Visualizing Algebraic Surfaces

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1 An Informal Introduction

In this section we briefly look at the natural construction of the mathematical objects we are interested in studying in the rest of this paper.

1.1 Real Algebraic Curves

From elementary mathematics one learns about real valued functions, $f(x)$, and how to graph these functions by setting $y = f(x)$ and plotting points in the (x, y)-plane. Now, the *graph of a function* is something of a peculiarity, because it comes with some restrictions. Not all curves in the (x, y)-plane correspond to functions. The method for verifying whether a certain "graph" corresponds to a function or not typically taught in school is the *vertical line test*.

Having an equation $y = f(x)$ we can form what we call the *equation of a curve at zero*. We define a new function in two variables

$$g(x, y) = y - f(x) = 0. \tag{1}$$

If the function $f(x)$ is a polynomial in the variable x with certain coefficients we call the function graph of $f(x)$ *algebraic*.

Generally speaking, we call a curve defined by eq. (1) an *algebraic curve* if the function $g(x, y)$ is a polynomial in two variables, x and y. Mathematically, this can be expressed as

$$g(x, y) = \sum_{i,j} a_{i_j} x^i y^i. \tag{2}$$

However, if this distinction between a graph and a curve is to be justified there has to be some curves that are not graphs. One of the first examples one encounter of a curve not corresponding to a function is what you get when you set $y^2 = x^3$. This curve does not pass the vertical-line-test and is therefore something different from a graph. Similarly, the equation $y^2 = x^3 - x^2$ defines a curve that again is not a graph. These are shown in section 1.1. These curves exhibit *singular points* at $(0,0)$.

(a) $g(x,y) = y^2 - x^3 = 0$ (b) $g(x,y) = y^2 - x^3 + x^2 = 0$

Figure 1: Examples of curves that are *not* graphs given by their functions $g(x,y)$
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With what we have so far, we can move up a dimension. Instead of considering curves in the (x,y)-plane we can look at surfaces in the (x,y,z)-space. We, as we did with $g(x,y)$ above, define a function $h(x,y,z) = z - g(x,y)$. The surfaces are given by equations on the form $h(x,y,z) = 0$. These surfaces are called algebraic if $h(x,y,z)$ is a polynomial in the variables x,y,z. Again, mathematically, this is expressed as

$$h(x,y,z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k. \tag{3}$$

1.2 Going Complex

The surfaces considered so far are over the real numbers, i.e, the coefficients a_{ijk} are real numbers. We can instead work over the complex numbers where the surfaces are given by an equivalent equation, but where the coefficients now are complex numbers[1]:

$$\{(x,y,z) \in \mathbb{C} \mid h(x,y,z) = 0\}. \tag{4}$$

Again, doing the same trick (one-trick ponies) we can now define $w = h(x,y,z)$ and look at the equations $w - h(x,y,z) = 0$. Now, we are stuck with an equation in four variables. With only three degrees of freedom to work with when visualizing these objects we encounter an obstacle. How do we know what these surfaces look like? This brings us to the world of *projective geometry*.

2 Projective Geometry

[1] We will come back to why we make this transition when we talk about sets being algebraically closed or not.