ASSIGNMENT 1 — MAT-INF4160

IVAR HAUGALØKKEN STANGEBY

Problem 1. Recall that a *Bernstein polynomial* is a polynomial of the form

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

where we typically let x live in the unit interval $[0,1]$. We define n to be *degree* of the polynomial, and we call B_i^n the *i'th Bernstein polynomial* of degree n where $i = 0, \ldots, n$. Note that by the binomial theorem the $n+1$ Bernstein polynomials of degree n sum to 1 on the unit interval.

By application of the product rule and substitution rule we can differentiate $B_i^n(x)$ with respect to x which yields

$$\frac{d}{dx} B_i^n(x) = n \left(B_{i-1}^{n-1}(x) - B_i^{n-1}(x) \right)$$

We seek to show that $B_i^n(x)$ has a unique maximum in the interval $[0,1]$ with this maximum being $x = i/n$. Note that by the extreme value theorem, since B_i^n is continuous and $[0,1]$ is compact, we know that B_i^n does indeed attain a maximum in $[0,1]$ but it is not neccessarily unique.

In order to find the maximum, we look at where the derivative is zero. Consider the derivative of B_i^n where $n \neq 0$. It then suffices to look at the difference between lower-degree Bernstein polynomials:

$$\binom{n-1}{i-1} x^{i-1}(1-x)^{n-i} - \binom{n-1}{i} x^i (1-x)^{n-i-1} = 0$$

assuming $x \neq 0$ and $x \neq 1$ and we can cancel common terms which yields

$$\frac{1}{n-1} - x\left(\frac{1}{n-i} + \frac{1}{i}\right) = 0$$

where solving for x gives us $x = i/n$. We have now shown that under the condition that $x \neq 0$ and $x \neq 1$ the statement holds. Plugging in $x = 0$ and $x = 1$ verifies the claim for those two numbers as well.

Problem 2. [1]

We define the *Bernstein approximation* of a function $f : [0,1] \to \mathbb{R}$ of order n as the polynomial

$$g(x) = \sum_{i=0}^{n} f(i/n)\, B_i^n(x).$$

Date: September 7, 2016.

[1] The solution here is based on the answer given to my question posted at [1] as well as the answer given at [2]. My initial idea to look at $m+1$'th derivatives of g did not lead to anything conclusive.

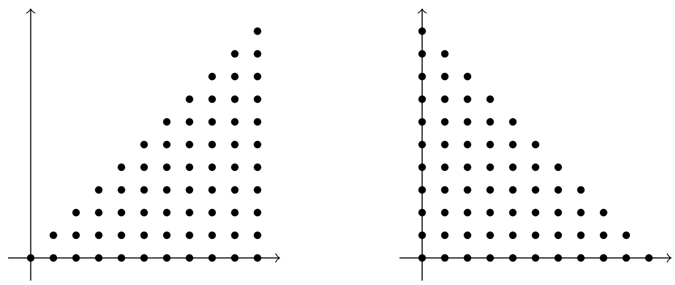

FIGURE 1. Summing over a triangular index set $\mathcal{I}$ in two equivalent ways.

We want to show that if f is a polynomial of degree $m \leq n$, then g is also a polynomial of degree m. Note first that the *Bernstein operator* $\mathcal{B}[f]$ sending f to g is *linear*, i.e.,

$$\mathcal{B}\left[x_1 f_1 + x_2 f_2\right] = x_1 \mathcal{B}[f_1] + x_2 \mathcal{B}[f_2].$$

It therefore suffices to show that the claim holds for $f(x) = x^m$, and this will simplify the problem a great deal. The overarching strategy here seems to be to manipulate the expression for g in such a way that we can consider the coefficients D_i in front of x^i and show that D_i is in fact zero for $i > m$.

We start by expanding the expression for $g(x)$ with f taken to be $f(x) = x^m$. This yields

$$g(x) = \frac{1}{n^m} \sum_{i=0}^{n} i^m \binom{n}{i} x^i (1-x)^{n-i}.$$

Expanding $(1-x)^{n-i}$ using the binomial theorem

$$= \frac{1}{n^m} \sum_{i=0}^{n} \sum_{j=0}^{n-i} i^m \binom{n}{i} \binom{n-i}{j} x^{i+j} (-1)^j$$

We can now rewrite the double sum in terms of new summation indices. If we let $\mathcal{I}$ denote our index set, then $\mathcal{I} = \{(i,j) \mid i = 0, \ldots n; j = 0, \ldots i\}$. This can be rewritten as $\mathcal{I} = \{(l, k-l) \mid k = 0, \ldots, n; l = 0, \ldots, k\}$. That is, setting $i = l$, $j = k - l$. We can with this in mind, rewrite $g(x)$ as

$$g(x) = \frac{1}{n^m} \sum_{k=0}^{n} \sum_{l=0}^{k} l^m \binom{n}{k} \binom{k}{l} (-1)^{k-l} x^k$$

and separating the independent terms yields

$$= \frac{1}{n^m} \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \sum_{l=0}^{k} l^m \binom{k}{l} (-1)^l.$$

It now makes sense to talk about the coefficient of x^k, so let $D_k = \sum_{l=0}^{k} l^m \binom{k}{l} (-1)^l$. Assume that $k > m$. In order to show that $D_k = 0$ for $k > m$, define the operator $\mathcal{L}$ by

$$\mathcal{L}[f](x) = x f'(x).$$

Furthermore, define the auxilliary functions $p(x) = (x-1)^k$ and $h(x) = x^l$. First observe that $\mathcal{L}^m[h](x) = l^m x^l$ and that since $k > m$, $(x-1)$ is a factor in $\mathcal{L}^m[p](x)$ so consequently, $\mathcal{L}^m[p](1) = 0$. Expanding $\mathcal{L}^m[p](1)$ we get

$$\begin{aligned} 0 &= \mathcal{L}^m[p](1) = \mathcal{L}^m[(x-1)^k](1) \\ &= \mathcal{L}^m\left[\sum_{i=0}^{k}\binom{k}{i}x^k(-1)^{k-i}\right](1). \end{aligned}$$

Using the fact that $\mathcal{L}$ is linear, we have

$$\begin{aligned} &= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathcal{L}^m\left[x^i\right](1) \\ &= (-1)^k\sum_{i=0}^{k}\binom{k}{i}(-1)^l l^m \\ &= (-1)^k D_k = 0 \end{aligned}$$

so D_k is zero for $k > m$.

Problem 3. We wish to show that $\Delta^i c_0 = \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}c_k$. We instead prove a more general claim, namely that

$$\Delta^i c_j = \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}c_{k+j},$$

and the initial claim will follow with $j = 0$. We proceed by induction. The base case consists of showing that the claim holds for $i = 1$. In this case we have $\Delta c_j = c_{j+1} - c_j$ so this holds. Assume inductively that the claim holds for the natural number i. We then have

$$\Delta^{i+1}c_j = \Delta^i c_{j+1} - \Delta^i c_j = \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\left(c_{k+j+1} - c_{k+j}\right).$$

Through a change of summation index this is the equivalent to

$$= \sum_{k=1}^{i+1}(-1)^{i-k}\binom{i}{k-1}c_{k+j} + \sum_{k=0}^{i}(-1)^{i+1-k}\binom{i}{k}c_{k+j}$$

We now extract the last term from the first sum, and the first term from the last sum, which yields

$$= c_{i+1+j} + \left(\sum_{k=1}^{i}\left(\binom{i}{k-1} + \binom{i}{k}\right)(-1)^{i+1-k}c_{k+j}\right) + (-1)^{i+1}c_j.$$

Using the fact that $\binom{i}{k-1} + \binom{i}{k} = \binom{i+1}{k}$ we can write this as

$$= c_{i+1+j} + \left(\sum_{k=1}^{i}\binom{i+1}{k}(-1)^{i+1-k}c_{k+j}\right) + (-1)^{i+1}c_j.$$

Putting it all together, we end up with

$$= \sum_{k=0}^{i+1}(-1)^{i+1-k}\binom{i+1}{k}c_{k+j},$$

as we wanted to show. This closes the induction, so if we set $j = 0$, we see that the initial claim is true: $\Delta^i c_0 = \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}c_k$.

Problem 4. We wish to show that

$$n(n-1)\dots(n-k)t^{k+1} = \sum_{i=0}^{n} i(i-1)\dots(i-k)B_i^n(t). \tag{1}$$

Note that this always holds for $k \geq n$, so we assume that k is strictly smaller than n.

References

[1] John Dawkins (http://math.stackexchange.com/users/189130/john-dawkins). *Prove that the Bernstein operator preserves function degree. (If f has degree m, then $\mathcal{B}(f)$ has degree m).* Mathematics Stack Exchange. URL: `http://math.stackexchange.com/q/1915642`.

[2] copper.hat (http://math.stackexchange.com/users/27978/copper-hat). *How come the Bernstein operator creates a polynomial of the same degree as its input function?* Mathematics Stack Exchange. URL: `http://math.stackexchange.com/q/377929`.