Algebraic Surfaces

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Abstract

In this paper we take a closer look at algebraic surfaces. With an emphasis on the visual aspects we look at singularities of surfaces, their symmetries, as well as techniques for resolving singularities. Unless specificed, the images are rendered in the software SURFER.

Contents

1 Preliminaries

In algebraic geometry the objects of study are, amongst other, the set of solutions to polynomial equations. Finding the zeroes of a quadratic equation

$$ax^2 + bx + c = 0$$

is something most people who have been exposed to elementary mathematics have done. Later one learns how to find the zeroes of more general functions $f \in K[x_1, x_2, \dots, x_n]$ of either one or several variables, and you often find functions that have no zeroes at all.

One of the central objects in algebraic geometry is the *affine algebraic variety* and the *projective variety*. These are defined as the set of common solutions to the zero-equations of a set of polynomials in some *polynomial ring*.

Definition 1 (Polynomial ring)**.** A *polynomial ring* in n variables $K[x_1, x_2, \dots, x_n]$ over a field K is the set of all functions on the form

$$f(x) = \sum_{\alpha} p_\alpha \prod_{i=1}^{n} x_n^{\alpha_i}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index and $p_\alpha = p_{\alpha_1 \dots \alpha_n}$ is a coefficient in K.

An example of a polynomial ring is $\mathbb{R}[x, y]$. Elements include $f_1(x, y) = x^2y + y + 1$ and $f_2(x, y) = \pi$. Given a set of polynomials $\{f_1, f_2, \dots, f_m\}$ in a polynomial ring $K[x_1, x_2, \dots, x_n]$ we can look at their common zeroes. Specifically for polynomials in *affine space* we make the following definition.

Definition 2 (Affine algebraic variety)**.** Given a set of polynomials $\{f_1, f_2, \dots, f_m\}$ from a ring of polynomials, the set of points $(x_1, x_2, \dots, x_n)$ in the affine space $\mathbb{A}^n$ that satisfy

$$f_i(x_1, \dots, x_n) = 0 \quad \text{for} \quad i = 1, \dots, m$$

is called an *affine algebraic variety* and is denoted $\mathbb{V}(f_1, f_2, \dots, f_m)$.

Given the polynomial $f(x) = x^2 - 1$, the affine algebraic variety $\mathbb{V}(f)$ is then the set of points $\{1, -1\}$ since $f(-1) = f(1) = 0$. Given an additional polynomial $g(x) = x - 1$ we have $\mathbb{V}(f, g) = \{1\}$. Finally, if we also consider the polynomial $h(x) = x^2$, then $\mathbb{V}(f, g, h) = \emptyset$. We therefore see that the variety of a set of function is an intersection of the varieties of the individual functions.

In this paper we will also deal a lot with singularities of our algebraic surfaces. We make a formal definition.

Definition 3 (Singularity)**.** A singularity of the variety $V = \mathbb{V}(f_1, f_2, \dots, f_m)$ is a point p such that the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (p)$$

has rank strictly less than $\min(m, n)$.

A very useful consequence of this definition deal with varieties of single polynomials.

Corollary 1. A point p is a singularity of the variety $\mathbb{V}(f)$ if and only if the partial derivatives of f vanish identically at p. Mathematically this is expressed as $\partial f(p)/\partial x_i = 0$ for $i = 1, 2, \dots, n$.

Proof. We recall that the *rank* of a matrix A is the dimension of its column space. Note that $\min(m, n)$ is in this case equal to 1 since we have n variables but only one function. For the Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \end{bmatrix} (p)$$

to have rank strictly less then one, its column space has to have dimension zero. In linear algebra terms, this simply means that the matrix have no pivot columns, and hence each column must be zero. Therefore, for p to be a singularity of $\mathbb{V}(f)$ all its partial derivatives must vanish identically. $\square$

2 Algebraic Surfaces

In this section we try to motivate the definition of an algebraic surface and look at some examples, and hopefully give an intuitive explanation of what exactly we are looking at. Looking at real algebraic curves is a good entry point.

2.1 An Informal Introduction

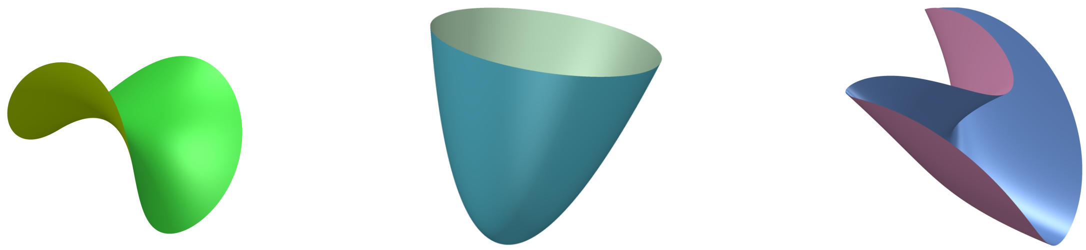
From elementary mathematics one learns about real valued functions, $f(x)$, and how to graph these functions by setting $y = f(x)$ and plotting points in the (x, y)-plane. Now, the *graph of a function* is something of a pecuiliarity, because it comes with some restrictions. Not all curves in the (x, y)-plane correspond to a specific function. The method for verifying whether a certain "graph" correspond to a function or not, typically taught in school, is the *vertical line test*.

Having an equation $y = f(x)$ we can form what we call the *equation of a curve at zero*. We define a new function in two variables

$$g(x, y) = y - f(x) = 0.$$

If the function $f(x)$ is a polynomial in the variable x we call the function graph of $f(x)$ *algebraic*. If the distinction between a graph and a curve is to be justified there has to be some curves that are not graphs. If we look at what happens when you set $y^2 = x^3$, then the plotted curve does not pass the vertical line test, and is therefore something different from a function graph. Similarily, the equation $y^2 = x^3 - x^2$ defines a curve that is again not a graph. These curves also happen to exhibit *singular points* at $(0, 0)$.

With what we have so far, we can move up a dimension. Instead of considering curves in the (x, y)-plane we can look at surfaces in (x, y, z)-space. Following the same procedure as above, we define a new function $h(x, y, z) = z - g(x, y)$. The surfaces are given by equations on the form $h(x, y, z) = 0$. These surfaces are *algebraic* if $h(x, y, z)$ is a polynomial in the variables x, y, z.



(a) $h_1(x,y,z) = 0$. (b) $h_2(x,y,z) = 0$. (c) $h_3(x,y,z) = 0$.

Figure 1: The algebraic surfaces from example 1.

Example 1. The following equations satisfy the definition of an algebraic surface

1. $h_1(x,y,z) = z - xy = 0$,
2. $h_2(x,y,z) = z - x^2 - y^2 = 0$,
3. $h_3(x,y,z) = z - (2x^2 - y)(y - x^2)$,

where as the following does not satisfy the definition:

1. $g_1(x,y,z) = \sin(xy) + z$,
2. $g_2(x,y,z) = \tan^2(x) + \pi zy$.

The examples of algebraic surfaces are shown visualized in fig. 1.

We complete this section by formally defining an algebraic surface.

Definition 4 (Algebraic surface)**.** An *algebraic surface* is an algebraic variety of dimension two.[1]

We define the *degree* of the algebraic surface to be highest combined power in the defining polynomials terms, and denote this d, i.e.,

$$d = \max_{\alpha} \{\alpha_1 + \alpha_2 + \ldots + \alpha_n\}$$

where the α_i is the power of x_i in a term.

3 Singularities

The singularities of an algebraic surface is one of the more important elements that give the surfaces its visual appeal. I am sure many people would agree with me that surfaces with many singularities, and with singularities in interesting configurations, are more exciting to look at then the surfaces that show no singularities.

We first take a look at the number of singularities that various surfaces have.

3.1 The Number of Singularities

For an algebraic surface of degree d we let $\mu(d)$ denote the maximum number of singularities such a surface *can* exhibit. We first look at some examples that help illustrate this.

Example 2 (The plane has no singularities)**.** The defining polynomial of a plane in three dimensions is

$$f(x,y,z) = ax + by + cz + d.$$

The corresponding surface is then $V = \mathbb{V}(f)$. We want to find the singularities of V. Applying corollary 1 we only need consider the partial derivatives and find the points in V where they vanish. We solve the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

[1] The dimension of an algebraic variety is a bit hard to grasp, and since we only will deal with algebraic surfaces in this article we do not spend any time classifying the different kinds of varieties.

This gives $a = 0$, $b = 0$ and $c = 0$ which again forces $d = 0$ since otherwise our point (x, y, z) would not be a zero of f. This however, tells us that $f(x, y, z) = 0$ which does not define any surface. Hence, the plane has no singularities. The plane is an algebraic surface of degree $d = 1$, so consequently $\mu(1) = 0$.

Example 3 (The Cayley cubic)**.** The Cayley cubic is defined by the polynomial

$$f(x, y, z) = xyz + (1 - x - y - z)(yz + xz + xy).$$

Again, we are interested in the singularities of $V = \mathbb{V}(f)$. Once again, we apply corollary 1 and look at the partial derivatives and find their zeroes. This gives the four points $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$.

It can also be shown that there cannot be *more* than four singularities for a third degree surface. Hence $\mu(3) = 4$.

3.2 Symmetric singularities

4 Resolving Singularities

We call a variety that exhibits no singularities *smooth*. It is often possible to study a smooth copy of a variety with singularities if we move from affine space to projective space.

4.1 Projecive Space

In this section we define the notion of a *projective space*.

4.2 Blowups

Informally, when we blow up a point p in $\mathbb{A}^n$, we replace the point p by an entire copy of a projective space $\mathbb{P}^{n-1}$ and try to leave the rest of the space $\mathbb{A}^n$ unchanged. We will give a few examples of this.

Definition 5 (Blowup of $\mathbb{A}^n$ at p)**.** Let the blowup-surface B be the set of all pairs consisting of a point $p \in \mathbb{A}^n$ and a line from p through the origin. Then $B \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$. We write

$$\begin{aligned} B &= \{(p, L) \mid p \in \mathbb{A}^n, p \in L\} \\ &= \left\{(x_1, \ldots, x_n; y_1 : y_2 : \ldots : y_n) \mid x_i y_j = x_j y_i \quad 0 \leq i < j \leq n\right\}. \end{aligned}$$

In order to aid us with this, we have the following result:

Corollary 2. Given a point $x = (x_1, \ldots, x_n)$ in the space $\mathbb{A}^n$ and a point $y = (y_1 : \ldots : y_n)$ in the space $\mathbb{P}^{n-1}$ the point x is a multiple of y if and only if the matrix

$$A = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{bmatrix}$$

has rank less than or equal to one. Similarly, the rank is less than or equal to one only if each 2×2-minor $x_i y_j - x_j y_i = 0$.[2]

Proof. In order to verify this, we recall that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$, so we chose to work with the transposed matrix instead. Assume that x is a multiple of y, so $y = \lambda x$ for some scalar λ. Then

$$A^T = \begin{bmatrix} x_1 & \lambda x_1 \\ \vdots & \vdots \\ x_n & \lambda x_n \end{bmatrix}.$$

Assume that the point x is non-zero. Then we have exactly one pivot column in this matrix, hence $\dim(\operatorname{Col}(A^T)) = 1$. If $x = 0$, then we have no pivot columns, so $\dim(\operatorname{Col}(A^T)) = 0$. In any case, $\operatorname{rank}(A) = \operatorname{rank}(A^T) \leq 1$.

For the second claim, since A^T has rank 1, there cannot exists any non-zero minors $r \times r$ for $r > 1$. Hence all 2×2 minors must vanish. □

[2]Figure this out properly.

Example 4 (The Node)**.** The defining polynomial of the node is given by

$$f(x,y) = y^2 - x^2 - x^3, \tag{1}$$

so the curve is $C = \mathbb{V}(f)$. This curve lives in the affine plane $\mathbb{A}^2$. The blow-up B_p of C at the point $p = (0,0)$ lives in $\mathbb{A}^2 \times \mathbb{P}^1$. For a point $(x,y) \in C$ we figure out what line it passes through by applying corollary 2 with homogeneous coordinates (r,s).

$$\begin{bmatrix} x & y \\ r & s \end{bmatrix}$$

has rank less than or equal to one when $xs - yr = 0$ (our only 2×2-minor in this case). This gives $xs = yr$. Since r and s are homogeneous coordinates, we have $(r : s) = (1 : s)$ and $(r : s) = (r : 1)$. We can therefore eliminate either r or s in our equation.

Case 1 Setting $s = 1$ we get $x = yr$. Eliminating x from eq. (1) gives a new equation $g(y,r) = y^2 - y^2r^2 - y^3r^3 = y^2(1 - r^2 - yr^3)$. This can be visualized in the (y,r)-plane.

Case 2 Setting $r = 1$ we get $y = xs$. We can now eliminate y from eq. (1) which gives another new equation $h(x,s) = x^2s^2 - x^2 - x^3 = x^2(s^2 - 1 - x)$ which can be visualized in the (x,s)-plane.