary

4×4 ^ matrix

$$M \otimes N = \begin{pmatrix} m_{00} N & m_{01} N \\ m_{10} N & m_{11} N \end{pmatrix} = \begin{pmatrix} m_{00}n_{00} & m_{00}n_{01} & m_{01}n_{00} & m_{01}n_{01} & \cdots \\ m_{00}n_{10} & m_{00}n_{11} & m_{01}n_{10} & m_{01}n_{11} & \\ \vdots & & \ddots & & \end{pmatrix}$$



$$X \otimes \mathbb{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$X \otimes \mathbb{1} \otimes \mathbb{1} = \qquad\qquad 2^3 = 8$$

To describe 1 qubit, need 2 states, 2×2 gates

2 qubits, need 2×2 states, 4×4 gates

⋮

n qubits 2^n states, $2^n \times 2^n$ gates

Also two-qubit gates that are not tensor-product gates.

$$\underbrace{\text{CNOT}}_{\text{entangling gate}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \neq M \otimes N$$