Simulation of SNI for Bias Estimation

We cannot compute $\hat{\mathcal{N}}_{st}^{-1}\mathcal{N}_{st}$ exactly, and using Monte Carlo to directly estimate $\hat{\mathcal{N}}_{st}^{-1}\mathcal{N}_{st}$ introduces statistical errors due to the finite size of samples M, which may obscure the information about the bias. Therefore, we use a small trick in the SNI.py: we simulate the relative error between $\hat{\mathcal{N}}_{st}$ and the true $\mathcal{N}_{st}$, as shown below:

$$
\begin{aligned}
\hat{\mathcal{N}}_{st}^{-1}\mathcal{N}_{st} &= \frac{1}{(1-\hat{P})\mathcal{I} + \hat{P}\mathcal{E}}\{(1-P)\mathcal{I} + P\mathcal{E}\} \\
&= \frac{1}{(1-\hat{P})\mathcal{I} + \hat{P}\mathcal{E}}\{(1-\hat{P})\mathcal{I} + \hat{P}\mathcal{E} + (P-\hat{P})(\mathcal{E}-\mathcal{I})\} \\
&= \mathcal{I} + \frac{1}{(1-\hat{P})\mathcal{I} + \hat{P}\mathcal{E}}\{(P-\hat{P})(\mathcal{E}-\mathcal{I})\} \\
&= \mathcal{I} + \sum_{k=0}^{\infty}\frac{(-1)^k\hat{P}^k}{(1-\hat{P})^{k+1}}\mathcal{E}^k\{(P-\hat{P})(\mathcal{E}-\mathcal{I})\} \\
&= \mathcal{I} + (P-\hat{P})\sum_{k=0}^{\infty}\{\frac{(-1)^k\hat{P}^k}{(1-\hat{P})^{k+1}}\mathcal{E}^{k+1} - \frac{(-1)^k\hat{P}^k}{(1-\hat{P})^{k+1}}\mathcal{E}^k\} \\
&= (1 + \frac{P-\hat{P}}{1-\hat{P}})\mathcal{I} + (P-\hat{P})\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\hat{P}^{k-1}}{(1-\hat{P})^{k+1}}\mathcal{E}^k
\end{aligned}
\tag{1}
$$

We can use Monte Carlo simulation to estimate the second term, $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\hat{P}^{k-1}}{(1-\hat{P})^{k+1}}\mathcal{E}^k$, which yield the estimate of $\langle O \rangle_{circuit}$, ultimately allows us to approximate the absolute value of the bias:

$$
|Bias| = |\frac{P-\hat{P}}{1-\hat{P}}\langle O \rangle_I + (P-\hat{P})\langle O \rangle_{circuit}|
\tag{2}
$$