Notes and exercises in MAT1140

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Chapter 1

Logic

1.1 True or False

We first want to examine some thought experiments. Examine each one and figure out what the statement means, and whether it is true or false.

1. The points $(-1, 1), (2, -1)$ and $(3, 0)$ lie on a line.

   We know from linear algebra, that if these points are to lie on the same line, they all have to satisfy the same linear equation $ax + b$. These do not, so the statement is false.

2. If x is an integer, then $x^2 \geq x$.

   This is an implication, on the form "if A, then B". We are interested in the world where x is an integer, and want to examine whether this means that $x^2 \geq x$ must hold. We have three cases, $x > 0, x = 0$ and $x < 0$. It holds for each case, so the statement is true. To prove this rigorously, we should use axiomatic set theory and the trichotomy of the integers.

3. If x is an integer, then $x^3 \geq x$.

   This is a statement on the same form as above. This one however, is false, and we can show this just by finding a counter-example. We know that any negative number raised to an odd power is also negative. Hence, if we let $x < 0$ and $|x| > 1$ we must have that $x^3 < x$. Thus, the statement is false.

4. For all real numbers x, $x^3 = x$.

   This again, is an implication, if one restates the statement. If x is a real number, then $x^3 = x$. Knowing that all integers are real numbers we can use the knowledge from the previous one to instantly see that this must be false. Just pick any real number other than 0 and 1 really.

5. There exists a real number x such that $x^3 = x$.

   This statement is in some sense a weaker version of the statement above. It only asserts the existence of at least one. Hence, all we have to do is find a real number that satisfies the equation. If we let $x = 1$, then we have $1^3 = 1$, hence the statement is true.

6. $\sqrt{2}$ is an irrational number.

   To show this statement, we must remind ourselves what an irrational number is. An irrational number is a number on the form a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$. The statements essentially asserts that there are no numbers on the form $x = a/b$ such that $x^2 = 2$. We can assume for contradiction that there does exist some rational number x such that

$x^2 = 2$. We can then write $x = \frac{a}{b}$, and we can also assume that the numbers are in their lowest terms. That any common factors have been canceled out. If we square x, we get $x^2 = \frac{a^2}{b^2}$. Remember, that we assumed that the square of x was equal to 2, so we have that $2 = \frac{a^2}{b^2}$. Multiplying through by b^2 we get that $2b^2 = a^2$, hence a^2 must be an even number. But, in that case a must be an even number, since the square of an even number is even.[1] Thus, we can write a on the form $2n$ for some integer n. If we substitute this into our previous equation we get that $2b^2 = 4n^2$, and the 2's cancel and we get $b^2 = 2n^2$, hence b must be an even number. But this means that both b and a have the common factor of 2, and since we assumed that a and b had no common factors this is the contradiction we wanted. Hence, $\sqrt{2}$ is an irrational number.

7. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.

   A number is odd if there exists some integer n such that the number can be written on the form $2n + 1$. We therefore let $n, m \in \mathbb{Z}$ such that $x + y = 2n + 1$ and $y + z = 2m + 1$. Solving these equations for x and z respectively, we get that $x = 2n + 1 - y$ and $z = 2m + 1 - y$. Adding we see that $x + z = 2(n + m + 1 - y)$. Hence $x + z$ is an *even* number, not an odd one.

8. If x is an even integer, then x^2 is an even integer.

   Since x is an even integer, we can write it on the form $x = 2n$ for some integer n. If we square x, we see that $x^2 = 4n^2$, which we can rewrite as $2(2n^2)$. Hence x^2 is an even integer. This fact was used in the proof for the irrationality of the square root of two above.

9. Every positive integer is the sum of distinct powers of two.

   This one is tricky to solve, and I have no idea whether it is true or not. By testing some cases one can easily see that most numbers follow this, and I have not found any counter-example.

10. Every positive integer is the sum of distinct powers of three.

    Same as above, no idea.

11. If x is an integer, then x is even or x is odd.

    An even number is on the form $2n$ for some integer n, and hence, must be an integer. The same goes for odd numbers on the form $2n + 1$. Hence, if x is even it must be an integer, and if x is odd it must be an integer. It now remains to show that if x is an integer, then it cannot be neither even nor odd. If we assume that x is an integer, but is neither even nor odd there must in some sense be a decimal part to x. And in that case it cannot be an integer.

12. If x is an integer, then x cannot be both even and odd.

    If we assume x to be an integer that can be written on the form $x = 2n$ and $x = 2m + 1$ for some integers $n, m \in \mathbb{Z}$, then we can add the two expressions together to achieve

    $$2x = 2n + 2m + 1 = 2(n + m) + 1.$$

    If we solve this for x, we get that $x = 2(n + m) + \frac{1}{2}$. Hence x is not an integer, which contradicts our initial assumption. This concludes the proof.

[1] This is shown in no. 8.

13. Every even integer greater than 2 can be expressed as the sum of two prime numbers.

    This is one of the unsolved problems in mathematics. I do not know how to solve this one, but later in this chapter I will attempt to rewrite the statement into equivalent ones to get different perspectives on the problem.

14. There are infinitely many prime numbers.

    A prime number is a whole number that is only divisible by 1 and itself. If we assume that there are a finite number of prime numbers $p_0, p_1, \ldots, p_n$ then we know that these have no common factors. We can now construct a new prime number, called q like this:

    $$q = p_0 \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$$

    This new number q cannot possibly have any common factors with the other prime numbers, hence it is divisible only by itself and one. We can repeat this process indefinitely to generate an infinite amount of prime numbers. Hence the statement is true.

15. For any positive real number x there exists a positive real number y such that $y^2 = x$.

    This statement essentially asserts that given any real number x, we can always find another real number y such that if you square this number y you get x. Not quite sure how to prove this.

16. Given three distinct points in space, there is one and only one plane passing through them.

    This is fairly obvious, and this is a typical uniqueness-proof in mathematics. By assuming that there are two distinct planes passing through the same points, one can show that these two planes must actually be equal.

I am now to divide my answers into four categories, namely the following:

a) I am confident that the justification I gave is conclusive.

   1, 3, 4, 5, 6, 7, 8, 12, 14

b) I am not confident that the justification I gave is conclusive.

   2

c) I am confident that the justification I gave is *not* conclusive. (If you gave no justification at all, your answer falls into this category.)

   9, 11, 13, 15, 16

d) I could not decide whether the statement was true or false.

   None

1.2 Statements and Predicates

A **statement** is a sentence that is either true or false, but not ambiguous.

Exercise 1.2.1. Give some examples of sentences that are statements and some examples of sentences that are not statements.

1. *World War 2 ended in 1945* is a statement.

2. *The university campus was tranquil during the summer* is not a statement. What university are we talking about?

3. *The Eiffel tower is made of metal* is a statement.

4. *The building is made of bricks* is not a statement. Which building?

A sentence with one or more free variables in it that becomes a statement when the free variables takes on a particular value is called a **predicate**.

Exercise 1.2.2. Give examples of mathematical predicates that have two and three free variables.

1. *He ate bacon and eggs for breakfast* is a predicate with one free variable, "He".

2. *He said to her that he was leaving* is a predicate with two free variables, "He" and "She".

3. $x^2 + 3y = 12z$ is a predicate with three free variables, namely x, y and z.

1.3 Quantification

We can turn a predicate into a statement by substituting particular values for its free variables. There are at least two other ways in which predicates can be used to build statements. We make a claim about which values of the free variable turn the predicate into a true statement.

The phrases **for all** and **there exists** are called **quantifiers**, and the process of using quantifiers to make statements out of predicates is called **quantification**.

Exercise 1.3.1. Suppose we understand the free variable z to refer to fish.

If we let $A(z)$ stand for the predicate *z lives in the sea*, then the statement "for all z, $A(z)$" is true.

If we let $B(z)$ stand for the predicate *z is blue*, then the statement "for all z, $B(z)$" is false but "there exists z such that $B(z)$" is true.

"For all" is called the **universal quantifier** and has the symbol $\forall$. "There exists ... such that" is called the **existential quantifier** and has the symbol $\exists$.

You might find it curious that a sentence can contain a variable, as quantified statements do, and yet be a statement. We call the variables in a statement with quantifiers **bound variables**. This essentially means that when the variable is bound there is no ambiguity. Note that the order of quantification matters greatly.

Exercise 1.3.2. Consider the following two statements.

1. There exists x and there exists y such that $y^2 = x$.

2. There exists y and there exists x such that $y^2 = x$.

Did quantifying over y and then x rather than the other way around change the meaning of the statement? What if the quantifiers had been *for all* instead of *there exists*?

It seems like the order of quantifiers do not matter as long as the quantifiers are the same. In other words, having two "for all" quantifiers it does not matter what order and similarly for two "there exists" quantifiers. However, if we have one "for all" and one "there exists" changing the order definitely changes the meaning of the statement. Examine the statements

1. There exists x such that for all y, $y^2 = x$.

2. For all x there exists y such that $y^2 = x$.

The first one asserts the existence of one single x such that the equation holds for all y, which is obviously false. The second one however asserts the fact that no matter what number x you pick you can always find a number y that has x as its square.

Exercise 1.3.3. Consider the predicate about integers "$x = 2y$". There are six distinct ways of quantifying this statement. Find all six and determine the truth value.

From the previous exercise, we can change the order of quantifiers without altering the meaning of the statement. Thus, we have six distinct ways of quantifying the statements.

1. $\forall x \exists y$ such that $x = 2y$. Chose $x = 1$, this is false.
2. $\exists x \forall y$ such that $x = 2y$. False.
3. $\exists x \exists y$ such that $x = 2y$. True, $x = 2$ and $y = 1$.
4. $\forall x \forall y$ such that $x = 2y$. False.
5. $\forall y \exists x$ such that $x = 2y$. True.
6. $\exists y \forall x$ such that $x = 2y$. False.

1.4 Mathematical Statements

The vast majority of mathematical statements can be written in the form "If A, then B" where A and B are predicates. The question now is, if A and B are predicates then surely "If A, then B" should be a predicate as well, not a statement. However, it is standard practice to assume universal quantification over the variables. In other words, the statement is to be read as "For all x, if $A(x)$, then $B(x)$".

Definition 1.4.1. A statement on the form "If A, then B", where A and B are statements or predicates is called an **implication**. A is called the **hypothesis** of the statement and B is called the **conclusion**.

Exercise 1.4.3. Identify the hypotheses and conclusions in each of the implications given in the previous example. (pg. 14)

1. Hypothesis: $x + y$ is odd and $y + z$ Conclusion: $x + z$ is odd
2. Hypothesis: x is an integer Conclusion: x either even or odd, but not both
3. Hypothesis: $x^2 < 17$ Conclusion: x is a positive real number.
4. Hypothesis: x is an integer Conclusion: $x^2 \geq 2$
5. Hypothesis: f is a polynomial of odd degree Conclusion: f has at least one real root

Remember that most mathematical statements can be rephrased as implications!

1.5 Mathematical Implication

We first start by talking about what it means for an implication to be true.

Example 1.5.1. If x is an integer, then $x^2 \geq 2$.

Proof. If $x = 0$, then $x^2 = x$, so certainly $x^2 \geq x$. The same is true if $x = 1$. If $x \geq 1$, then $x^2 > 1 \cdot x = x$ If $x < 0$ then $x^2 > 0 > x$. This accounts for all integer values of x. □

Here we studied all the values of the variable x that mate the hypothesis true. We then showed that in those cases the conclusion also is true. We did not care for the values of x that were not integers because they are not relevant for our case.

Exercise 1.5.2. We now consider the statement "If x is an integer, then $x^3 \geq x$."

1. Show that "If x is an integer, then $x^3 \geq x$" is false.
2. Thinking in terms of hypotheses and conclusions, explain what you did to show that the statement is false.

To show that this statement is false, we simply have to find one counter-example where we have an integer, but its cube is less than itself. If we for example let $x = -2$ we have that $x^3 = -8$ hence the statement is clearly false.

In terms of hypotheses and conclusions, we only considered the values for x that make the hypothesis true. Namely the integers. Then proceeded to find a value for x that satisfied the hypothesis yet did not satisfy the conclusion. Hence a contradiction was found and the statement was proved to be false.

A value of x that makes the hypothesis A true and the conclusion B false is called a **counterexample**.

Exercise 1.5.3. Occasionally you will see "If A, then B" written as "A is sufficient for B" or "B is necessary for A"or "B, if A" or "A only if B". Explain why it is sensible to say that each of these means the same thing.

"A is sufficient for B" means that if we have that A is true, then B also must be true because that A is true is sufficient. This is equivalent to the implication. "B is necessary for A" means that if we have A then we also must have B. They essentially all mean the same thing.[2]

An implication in which the hypothesis is false is often said to be **vacously true**. We define the implication to be true when the hypothesis is false because values that make the hypothesis false can never generate a counterexample.

1.6 Compund Statements and Truth Tables

Symbolically we write "If A, then B" as $A \implies B$, which is read "**A implies B**".

Table 1.1: Truth table for implication

A	B	A $\implies$ B
T	T	T
T	F	F
F	T	T
F	F	T

The above table is an example of a **truth table**. In addition to implication we can combine statements A and B in a number of ways.

1. "A and B" is called the **conjunction** of A and B. This is denoted $A \wedge B$.
2. "A or B" is called the **disjunction** of A and B. We denote it by $A \vee B$.
3. "Not A" is called the **negation** of A. We denote it $\sim A$.

[2] I have no idea how to properly explain this

4. "A if and only if B" is called the **equivalence** of A and B. We denote it by $A \iff B$. ("If and only if" is often abbreviated **iff**.)

Definition 1.6.1. Suppose that A and B are statements. The following truth table gives the truth values of all the previously mentioned compound statements.

Table 1.2: Compound statements

		A implies B	A and B	A or B	not A	A iff B
A	B	$A \implies B$	$A \wedge B$	$A \vee B$	$\sim A$	$A \iff B$
T	T	T	T	T	F	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	T	T	T

Exercise 1.6.2. Examine table 1.2. Given the colloquial meaning of terms "and", "or", "not" and "equivalent" explain why the truth values in the table make sense.

1. "and" When we say that we want A *and* B we want to have both at the same time. Hence if either A or B is false then we do not have both and hence the statement is false. This coincides with the truth table.

2. "or" When we say "this or that" we mean "either this, or that, or both" therefore we only require at least one of A or B to be true for the statement "A or B" to be true.

3. "not" Not is simply the negation. Hence if we have something, then the negation of that is not having something.

4. "equivalent" Two statements are equivalent if one can swap one for the other without losing any meaning. In other words, we must have both A and B true or A and B false for the equivalence to be true.

Example 1.6.3. Given that A and B are statements we can look at the compound statements

1. $B \wedge \sim B$: If we construct the truth table for this compound statement we see that the statement is always false. A compound statement that is always false is regardless of the truth value of the simpler statements involved is called a **contradiction**.

2. $(A \wedge \sim B) \iff \sim(A \implies B)$: This compound statement is true no matter what the truth value of the simpler statements involved is called a **tautology**.

Exercise 1.6.4. Verify that

$$(A \implies (B \vee C)) \iff ((A \wedge \sim B) \implies C)$$

is a tautology.

A	B	C	$B \vee C$	$\sim B$	$A \wedge \sim B$	$A \implies (B \vee C)$	$(A \wedge \sim B) \implies C$
T	T	T	T	F	F	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	F	T	T	F	F
F	T	T	T	F	F	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	F	T	F	T	T

We here see that the statements $(A \implies (B \vee C))$ and $((A \wedge \sim B) \implies C)$ have exactly the same truth values. We know that an equivalence only hold if the two statements have the same truth values for all possible values of the variables. Hence the equivalence is true no matter what, and we call it a tautology.

1.7 Learning from Truth Tables

Exercise 1.7.1. Consider the following statements.

1. $(A \implies (B \wedge C)) \implies (A \implies B)$.
2. $(A \wedge (A \implies B)) \implies B$.
3. $((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$.

Each of these statements are tautologies. First verify using truth tables that the statements are tautological. Then figure out what logical principles each statement embodies and what they tell us about proving theorems.[3]

1. $(A \implies (B \wedge C)) \implies (A \implies B)$. This statement tells us that if A implies that B and C are true, then that implies that A implies that B is true. This means, in terms of proving theorems that we can show that the hypothesis implies a stronger conclusion $(B \wedge C)$ than the one we want to show, and thus deduce that it must also hold for the weaker conclusion which we wanted to show (B).
2. $(A \wedge (A \implies B)) \implies B$. If we know that $A \implies B$ is true, and that A is true, then we can conclude with B also being true.
3. $((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$. This statement tells us that implication has the transitive property. If both A implies B and B implies C, then it must follow that A implies C. In terms of proving theorems, if we want to show that A implies C, then and we know that B implies C we can instead chose to prove that A implies B.

Definition 1.7.2. The implication $B \implies A$ is called the **converse** of $A \implies B$.

[3] I am going to skip creating the truth tables as they take some time.

Exercise 1.7.3. Construct a truth table to show that it is possible for $A \implies B$ to be true while its converse $B \implies A$ is false, and vice versa.

A	B	A $\implies$ B	B $\implies$ A
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

We here see that the implication and its converse are only equal in the two cases where A and B have the same truth value. Hence, we cannot simply flip the implication sign without changing the meaning of the statement. Thus, we have to treat a statement and its converse as two distinct mathematical claims.

Exercise 1.7.5. Find an example of a true statement whose converse is false and one whose converse is true.

1. "If he lives in Oslo, then he lives in Norway" is true but has a false converse.
2. "If $x^3 = 1$, then $x = 1$" is true, and has a true converse.

Let us consider the truth table for $A \iff B$:

Table 1.3: Equivalence

A	B	A $\iff$ B
T	T	T
T	F	F
F	T	F
F	F	T

We see that A and B must have the same truth value for the equivalence to be true. If $A \iff B$ is true, then we say that A and B are **equivalent**. If we manage to prove A we know that B is true as well and conversely.

Equivalence of statements are important because when proving a statement we can instead chose to prove an equivalent statement. Consider a statement on the form "If A, then B or C", we can instead chose to prove either

1. If A and not B, then C.
2. If A and not C, then B.

Exercise 1.7.6. Show by constructing a truth table that

$$(A \iff B) \iff ((A \implies B) \land (B \implies A))$$

is a tautology.

A	B	A $\iff$ B	A ($\implies$ B) $\land$ (B $\implies$ A))
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Hence, we can when proving equivalence of statements prove the one sided implication and its converse instead. This is an important thing to remember.

Exercise 1.7.7. Construct a truth table for the following four statements:
$\sim(A \wedge B)$ $\quad \sim A \wedge \sim B$ $\quad \sim(A \vee B)$ $\quad \sim A \vee \sim B$

A	B	$\sim(A \wedge B)$	$\sim A \wedge \sim B$	$\sim(A \vee B)$	$\sim A \vee \sim B$
T	T	F	F	F	F
T	F	T	F	F	T
F	T	T	F	F	T
F	F	T	T	T	T

We can from this see that the negation of a conjunction is the disjunction of the negations and the negation of a disjunction is the conjunction of the negations. These are what we call De'Morgans laws.

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