Simulation of SNI for Bias Estimation

We cannot compute $\hat{\mathcal{N}}_{st}^{-1}$ exactly, and using Monte Carlo to directly estimate $\hat{\mathcal{N}}_{st}^{-1}\mathcal{N}_{st}$ introduces statistical errors due to the finite size of samples M, which may obscure the information about the bias. Therefore, we use a small trick in the SNI.py: we simulate the relative error between $\hat{\mathcal{N}}_{st}$ and the true $\mathcal{N}_{st}$, as shown below:

$$\begin{aligned}
\hat{\mathcal{N}_{st}}^{-1}\mathcal{N}_{st} &= \frac{1}{(1-\hat{P})\mathcal{I}+\hat{P}\mathcal{E}}\{(1-P)\mathcal{I}+P\mathcal{E}\} \\
&= \frac{1}{(1-\hat{P})\mathcal{I}+\hat{P}\mathcal{E}}\{(1-\hat{P})\mathcal{I}+\hat{P}\mathcal{E}+(P-\hat{P})(\mathcal{E}-\mathcal{I})\} \\
&= \mathcal{I}+\frac{1}{(1-\hat{P})\mathcal{I}+\hat{P}\mathcal{E}}\{(P-\hat{P})(\mathcal{E}-\mathcal{I})\} \\
&= \mathcal{I}+\sum_{k=0}^{\infty}\frac{(-1)^k\hat{P}^k}{(1-\hat{P})^{k+1}}\mathcal{E}^k\{(P-\hat{P})(\mathcal{E}-\mathcal{I})\} \\
&= \mathcal{I}+(P-\hat{P})\sum_{k=0}^{\infty}\{\frac{(-1)^k\hat{P}^k}{(1-\hat{P})^{k+1}}\mathcal{E}^{k+1}-\frac{(-1)^k\hat{P}^k}{(1-\hat{P})^{k+1}}\mathcal{E}^k\} \\
&= (1+\frac{P-\hat{P}}{1-\hat{P}})\mathcal{I}+(P-\hat{P})\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\hat{P}^{k-1}}{(1-\hat{P})^{k+1}}\mathcal{E}^k
\end{aligned} \tag{1}$$

We can use Monte Carlo simulation to estimate the second term, $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\hat{P}^{k-1}}{(1-\hat{P})^{k+1}}\mathcal{E}^k$, which yield the estimate of $\langle O\rangle_{circuit}$, ultimately allows us to approximate the absolute value of the bias:

$$|Bias| = |\frac{P-\hat{P}}{1-\hat{P}}\langle O\rangle_I+(P-\hat{P})\langle O\rangle_{circuit}| \tag{2}$$