ASSIGNMENT 1 — MAT-INF4160

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Problem 1. Recall that a *Bernstein polynomial* is a polynomial of the form

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-1}$$

where we typically let x live in the unit interval $[0,1]$. We define n to be *degree* of the polynomial, and we call B_i^n the *i'th Bernstein polynomial* of degree n where $i = 0, \dots, n$. Note that by the binomial theorem the $n+1$ Bernstein polynomials of degree n sum to 1 on the unit interval.

By application of the product rule and substitution rule we can differentiate $B_i^n(x)$ with respect to x which yields

$$\frac{d}{dx} B_i^n(x) = n \left(B_{i-1}^{n-1}(x) - B_i^{n-1}(x) \right)$$

We seek to show that $B_i^n(x)$ has a unique maximum in the interval $[0,1]$ with this maximum being $x = i/n$. Note that by the extreme value theorem, since B_i^n is continuous and $[0,1]$ is compact, we know that B_i^n does indeed attain a maximum in $[0,1]$ but it is not neccessarily unique.

In order to find the maximum, we look at where the derivative is zero. Consider the derivative of B_i^n where $n \neq 0$. It then suffices to look at the difference between lower-degree Bernstein polynomials:

$$\binom{n-1}{i-1} x^{i-1} (1-x)^{n-i} - \binom{n-1}{i} x^i (1-x)^{n-i-1} = 0$$

assuming $x \neq 0$ and $x \neq 1$ and we can cancel common terms which yields

$$\frac{1}{n-1} - x \left(\frac{1}{n-i} + \frac{1}{i} \right) = 0$$

where solving for x gives us $x = i/n$. We have now shown that under the condition that $x \neq 0$ and $x \neq 1$ the statement holds. Plugging in $x = 0$ and $x = 1$ verifies the claim for those two numbers as well.

Problem 2. We define the *Bernstein approximation* of a function $f : [0,1] \to \mathbb{R}$ of order n as the polynomial

$$g(x) = \sum_{i=0}^{n} f(i/n)\, B_i^n(x).$$

We want to show that if f is a polynomial of degree $m \leq n$, then g is also a polynomial of degree m. Note that differentiating a function of degree m, $m+1$ times, then you are left with zero. So, the question we need to ask is whether $g^{(m+1)}(x) = 0$. Make note of the fact that $g(x)$ is a form of Bézier curve where

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the control points are given as $c_i = f(i/n)$. We therefore, for convenience sake, represent g as

$$g(x) = \sum_{i=0}^{n} c_i B_i^n(x).$$

We have a formula for the $(m+1)$'th derivative of g on the interval $[0, 1]$, namely

$$g^{(m+1)}(x) = \frac{n!}{(n-(m+1))!} \sum_{i=0}^{n-(m+1)} \Delta^{m+1} c_i B_i^{n-(m+1)}(x) \tag{1}$$

where $\Delta^k c_i = \Delta^{k-1} c_{i+1} - \Delta^{k-1} c_i$ is the k'th forward difference of c_i. We can find a closed form expression for Δc_i which is given by

$$\Delta c_i = f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) = \sum_{j=1}^{m} a_j \frac{(i+1)^{j-1}}{n^j},$$

obtained by expanding the expression and using the fact that $(i+1)^j - i^j = (i+1)^{j-1}$. This can be extended to $\Delta^k c_i$ by applying the same tricks. This yields a closed form

$$\Delta^k c_i = \sum_{j=k}^{m} a_j \frac{(i+k)^{j-k}}{n^j}$$

it here follows that when $k = m+1$ we have $\Delta^k = 0$, hence eq. (1) is a sum over zero. Hence $g(x)$ when differentiated $m+1$ times is equal to zero, which means that $g(x)$ is a polynomial of degree m.