UNIVERSITY OF OSLO

MAT1120 - Assignment 2

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PROBLEM 1

Assume that $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ is an $(n \times n)$-matrix. Verify that U is ***semi-orthogonal*** if and only if $\mathbf{u}_j \neq \mathbf{0}$ for $j = 1, \ldots, n$ and all the $\mathbf{u}_j$'s are orthogonal to each other. In other words, $\mathbf{u}_j \cdot \mathbf{u}_i = 0$ when $j \neq i$.

Definition. A quadratic matrix U is called **semi-orthogonal** if $U^T U$ is a **diagonal** matrix with only positive entries along the main diagonal.

Definition. A matrix U is called **diagonal** if U has non-zero entries only along the main diagonal.
We now want to show that the condition is both sufficient and neccessary for the matrix to be ***semi-orthogonal***.
Let us first form the matrix $B = U^T U$.

$$B = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \ldots & \mathbf{u}_1 \cdot \mathbf{u}_n \\ \vdots & \ddots & \vdots \\ \mathbf{u}_n \cdot \mathbf{u}_1 & \ldots & \mathbf{u}_n \cdot \mathbf{u}_n \end{bmatrix}.$$

Assume now that each column in U is orthogonal to all the other columns. We can then directly see that all the entries in B that are not along the main diagonal must be zero. It is now sufficient to show that the entries along the main diagonal must be strictly positive. Let $\mathbf{u} \in U$ be an arbitrary vector from U.

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + \cdots + u_n^2.$$

Each term is squared, thus $\mathbf{u} \cdot \mathbf{u}$ is strictly positive.

If we instead were to assume that each column in U was **not** orthogonal to the other columns, we can directly from the matrix B see that we get non-zero entries that are not along the main diagonal, which contradict the condition that B must be a diagonal matrix. We can then conclude that U is ***semi-orthogonal*** if and only if the columns of U are orthogonal, and not the zero vector.
Further assume that U is ***semi-orthogonal*** and let

$$\mathbf{u}'_j = \frac{1}{\mathbf{u}_j \cdot \mathbf{u}_j}\mathbf{u}_j, \qquad j = 1, \ldots, n.$$

Argue that U is invertible and that

$$U^{-1} = \left[\mathbf{u}'_1 \ldots \mathbf{u}'_n\right]^T.$$

Let us look at the columns of $U, \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. If the set of columns is an orthogonal set of non-zero vectors then the set is linearly independent. Since we assumed that U is ***semi-orthogonal***, then it follows that the set of columns is linearly independent.
By the <u>Invertible Matrix Theorem</u>, U is invertible.
We now want to show that the inverse of U is as given above. Let us check if $UU^{-1} = I$.

$$UU^{-1} = \begin{bmatrix}\mathbf{u}_1 & \ldots & \mathbf{u}_n\end{bmatrix}\begin{pmatrix}\mathbf{u}'_1 \\ \vdots \\ \mathbf{u}'_n\end{pmatrix} = \begin{bmatrix}\mathbf{u}_1 \cdot \mathbf{u}'_1 & \ldots & \mathbf{u}_1 \cdot \mathbf{u}'_n \\ \vdots & \ddots & \vdots \\ \mathbf{u}_n \cdot \mathbf{u}'_1 & \ldots & \mathbf{u}_n \cdot \mathbf{u}'_n\end{bmatrix}$$

Let $\mathbf{x}, \mathbf{y}' \in \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$.

- $\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{x} \cdot \mathbf{y}' = 1$
- $\mathbf{x} \neq \mathbf{y} \Rightarrow \mathbf{x} \cdot \mathbf{y}' = 0$

We have that $\mathbf{x} = \mathbf{y}$ only along the main diagonal, therefore $UU^{-1} = I$.

PROBLEM 2

Divide $[0, \pi]$ into 3 intervals of length $\frac{\pi}{3}$ and let t_1, t_2 and t_3 be the midpoints of these intervals.
Let

$$C_3 = \begin{bmatrix}1 & \cos(t_1) & \cos(2t_1) \\ 1 & \cos(t_2) & \cos(2t_2) \\ 1 & \cos(t_3) & \cos(2t_3)\end{bmatrix} \qquad S_3 = \begin{bmatrix}\sin(t_1) & \sin(2t_1) & \sin(3t_1) \\ \sin(t_2) & \sin(2t_2) & \sin(3t_2) \\ \sin(t_3) & \sin(2t_3) & \sin(3t_3)\end{bmatrix}$$

Plugging in the values of t_1, t_2 and t_3 gives:

$$C_3 = \begin{bmatrix}1 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \qquad S_3 = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ 1 & 0 & -1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1\end{bmatrix}$$

These two matrices are ***semi-orthogonal*** because $C_3^T C_3$ and $S_3^T S_3$ are diagonal matrices with only positive entries along the main diagonal.

$$C_3^T C_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \qquad S_3^T S_3 = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Based on this we know that the inverses of these matrices are given by $U^{-1} = \begin{bmatrix} \mathbf{u}'_1 & \dots & \mathbf{u}'_n \end{bmatrix}^T$ This gives us the inverses:

$$C_3^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad S_3^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

PROBLEM 3

According to my python program both the matrices C and S are ***semi-orthogonal***, with the following inverses:

$$C^{-1} = \begin{bmatrix} 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 & 0.125 \\ 0.245 & 0.208 & 0.139 & 0.049 & -0.049 & -0.139 & -0.208 & -0.245 \\ 0.231 & 0.096 & -0.096 & -0.231 & -0.231 & -0.096 & 0.096 & 0.231 \\ 0.208 & -0.049 & -0.245 & -0.139 & 0.139 & 0.245 & 0.049 & -0.208 \\ 0.177 & -0.177 & -0.177 & 0.177 & 0.177 & -0.177 & -0.177 & 0.177 \\ 0.139 & -0.245 & 0.049 & 0.208 & -0.208 & -0.049 & 0.245 & -0.139 \\ 0.096 & -0.231 & 0.231 & -0.096 & -0.096 & 0.231 & -0.231 & 0.096 \\ 0.049 & -0.139 & 0.208 & -0.245 & 0.245 & -0.208 & 0.139 & -0.049 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 0.049 & 0.139 & 0.208 & 0.245 & 0.245 & 0.208 & 0.139 & 0.049 \\ 0.096 & 0.231 & 0.231 & 0.096 & -0.096 & -0.231 & -0.231 & -0.096 \\ 0.139 & 0.245 & 0.049 & -0.208 & -0.208 & 0.049 & 0.245 & 0.139 \\ 0.177 & 0.177 & -0.177 & -0.177 & 0.177 & 0.177 & -0.177 & -0.177 \\ 0.208 & 0.049 & -0.245 & 0.139 & 0.139 & -0.245 & 0.049 & 0.208 \\ 0.231 & -0.096 & -0.096 & 0.231 & -0.231 & 0.096 & 0.096 & -0.231 \\ 0.245 & -0.208 & 0.139 & -0.049 & -0.049 & 0.139 & -0.208 & 0.245 \\ 0.125 & -0.125 & 0.125 & -0.125 & 0.125 & -0.125 & 0.125 & -0.125 \end{bmatrix}$$

Multiplying the matrices with the respective inverses yields the identity matrix in both cases.

PROBLEM 4

Let $\mathscr{F}([0,2\pi],\mathbb{R})$ be the vector space consisting of all real functions defined on $[0,\pi]$ and let $\mathscr{W}_c$ be its subspace that is spanned by the set

$$\mathscr{C} = \{1, \cos(t), \cos(2t), \dots, \cos(7t)\}$$

We showed in the previous problem that C is semi-orthogonal. It then follows that C is invertible, which means that the columns of C neccessarily has to be linearly independent. This, in turn, means that $\mathscr{C}$ is a basis for $\mathscr{W}_C$. There is a result regarding signal spaces that tells ut that it is sufficient to show that this holds for the discrete values $t = t_1, \ldots, t_8$

PROBLEM 5

Let $T_c : \mathscr{W}_c \to \mathbb{R}^8$ be defined by:

$$T_c(h) = (h(t_1), \ldots, h(t_8))$$

for $h \in \mathscr{W}_c$, and let $\epsilon = \{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$.

Verify that C is the matrix of T_c with respect to the bases $\mathscr{C}$ and ϵ, and show that T_c is an isomorphism.
C is a matrix with those properties if $T_c(\mathbf{x})$ maps $[\mathbf{x}]_{\mathscr{C}}$ to $[T_c(\mathbf{x})]_\epsilon$.
Let h be any arbitrary vector in $\mathscr{W}_c$ with respect to the basis $\mathscr{C}$

$$h = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_8 \end{bmatrix}$$

.
Multiplying from the left by the matrix C, we get

$$Ch = \begin{bmatrix} 1 & \ldots & \cos(7t_1) \\ \vdots & \ldots & \vdots \\ 1 & \ldots & \cos(7t_8) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_8 \end{bmatrix} = [(h(t_1), \ldots, h(t_8))]_\epsilon$$

Since h was any arbitrary vector from $\mathscr{W}_c$, this means that C is the matrix of T_c with respect to the bases $\mathscr{C}$ and ϵ.
Since $\mathscr{C}$ is a basis for $\mathscr{W}_c$, and ϵ is a basis for $\mathbb{R}^8$ it follows that T_c is a bijective transformation. Since T_c is linear, this means T_c is an isomorph transformation. The matrix for T_c^{-1} is C^{-1}.

PROBLEM 6

Let $g \in \mathscr{F}([0,\pi], \mathbb{R})$ and set $\mathbf{y} = \left(g(t_1), \ldots, g(t_8)\right) \in \mathbb{R}^8$. In addition, let

$$g^c = T_c^{-1}(\mathbf{y}) \in \mathscr{W}_c$$

Show that g^c is an 8-midpoint interpolation of g on the interval $[0,\pi]$ that satisfy $\left[g^c\right]_C = C^{-1}\mathbf{y}$.
g^c is a function in $\mathscr{W}_c$ that is defined in such a way that g^c interpolate g in the midpoints $t_1, \ldots, t_8$. This means that g^c is an 8-midpoint interpolation of g on $[0,\pi]$ since,

$$Cg^c = CT_c^{-1}(\mathbf{y}) = CC^{-1}\mathbf{y} = \mathbf{y}$$
$$T_c(g^c) = C\left[g^c\right]_C = y \Leftrightarrow \left[g^c\right]_C = C^{-1}\mathbf{y}$$

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