Visualizing Algebraic Surfaces

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April 14, 2016

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1 An Informal Introduction

In this section we briefly look at the natural construction of the mathematical objects we are interested in studying in the rest of this paper.

1.1 Real Algebraic Curves

From elementary mathematics one learns about real valued functions, $f(x)$, and how to graph these functions by setting $y = f(x)$ and plotting points in the (x, y)-plane. Now, the *graph of a function* is something of a peculiarity, because it comes with some restrictions. Not all curves in the (x, y)-plane correspond to functions. The method for verifying whether a certain "graph" corresponds to a function or not typically taught in school is the *vertical line test*.

Having an equation $y = f(x)$ we can form what we call the *equation of a curve at zero*. We define a new function in two variables

$$g(x, y) = y - f(x) = 0. \tag{1}$$

If the function $f(x)$ is a polynomial in the variable x with certain coefficients we call the function graph of $f(x)$ *algebraic*.

Generally speaking, we call a curve defined by eq. (1) an *algebraic curve* if the function $g(x, y)$ is a polynomial in two variables, x and y. Mathematically, this can be expressed as

$$g(x, y) = \sum_{i,j} a_{i_j} x^i y^i. \tag{2}$$

However, if this distinction between a graph and a curve is to be justified there has to be some curves that are not graphs. One of the first examples one encounter of a curve not corresponding to a function is what you get when you set $y^2 = x^3$. This curve does not pass the vertical-line-test and is therefore something different from a graph. Similarly, the equation $y^2 = x^3 - x^2$ defines a curve that again is not a graph. These are shown in section 1.1. These curves exhibit *singular points* at $(0, 0)$.

(a) $g(x, y) = y^2 - x^3 = 0$ (b) $g(x, y) = y^2 - x^3 + x^2 = 0$

Figure 1: Examples of curves that are *not* graphs given by their functions $g(x, y)$
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With what we have so far, we can move up a dimension. Instead of considering curves in the (x, y)-plane we can look at surfaces in the (x, y, z)-space. We, as we did with $g(x, y)$ above, define a function $h(x, y, z) = z - g(x, y)$. The surfaces are given by equations on the form $h(x, y, z) = 0$. These surfaces are called algebraic if $h(x, y, z)$ is a polynomial in the variables x, y, z. Again, mathematically, this is expressed as

$$h(x, y, z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k. \tag{3}$$

1.2 Going Complex

The surfaces considered so far are over the real numbers, i.e, the coefficients a_{ijk} are real numbers. We can instead work over the complex numbers where the surfaces are given by an equivalent equation, but where the coefficients now are complex numbers[1]:

$$\{(x, y, z) \in \mathbb{C} \mid h(x, y, z) = 0\}. \tag{4}$$

Again, doing the same trick (one-trick ponies) we can now define $w = h(x, y, z)$ and look at the equations $w - h(x, y, z) = 0$. Now, we are stuck with an equation in four variables. With only three degrees of freedom to work with when visualizing these objects we encounter an obstacle. How do we know what these surfaces look like? This brings us to the world of *projective geometry.*

2 Projective Geometry

The way we visualize these surfaces in four variables is by *fixing* one of the variables. This is called *homogenizing* the polynomial $h(x, y, z)$. We first want to introduce some definitions.

[1] We will come back to why we make this transition when we talk about sets being algebraically closed or not.

Quite informally, projective space formalizes the notion of parallell lines intersecting at infinity. In order to get a better understanding of the general projective space $\mathbb{P}^n$ we will first construct the *projective plane* $\mathbb{P}^2(\mathbb{C})$.

2.1 The Projective Plane

According to [**Wik16**] there are three equivalent definitions:

1. The set of all lines in $\mathbb{C}^3$ passing through the origin $(0, 0, 0)$. Every such line meets the *sphere* of radius one centered in the origin exactly twice, say in $P = (x, y, z)$ and its antipodal point $(-x, -y, -z)$.

2. The points on the sphere S^2, where every point P and its antipodal points are not distinguished. For example, the point $(1, 0, 0)$ is identified with the point $(-1, 0, 0)$.

3. The set of equivalence classes of $\mathbb{C}^3 \setminus \{0\}$ where two points P and P' are considered equivalent if and only if there is a non-zero $\lambda \in \mathbb{C}$ such that $P = \lambda P'$.

The last definition is the one seemingly most used, and is therefore the one we will employ here. Note that the elements in the projective plane are equivalence classes of points in $\mathbb{C}^3$. We denote an element in $\mathbb{P}^2(\mathbb{C})$ as $[x : y : z]$. These elements are commonly referred to as *homogenous coordinates*. We state this formally in a definition.

Definition 1 (The projective plane)**.** Let $\sim$ be the equivalence relation on $\mathbb{C}^3$ defined by

$$(x, y, z) \sim (x', y', z') \iff (x', y', z') = \lambda(x, y, z),$$

where λ is some non-zero scalar in $\mathbb{C}$. We denote the equivalence class of (x, y, z) as $[x : y : z]$ and we define the *projective plane* as the set

$$\mathbb{P}^2(\mathbb{C}) = \left\{ [x : y : z] \mid (x, y, z) \in \mathbb{C}^3 \setminus \{0\} \right\}.$$

We can now look at an interesting subset of $\mathbb{P}^2(\mathbb{C})$, namely the set of points $[x : y : z]$ where $z \neq 0$. Remembering the identification from the above definition, these are equivalent to $\left[\frac{x}{z} : \frac{y}{z} : 1\right]$ where $\lambda = 1/z$. If we consider definition item 2 from above, this corresponds to the set of points on the northern hemisphere of S^2 not including the circle of intersection in the (x, y)-plane. Similarly, we can look at the set of points $[x : y : 0]$. This correponds, in S^2, to the circle of intersection in the (x, y)-plane.

We can now take the notion of the projective plane and generalize this to n dimensions.

2.2 Projective Space

The general definition of projective n-space is completely anologous to the definition of the projective plane, the set of lines in $\mathbb{C}^{n+1}$ passing through the origin, however we include a formal definition for completeness.

Definition 2 (The projective space of dimension n)**.** Let $\sim$ be an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$(x_1, \ldots, x_{n+1}) \sim \left(x'_1, \ldots, {}'_{n+1}\right) \iff \left(x'_1, \ldots, x'_{n+1}\right) = \lambda \left(x_1, \ldots, x_{n+1}\right)$$

where λ is some non-zero complex number. We then denote the equivalence class of $(x_1, \ldots, x_{n+1})$ by $[x_1 : \ldots : x_{n+1}]$ and define the *projective space of dimension* n as the set

$$\mathbb{P}^n\mathbb{C} = \left\{[x_1 : \ldots : x_{n+1}] \mid (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}\right\}.$$