Visualizing Algebraic Surfaces

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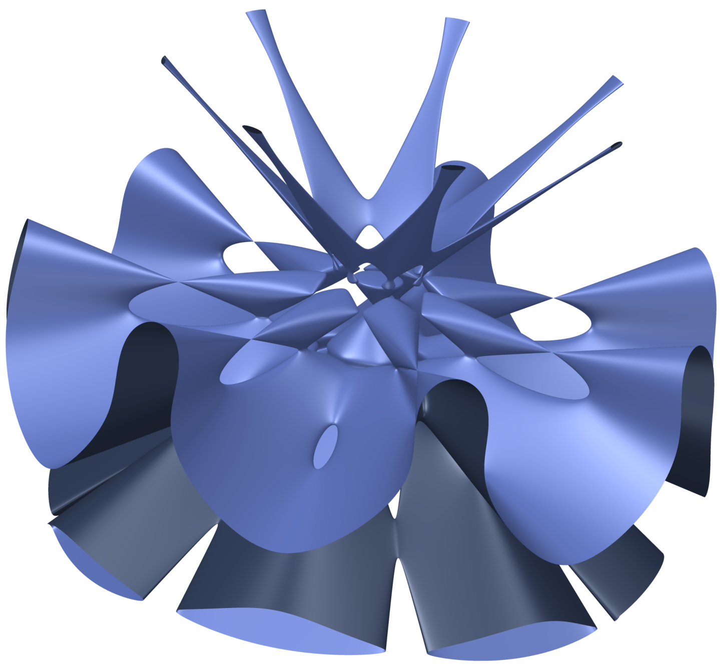

Abstract

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We start our study of algebraic surfaces by looking at *projective space*. Quite informally, projective space formalizes the notion of parallell lines intersecting at infinity. In order to get a better understanding of the general projective space $\mathbb{P}^n$ we will first construct the *real projective plane* $\mathbb{P}^2(\mathbb{R})$.

2.1 The real projective plane

According to [Wik16] there are three equivalent definitions:

I. The set of all lines in $\mathbb{R}^3$ passing through the origin $(0, 0, 0)$. Every such line meets the *sphere* of radius one centered in the origin exactly twice, say in $P = (x, y, z)$ and its antipodal point $(-x, -y, -z)$.

II. The points on the sphere S^2, where every point P and its antipodal points are not distinguished. For example, the point $(1, 0, 0)$ is identified with the point $(-1, 0, 0)$.

III. The set of equivalence classes of $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ where two points P and P' are considered equivalent if and only if there is a positive $\lambda \in \mathbb{R}$ such that $P = \lambda P'$.

The last definition is the one seemingly most commonly used, and is therefore the one we will employ here. Note that the elements in the real projective plane are equivalence classes of points in $\mathbb{R}^3$. We denote an element in $\mathbb{P}^2(\mathbb{R})$ as $[x : y : z]$. These elements are commonly referred to as *homogenous coordinates*. We state this formally in a definition.

Definition 1 (The real projective plane)**.** Let $\sim$ be the equivalence relation on $\mathbb{R}^3$ defined by

$$(x, y, z) \sim (x', y', z') \iff (x, y, z) = \lambda(x, y, z),$$

where λ is some positive real number. Then we denote the equivalence class of (x, y, z) as $[x : y : z]$ and we define *the real projective plane* as the set

$$\mathbb{P}^2(\mathbb{R}) = \left\{[x : y : z] \mid (x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}\right\}.$$

We can also look at a subset of $\mathbb{P}^2(\mathbb{R})$, namely the set of points $[x : y : z]$ where $z \neq 0$. Remembering the identification from the above definition, these are equivalent to $[\frac{x}{z} : \frac{y}{z} : 1]$ with $\lambda = 1/z$. If we consider definition II from above, this corresponds to the set of points on the northern hemisphere of S^2, not including the circle of intersection in the (x, y)-plane. Similarly, we can look at the set of points $[x : y : 0]$. This corresponds, in S^2, to the circle of intersection in the (x, y)-plane.

We can now take the notion of the projective plane and generalize to n dimensions.

2.2 Projective Space

The general definition of projective n-space is completely anologous to the definition of the projective plane, the set of lines in $\mathbb{R}^{n+1}$ passing through the origin, however we include a formal definition for completeness.

Definition 2 (The real projective space of dimension n)**.** Let $\sim$ be an equivalence relation on $\mathbb{R}^{n+1} \setminus \{(0, \ldots, 0)\}$ defined by

$$(x_1, \ldots, x_{n+1}) \sim \left(x'_1, \ldots, x'_{n+1}\right) \iff (x_1, \ldots, x_{n+1}) = \lambda \left(x'_1, \ldots, x'_{n+1}\right)$$

where λ is some positive real number. We then denote the equivalence class of $(x_1, \ldots, x_{n+1})$ by $[x_1 : \ldots : x_{n+1}]$ and define *the real projective space of dimension* n as the set

$$\mathbb{P}^n(\mathbb{R}) = \left\{[x_1 : \ldots : x_{n+1}] \mid (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{(0, \ldots, 0)\}\right\}.$$

3 Algebraic Surfaces

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References

[Wik16] Wikipedia. *Projective space — Wikipedia, The Free Encyclopedia*. [Online; accessed 19-February-2016]. 2016. URL: `https://en.wikipedia.org/wiki/Projective_space`.