# Stick-Slip Friction

## FYS-MEK1110 - Assignment 4

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**Initial values:**
$x(t_0) = 0$ at $t_0 = 0$
$x_b(t_0) = x(t_0) + b$ at $t_0 = 0$

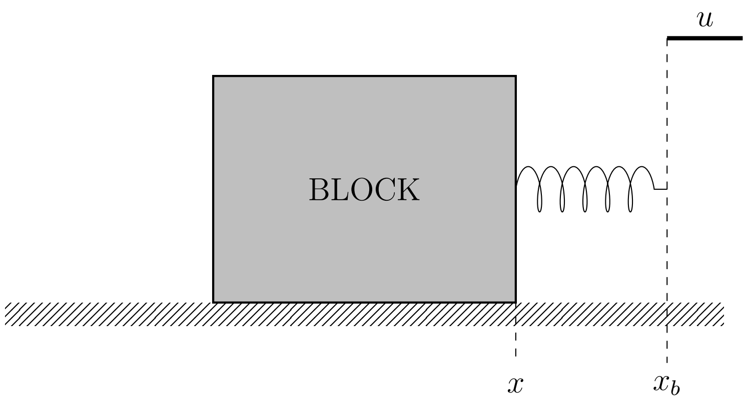
**Properties of block:**

- Mass: $m$
- Spring connection point $x$
- Static friction: $\mu_s$
- Dynamic friction: $\mu_d$

**Properties of spring:**

- Spring constant: $k$
- Equilibrium length: $b$
- Block connection point: $x$
- Velocity: $u$

Figure 1: Enviroment



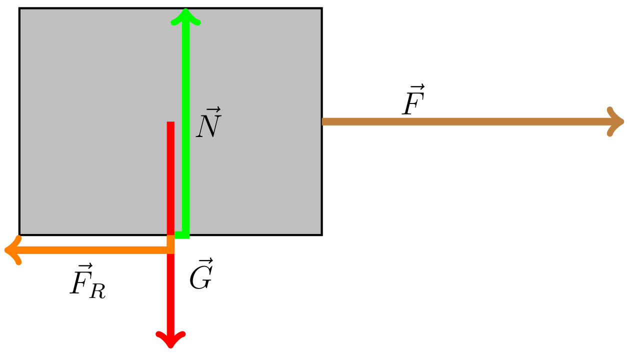
### 1.a - Draw a free body diagram for the block.

We have several forces acting on the block:

- $\vec{F}$: The force from the spring
- $\vec{F}_R$: The frictional force, static if stationary and dynamic if in motion
- $\vec{N}$: The normal force from the ground
- $\vec{G}$: The gravitational force

Here assuming the block in motion. The fact that the loose end of the spring is moving at a constant velocity $u$ does not directly translate into a constant velocity $u$ for the block. This gives $|\vec{F}| \geq |\vec{F}_R|$

Figure 2: Free Body Diagram




### 1.b - Find the position of $x_b$ as a function of time.

We know that we move the free end of the spring $x_b$ with a constant velocity $u$. This means that we can find $x_b(t)$ by integrating $u$ with respect to $t$.

$$x_b(t) = x_b(t_0) + \int_{t_0}^{t} u(t)\,dt = x_0 + b + u\int_{t_0}^{t} dt = b + u\int_{0}^{t} dt = b + ut$$

### 1.c - Show that the force on the block from the spring is $\vec{F} = k(x_b - x - b)\hat{\imath}$.

We are going to model the spring with a full spring force model. Because we are placing the origin at the attachment point of the spring $x$ the full model simplifies to

$$\vec{F} = k(r - L_0)\frac{\vec{r}}{r}$$

where $r$ is the length of the spring and $L_0$ is the springs equilibrium length. $r = x_b - x$ and $L_0 = b$ Also, $\frac{\vec{r}}{r}$ is the unit vector pointing in the direction of the spring, which in our case is in the $x$-direction, in other words $\hat{i}$.
This gives us a complete expression for the spring force:

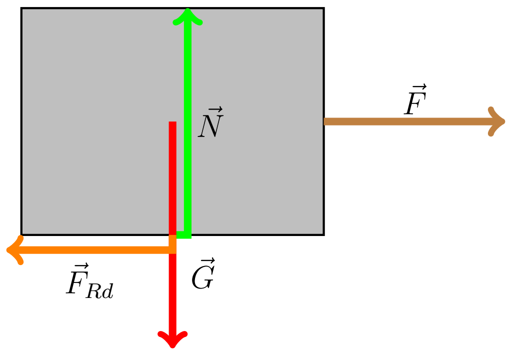
$$\vec{F} = k(x_b - x - b)\hat{i}$$

**1.d - Identify the forces acting on the block and draw a free-body diagram for the block in the stationary state.**

Forces acting on the block in the stationary state:

- $\vec{F} = k(x_b - x - b)\hat{i}$: The force from the spring
- $\vec{F}_{Rd} = -\mu_d|\vec{N}| = -\mu_d mg\hat{i}$: The dynamic friction
- $\vec{N} = mg\hat{j}$: The normal force from the ground
- $\vec{G} = -mg\hat{j}$: The gravitational force

Figure 3: Free Body Diagram - Constant velocity




**1.e - Introduce force models for all the forces acting on the block and find the normal force, $N$, on the block.**

I have already introduced force models for all the forces acting on the block, $F, F_{Rs}, N$ and $G$. Since we know there is no vertical displacement we can use Newton's second law to find the normal force $N$.

$$\sum F_y = N + G = ma_y = 0$$
$$N - mg = 0$$
$$N = mg$$

### 1.f - Find the acceleration of the block in the stationary state.

The acceleration of the block is per definition zero, because the block is moving at a constant velocity. Using Newton's second law of motion for the horizontal forces.

$$\sum F_x = ma_x \Leftrightarrow a_x = \frac{\sum F_x}{m} = \frac{F + F_{Rd}}{m} = \frac{F - \mu_d mg}{m} = \frac{F}{m} - \mu_d g = 0$$

Which is perfectly reasonable since the block has a constant velocity.

### 1.g - Find the elongation $\Delta L$ of the spring in the stationary state.

We know for a fact that since the block is moving at a constant velocity, the frictional force has to equal the spring force.

$$F = k(x_b - x - b) = k\Delta L = F_{Rd} = \mu_d N = \mu_d mg$$
$$\Delta L = \frac{1}{k}\mu_d mg$$

### 1.h - Find the position $x(t)$ of the block as a function of time in the stationary state.

We know that the acceleration of the block is 0, since it is moving at a constant velocity. Therefore the displacement $x(t)$ must be linear.

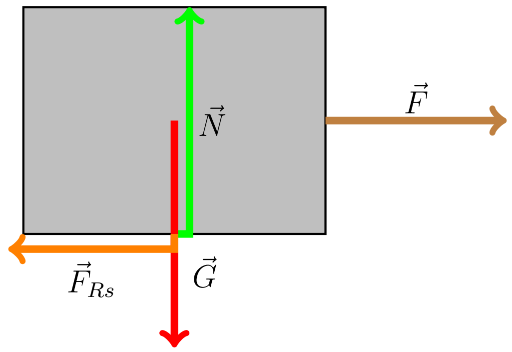
$$\Delta L(t) = x_b(t) - x(t) - b = \frac{1}{k}\mu_d mg$$
$$x(t) = \frac{1}{k}\mu_d mg + b - x_b(t) = \frac{1}{k}\mu_d mg + b - (b + ut) = \frac{1}{k}\mu_d mg - ut$$

### 1.i - Identify the forces acting on the block before the block starts moving.

Now assuming that the block starts at $x(t_0) = 0m$ with $v(t_0) = 0m/s$ at the time $t_0 = 0s$. Again, we still only have the forces $F_{Rs}, N, G$ and $F$. We also know that

$$F = F_{Rs} = k(x_b - x - b) \leq \mu_s N = \mu_s mg$$

Figure 4: Free Body Diagram - Before moving




The static friction is always equal to the spring force as long as the spring force does not exceed the maximum static friction. $F_{Rs} = F$

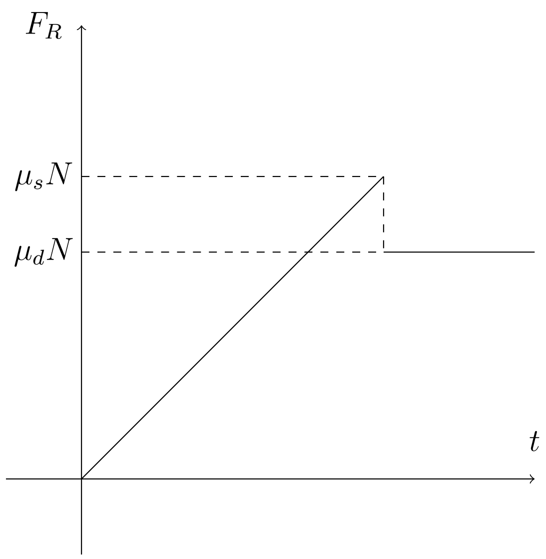
**1.j - Assume that the block starts at rest. Find the elongation $\Delta L$ of the spring the instant the block starts moving.**

The istant the block starts moving we have

$$F = k(x_b - x - b) = k\Delta L = \mu_s mg = F_{Rs}$$
$$\Delta L = \frac{1}{k}\mu_s mg$$

**1.k - Find the friction force as a function. Sketch the friction force as a function until some time after the block starts moving.**

Figure 5: Friction force sketch



### 1.l - The acceleration of the block immediately after it starts moving.

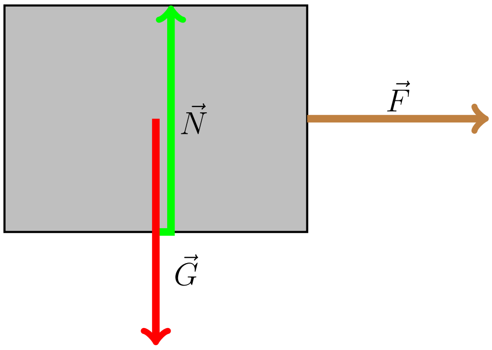
By using Newton's second law of motion we get:

$$\sum F = ma \Leftrightarrow a = \frac{F - F_R}{m} = \frac{k}{m}(x_b - x - b) - \mu_d g$$

### 1.m - Identify forces acting on the block.

With the frictional coefficients equal to zero, and the velocity $u = 0\text{m/s}$. The spring is in tension despite no velocity. We therefore have the following free body diagram:

Figure 6: Free Body Diagram - Zero friction




We have $G = mg$, $N = mg$ and $F = k(x_b - x - b)$.

### 1.m - Find an expression for the horizontal acceleration of the block

With the frictional coefficients equal to zero we find that the acceleration of the block is:

$$a = \frac{k}{m}(x_b - x - b)$$

**1.o - Show that $x(t) = \frac{v_0}{w} \sin wt$.**

Assuming that the block starts with velocity $v(t_0) = v(0) = v_0$. We can solve a non-homogenous differential equation:

$$
\begin{aligned}
a = x''(t) &= \frac{k}{m}(x_b - x(t) - b) \\
x''(t) + \frac{k}{m}x(t) + \frac{k}{m}(b) &= \frac{k}{m}x_b(t) \\
\text{Characteristic equation:} \\
r^2 + \frac{k}{m} \\
r = \sqrt{-\frac{k}{m}} \\
\beta = \sqrt{\frac{k}{m}} = \omega \\
a = 0 \\
\text{General solution:} \\
x(t) = Ce^{at}\cos(\beta t) + De^{at}\sin(\beta t) \\
x(t) = C\cos(\beta t) + D\sin(\beta t)\text{Initial value problem:} \\
x(0) = C\cos(0) + D\sin(0) = 0 \\
= C = 0 \\
x'(0) = 0
\end{aligned}
$$

**1.p - Write a numerical algorith for position and velocity.**

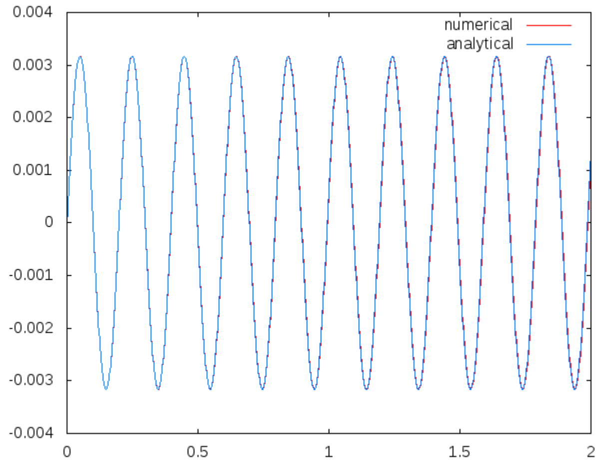
We know that the point $x_b$ does not move because it's velocity $u = 0$. This gives:

```
# EULER CROMER:

    v[i+1] = v[i] + dt * a[i]
    x[i+1] = x[i] + dt * v[i+1]
    a[i+1] = -k*(x[i] + b)
```

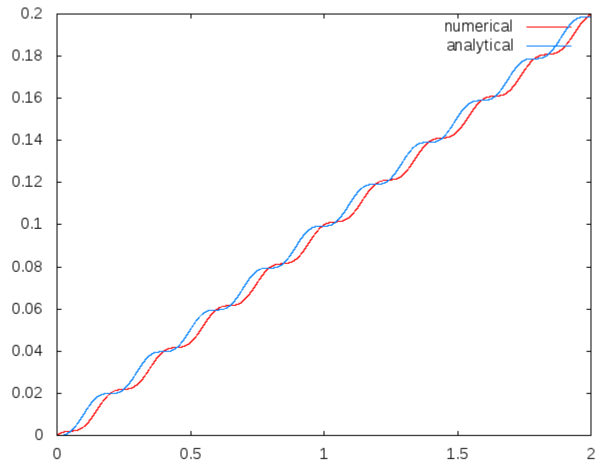
### 1.q - Implement the numerical algorithm

Figure 7: $u = 0$ m/s




### 1.r - $u = 0.1$ m/s and the block starts at rest.

Figure 8: $u = 0.1$ m/s




### 1.s - Including friction

Figure 9: u = 0.1, m = 0.1

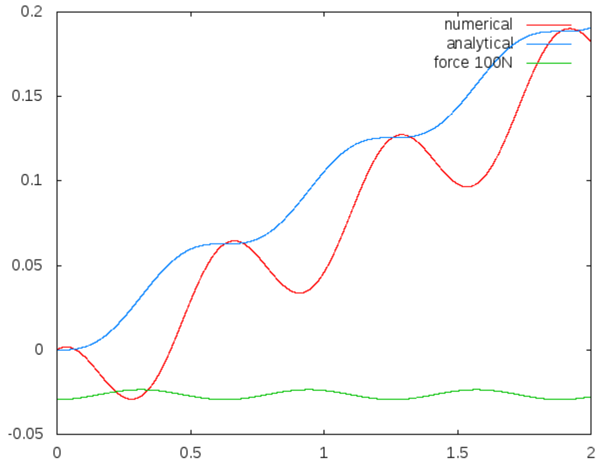
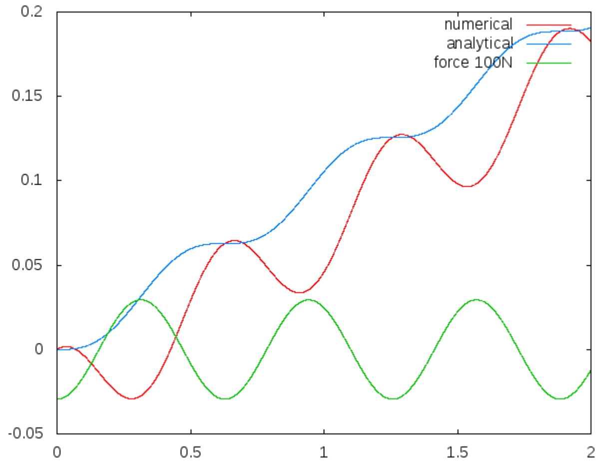



Figure 10: u = 0.1, m = 1.0



**Appendix - Python program.**

Only including the latest program. The ones used for the other exercises are just derived from this.

```
 from scitools.std import *

m = 0.1 #kg
k = 10. #N/m
b = 0.1 #m
T = 2
n = 10000
t = linspace(0, 2, n)
dt = float(T)/n
g = 9.81
mu_s = 0.6
mu_d = 0.3

def force(i):
    return k*(xb[i+1]-x[i+1]-b) - g*mu_d

def exact(t):
    w = sqrt(k/m)
    return u*t - (u / w) * sin(w*t)

def friction(i):


    if k*(xb[i] - x[i] - b) < mu_s*mg:
        return k*(xb[i]-x[i]-b)
    else:
        return mu_d*mg

def acceleration(i):
    return (k/m)*(xb[i+1] -x[i+1] - b) - mu_d*g




f = zeros(n)
v = zeros(n)
a = zeros(n)
x = zeros(n)
xb = zeros(n)
```

```
# Initial values

x[0] = 0
v[0] = 0.1
xb[0] = x[0] + b
a[0] = k*(xb[0] - x[0] - b)
u = 0.1

for i in range(n-1):

    v[i+1] = v[i] + dt * a[i]
    x[i+1] = x[i] + dt * v[i+1]
    xb[i+1] = xb[i] + dt*u
    a[i+1] = acceleration(i)
    f[i+1] = force(i)

# exact solution:


ex = exact(t)

plot(t, x, t, ex,t, f/100., legend=("numerical", "analytical", "force 100N
savefig("OppgaveS0.1.png")
```