Contents

1 Preliminaries

In algebraic geometry the central objects of study is the set of solutions to polynomial equations. Finding the zeros of, as an example, the quadratic equation

$$ax^2 + bx + c = 0$$

is something most people that have been exposed to elementary mathematics have done. Later on, one learns how to find the zeroes of more general functions, of either one or several variables over some general field of coefficients. This leads us to our first definition.

Definition 1 (Polynomial ring [Pol16])**.** A *polynomial ring* in n variables over a field K is the set of all functions on the form

$$f(x_1, \dots, x_n) = \sum_{\alpha} p_\alpha \prod_{i=1}^{n} x_n^{\alpha_i}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index, and $p_\alpha = p_{\alpha_1,\dots,\alpha_n}$ is an element in K. We denote this polynomial ring $K[x_1, x_2, \dots, x_n]$. $\diamondsuit$

As an example, the most familiar polynomial ring is $\mathbb{R}[x]$. Elements of this ring include

$$f_1(x) = x^2, \qquad f_2(x) = \pi, \qquad f_3(x) = x^{17} + x^3 - e^4, \qquad f_4(x) = x^2 + 1.$$

Note that some of these have zeroes in the real numbers $\mathbb{R}$ while some does not. Most notably is the function f_4 whose zeroes lie in the realm of complex numbers, $\mathbb{C}$.

Given a set of polynomials, $f_1, f_2, \dots, f_m$ in a polynomial ring $K[x_1, x_2, \dots, x_n]$ we can look at their common zeroes. It is in our interest to restrict our attention to polynomial rings where the field of coefficients is algebraically closed. We recall the definition:

Definition 2 (Algebraically closed field [FK03, p. 287])**.** A field K is *algebraically closed* if every non-constant polynomial in $K[x_1, x_2, \dots, x_n]$ has a zero in K. $\diamondsuit$

Given an algebraically closed field K, we define *affine n-space* over K to be the set of all n-tuples of elements of K. We denote affine n-space by $\mathbb{A}^n_K$ or $\mathbb{A}^n$ if the field of coefficients is clear from context. For our intents and purposes, we will be working over the algebraically closed field of complex numbers $\mathbb{C}$.

We now define the mathematical object that is of utmost importance in the study of zeroes of polynomials, namely the variety.

Definition 3 (Affine algebraic variety [Har77, p. 3])**.** Given a set of polynomials $f_1, f_2, \dots, f_m$ from a ring of polynomials, the set of points $(x_1, x_2, \dots, x_n)$ in the affine space $\mathbb{A}^n_K$ that satisfy

$$f_i(x_1, x_2, \dots, x_n) = 0 \quad \text{for } i = 1, 2, \dots, m$$

is called an *affine algebraic variety* and is denoted $\mathbb{V}(f_1, f_2, \dots, f_m)$. $\diamondsuit$

Given the polynomial $f(x) = x^2 - 1$, the affine algebraic variety $\mathbb{V}(f)$ is then the set $\{1, -1\}$ since $f(1) = f(-1) = 0$. Given an additional polynomial $g(x) = x - 1$ we have $\mathbb{V}(f, g) = \{1\}$ since $g(1) = 0$, but $g(-1) \neq 0$. Finally, if we also consider the polynomial $h(x) = x^2$, then $\mathbb{V}(f, g, h) = \emptyset$. We see as a simple fact, and state without proof the following.

Corollary 1. Given a set of polynomials $f_1, f_2, \dots, f_m$, the variety $\mathbb{V}(f_1, f_2, \dots, f_m)$ can be expressed as a finite intersection

$$\mathbb{V}(f_1, \dots, f_m) = \bigcap_{i=1}^{m} \mathbb{V}(f_i).$$

It is also of interest, especially when aiming to create and visualize interesting figures, that the following hold:

Corollary 2. Given a set of polynomials $f_1, f_2, \dots, f_m$, the affine algebraic variety of the product $f_1 f_2 \cdots f_m$ is expressible as a finite union

$$\mathbb{V}(f_1 f_2 \cdots f_m) = \bigcup_{i=1}^{m} \mathbb{V}(f_i).$$

With this in mind, we define one of the more interesting properties of varieties, and one we will work with a lot in this article, namely the singularities.

Definition 4 (Singularity [Whi08, p. 3])**.** A singularity of the variety $V = \mathbb{V}(f_1, f_2, \dots, f_m) \subset \mathbb{A}^n_K$ is a point $p \in V$ such that the Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (p)$$

has rank strictly less than $\min(m, n)$. $\diamondsuit$

A very useful consequence of this definition deal with the varieties of single polynomials and will aid us tremendously in the search for singularities.

Corollary 3. A point p is a singularity of the variety $V = \mathbb{V}(f) \subset \mathbb{A}^n_K$ if and only if the partial derivatives of f vanish identically at p.

Proof. We recall that the rank of a matrix is the dimension of its column space, that is for a matrix A, $\operatorname{rank}(A) = \dim(\operatorname{Col}(A))$. Assume first that p is a singularity of V. Then $\operatorname{rank}(A) < min(n, m) = 1$. This means that the matrix A has no independent columns, otherwise the columns of A would span a space of dimension greater than or equal to one. The Jacobian matrix reads in this case

$$A = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) & \dots & \frac{\partial f}{\partial x_n}(p) \end{bmatrix}.$$

Since the matrix has no independent columns, we cannot have any pivot columns, hence each entry must be zero. Consequently, all partial derivatives of f vanish at p. For the opposite implication, assume that the partials vanish identically at p. Then the Jacobian matrix reads

$$A = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix},$$

and consequently A has rank strictly less than one. Hence p is a point of singularity of V. $\square$

We now have the tools needed to start looking at algebraic surfaces.

2 Algebraic Surfaces

In this section, we try to motivate the definition of an algebraic surface, and look at some examples. Hopefully, this will give an intuitive introduction to exactly what we will we looking at. We start by looking at real algebraic curves, as a good entry point.

2.1 Algebraic Curves

From elementary mathematics, one learns about real valued functions, $f(x)$, and how to plot these functions by setting $y = f(x)$ and plotting points in the (x, y)-plane. This yields a curve of sorts, moving around in the plane.

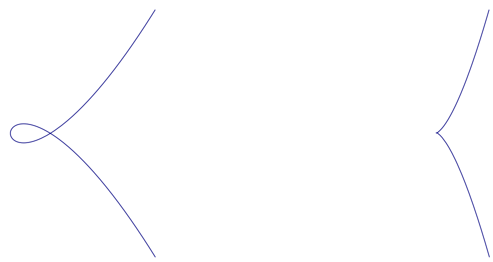
The graph of a function is something of a peculiarity, because it comes with some restrictions. Not all curves in the (x, y)-plane correspond to a specific function. One of the standard methods for verifying whether a certain curve corresponds to a function or not, typically taught in school, is the *vertical line test*. If you can draw a vertical line through two or more points on the curve, then the curve is not associated to any specific function. There is however a limit to how interesting function-curves can be.

Having an equation $y = f(x)$ we can form the *equation of a curve at zero*. We define a new function in two variables and equate it to zero. This yields

$$g(x, y) = y - f(x) = 0.$$

The set of points along the curve described by this equation is precisely the points that lie in the variety $\mathbb{V}(g(x, y))$. If the function $f(x)$ is a polynomial in the variable x, i.e., $f(x) \in K[x]$, we call the function curve of $f(x)$ algebraic. If the distinction between a function graph and curves is to be justified, then there has to be curves that are not function graphs.

Plotting the result when setting $y^2 - x^3$, the plotted curve does not pass the vertical line test and is therefore something different from a function graph. Similarity, the equation $y^2 = x^3 + x^2$ defines another curve that is not a function graph. These are shown in fig. 1, and are prime examples of curves that exhibit a singularity. We will get back to the curve shown in fig. 1a later.



(a) The curve given by the equation $y^2 - x^3 - x^2 = 0$. This curve is commonly known as "The Node". It exhibits a singularity at the point $p = (0, 0)$ which can be verified by computing the partial derivatives at p.

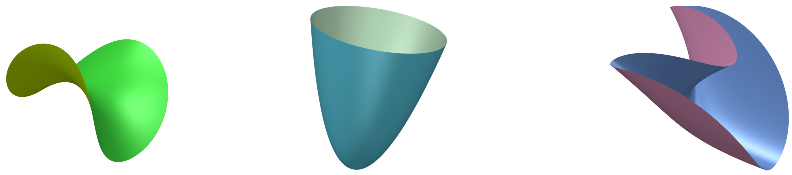
(b) The curve given by $y^2 - x^3 = 0$, or more commonly known as "The Cusp", is also an example of a singular algebraic curve. The point $p = (0, 0)$ is a point of singularity.

Figure 1: Curves not corresponding to functions.

2.2 Moving up a dimension

With what we have done so far, we can generalize and move up a dimension. Instead of considering curves in the (x, y)-plane, we can instead look at surfaces in the three-dimensional (x, y, z)-space. Following the same procedure as above, we define a new function $h(x, y, z) = z - g(x, y)$. Completely analogous to the definition made for curves, we call the surface *algebraic* if $h(x, y, z)$ is a polynomial in the variables x, y, z. That is $h(x, y, z) \in K[x, y, z]$. We give some examples.

Example 1. The following equations satisfy the definition of an algebraic surface:



(a) $h_1(x,y,z) = 0$ (b) $h_2(x,y,z) = 0$ (c) $h_3(x,y,z) = 0$

Figure 2: The surfaces that fell under the category algebraic from example 1.

1. $h_1(x,y,z) = z - xy = 0$,
2. $h_2(x,y,z) = z - x^2 - y^2 = 0$,
3. $h_3(x,y,z) = z - (2x^2 - y)(y - x^2) = 0$,

while the following does *not*:

4. $g_1(x,y,z) = \sin(xy) + z = 0$, and
5. $g_2(x,y,z) = \tan^2(x) + \pi zy$.

These examples are visualized in fig. 2. ♠

We can of course continue this process of introducing new functions and move up a dimension for each time indefinitely. We complete this section by formally defining an algebraic surface. In any case, the surfaces are defined by the variety of some defining polynomial.

Definition 5 (Algebraic surface)**.** An *algebraic surface* is an algebraic variety of dimension[1] two. We define the *degree* of an algebraic surface to be the highest combined power in the defining polynomials terms, and denote this d. That is,

$$d = \max_{\alpha} \{\alpha_1 + \alpha_2 + \dots \alpha_n\}$$

where α_i is the power of x_i in a term. ♢

We now take a brief look at some of the properties of these surfaces in relation to the singularities.

3 Singularities

The singularities of an algebraic surface is one of the important elements that give the surface its visual appeal. I am sure most people agree with me, when I say that a surface full of singularities is more intriguing than one with no singularities at all. The singularities, along with their configurations, symmetries and their number are things that one can explore further.

[1]The dimension of an algebraic variety is a bit hard to grasp, and since we only will be dealing with algebraic surfaces in this text we do not spend any time classifying the different kinds of varieties. The explanation of this is therefore omitted. See [Dim16].

3.1 The number of singularities

For an algebraic surface of degree d we let $\mu(d)$ denote the maximal number of singularities such a surface can have. Determining $\mu(d)$ for even relatively small d has proven to be surprisingly difficult. [Lab14] gives a very intuitive introduction to the world of singularity-hunting and the search for the largest possible $\mu(d)$. We shall repeat some of the results here. We start with a non-example of a singular surface.

Proposition 1 (Singularities of the plane)**.** Let $V = \mathbb{V}(f(x,y,z)) = \mathbb{V}(ax + by + cz + d)$ be a plane, then V contains no singular points.

Proof. Assume for contradiction that $p \in V$ is a singular point. Then the Jacobian matrix of $ax + by + cz + d$ evaluated at p must have rank strictly less than 1, or equivalently, by corollary 3, the partial derivatives vanish at p. The partial derivatives of our defining polynomial is given by

$$\frac{\partial f}{\partial x} = a, \qquad \frac{\partial f}{\partial y} = b, \qquad \frac{\partial f}{\partial z} = c,$$

however, none of these are dependent on the point we chose. Hence, for these partials to vanish at p, we need $a = b = c = 0$. However, our variety V then reduces to $\mathbb{V}(d)$ where d is some constant. If $d \neq 0$, then $\mathbb{V}(d) = \emptyset$, and therefore does not contain p at all, which contradicts our assumption. If $d = 0$ on the other hand, then we are left with the equation $0 = 0$ which does not describe a plane. Consequently, p cannot be a singular point of V, and the plane therefore exhibits no singularities. $\square$

The argument above tells us that for a general algebraic surface of degree $d = 1$, we can have no singularities, hence $\mu(1) = 0$.

We now look at a surface with four singularities.

Example 2 (The Cayley cubic)**.** The Cayley cubic, a surface of degree $d = 3$, is defined by the polynomial

$$f(x,y,z) = xyz + (1 - x - y - z)(yz + xz + xy). \tag{1}$$

We are interested in finding the singularities of this surface, if there are any. We first compute the partial derivatives and find that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -(y+z)(2x+y+z-1), \\
\frac{\partial f}{\partial y} &= -(x+z)(x+2y+z-1), \\
\frac{\partial f}{\partial z} &= -(x+y)(x+y+2z-1).
\end{aligned}$$

By inspection, we see that the three partials vanish in the point $(1, 0, 0)$. Due to the symmetric nature[2] of the defining equation, we see that $(0, 1, 0)$ and $(0, 0, 1)$ also are zeroes of these polynomials. Finally, the partials also vanish in the point $(0, 0, 0)$. We also see that $f(1, 0, 0) = f(0, 1, 0) = f(0, 0, 1) = f(0, 0, 0) = 0$ so these are indeed singularities of the Cayley cubic. The symmetries in the defining polynomial is seen reflected in the visualized surface shown in fig. 3.

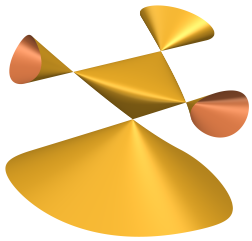


Figure 3: The Cayley cubic, a degree three surface given by the equation $f(x, y, z) = xyz + (1 - x - y - z)(yz + xz + xy)$. It is a good example of a surface whose set of singularities show a lot of symmetry. It also exhibit the maximal number of singularities for a degree 3 surface, namely 4.

♠

From [Lab14, p. 7] we have the following relation on the maximum number of singularities for $d \geq 3$,

$$\mu(d) \leq \frac{1}{2}(d(d-1)^2 - 3)$$

and plugging in $d = 3$ yields $\mu(d) \leq 4$. However, we have already presented a surface of degree $d = 3$ with four singularities, so in fact $\mu(d) = 4$. The theory of finding the maximal number of singularities of a given degree and constructing surfaces that exhibit all is a very technical matter. The thesis [Lab05] gives a good introduction to the construction and visualization of such surfaces.

We have now had a brief look at singularities of surfaces, and how to find them, we now consider how to remove, or *resolve*, such singularities.

4 Resolving singularities

We call a variety that exhibits no singularities *smooth*. It is often possible to study a smooth copy of a variety with singularities if we move from affine space

[2] The study of symmetric functions are also of great interest, which we will not go into detail here. However, as a brief mention, given a function $f(x_1, x_2, \ldots, x_n)$ and a permutation $\sigma \in S_n$, where S_n is the symmetry group on n letters, one can consider the set $\mathcal{S} = \left\{\sigma \in S_n \mid f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right) = f(x_1, x_2, \ldots, x_n)\right\}$ of permutations that fixes the function f. For the Cayley cubic discussed above, any $\sigma \in S_3$ fixes the function, and hence the zeros were easily found by symmetry.

to projective space. This process can be defined in many different ways, with the seemingly most prevalent one being through maps from various spaces to each other. In this paper we will try to keep it as intuitive as possible, by using the more geometrically based procedure. We first define the notion of a projective space.

4.1 Projective space

Informally, projective space generalizes the notion of parallel lines intersecting at infinity. The general definition of projective n-space is the set of lines in $\mathbb{A}^{n+1}$ passing through the origin. For our purposes we will take $\mathbb{A}^{n+1} = \mathbb{C}^{n+1}$.

Definition 6 (The projective space of dimension n)**.** Let $\sim$ be a relation on $\mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$(x_1, \ldots, x_{n+1}) \sim (x'_1, \ldots, x'_{n+1}) \iff (x'_1, \ldots, x'_{n+1}) = \lambda(x_1, \ldots, x_{n+1})$$

where λ is some non-zero complex number. This relation can be shown to be an equivalence relation. We denote the equivalence class of $(x_1, x_2, \ldots, x_{n+1})$ by $[x_1 : x_2 : \ldots : x_{n+1}]$ and define the *projective space of dimension n* as the set

$$\mathbb{P}^n_{\mathbb{C}} = \left\{ [x_1 : \ldots : x_{n+1}] \mid (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\} \right\}.$$

The elements in $\mathbb{P}^n_{\mathbb{C}}$ are called *homogeneous coordinates*. ♢

We now turn to the resolution of singularities. There are, as mentioned above, many ways of doing this, but we will here apply a geometrical approach.

4.2 Blowing up a point singularity.

We start by examining resolutions of single points. Informally, when we blow up a point p in $\mathbb{A}^n$, we replace the point p by an entire copy of a projective space $\mathbb{P}^{n-1}$ and try to leave the rest of the space $\mathbb{A}^n$ "unchanged". We will give a few examples, that hopefully will illustrate the geometrical aspect of this, but first a formal definition.

Definition 7 (Blowup of $\mathbb{A}^n$ at a point p.)**.** We let the *blowup-surface* $\mathrm{Bl}_p\,\mathbb{A}^n$ be the set of all pairs consisting of a point $p \in \mathbb{A}^n$ and a line from p through the origin. The surface is embedded in the space $\mathbb{A}^n \times \mathbb{P}^{n-1}$. We write

$$\mathrm{Bl}_p\,\mathbb{A}^n = \{(p, L) \mid p \in \mathbb{A}^n \text{ and } p \in L\}.$$

♢

Now, this is a fairly technical definition, so where do we go from here? When blowing up a surface at a point p we are essentially giving the surface "more room" so that the points creating the singularities can pass each other without hassle. In order to aid us with the above definition, we show the following claim that tells us under what circumstances the point p lie on the line L:

Proposition 2. Given a point $x = (x_1, x_2, \dots, x_n)$ in $\mathbb{A}^n$ and a point $y = [y_1 : y_2 : \dots : y_n]$ in the space $\mathbb{P}^{n-1}$, the point x is a multiple of y if and only if the matrix

$$A = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix}$$

has rank less than or equal to one. Similarly, the rank is less than or equal to one only if each 2×2-minor $x_i y_j - x_j y_i = 0$ for $1 \leq i < j \leq n$.

Proof. In order to verify this, we recall that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$, so we chose to work with the transposed matrix instead. Assume that x is a multiple of y, so $y = \lambda x$ for some scalar λ. Then

$$A^T = \begin{bmatrix} x_1 & \lambda x_1 \\ x_2 & \lambda x_2 \\ & \vdots \\ x_n & \lambda x_n \end{bmatrix}$$

is the associated matrix. If $x = 0$, then A^T is the zero matrix and has therefore rank 0. If $x \neq 0$, then we have exactly one pivot column in this matrix, hence $\dim(\operatorname{Col}(A^T)) = 1$. In any case, $\operatorname{rank}(A) = \operatorname{rank}(A^T) \leq 1$. For the opposite implication. If A has rank 0, then A is the zero matrix, and hence trivially, x is a multiple of y, both being zero. If A has rank 1, then there is one linearly independent column in A^T, hence the other column can be expressed as a multiple of the first. In other words, x is a multiple of y.

If there were a non-zero minor $r \times r$ for $r > 1$, then this suggests that there are r linearly independent column vectors, but this can't occur since $\operatorname{rank}(A^T) \leq 1$. $\square$

With the above result, we can rewrite the definition of the blowup-surface $\operatorname{Bl}_p \mathbb{A}^n$ as

$$\operatorname{Bl}_p \mathbb{A}^n = \{(x_1, x_2, \dots, x_n; y_1 : y_2 : \dots : y_n) \mid x_i y_j = x_j y_i \text{ for } 1 \leq i < j \leq n\}$$

which is computationally easier to work with.

We now give some examples of the computations involved when finding blowup-surfaces. We start with the resolution of a singularity in an algebraic curve.

Example 3 (The node)**.** The Node, shown in fig. 1a, is a singular algebraic curve with a singularity in the point $p = (0, 0)$. The defining polynomial of the node is given by

$$f(x, y) = y^2 - x^2 - x^3, \tag{2}$$

with the curve $\mathcal{C} = \mathbb{V}(f(x, y, z))$. This curve lies in the affine plane $\mathbb{A}^2$, while the resulting blowup-surface of $\mathcal{C}$ at the point $(0, 0)$ lies in the space $\mathbb{A}^2 \times \mathbb{P}^1$. For a point $(x, y) \in \mathcal{C}$ we need to figure out what line passes through both

(x, y) and the origin $(0, 0)$. However, proposition 2 tells us exactly under what circumstances this occurs. Let $[u : v]$ be a point in $\mathbb{P}^1$. Then, the matrix

$$A = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$$

has rank less than or equal to one only when $xv - yu = 0$, our only 2×2-minor in this case. This gives us the equation $xv = yu$. We chose the point (u, v) as a representative for the equivalence class $[u : v]$ however, we might as well chose the point $(u, 1)$ or $(1, v)$, since we can set either $u = 1$ or $v = 1$ and then multiply the representative by $1/u$ or $1/v$ respectively. This makes us able to eliminate either u or v from $xv = yu$. This gives us two different cases:

Case 1: Setting $u = 1$ we get the equation $y = xv$. This can be substituted back into eq. (2) in order to eliminate y. This gives us a new equation, namely

$$g(x, v) = x^2(v^2 - 1 - x).$$

This function $g(x, v)$ is what we call an *affine chart* of the blowup-surface $\mathrm{Bl}_p\,\mathbb{A}^2$. It can be visualized in the (x, v)-plane.

Case 2: Setting $v = 1$ yields the equation $x = yu$ which makes us able to eliminate x from eq. (2). This gives us a new equation

$$h(y, u) = y^2(1 - u^2 - u^2 y).$$

This is the other affine chart of $\mathrm{Bl}_p\,\mathbb{A}^2$ and can be visualized in the (y, u)-plane. The affine charts given by $h(x, y, z)$ and $g(x, y, z)$ are shown visualized in fig. 4.

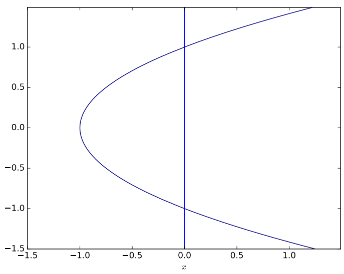
The relation between these two charts, and the original surface can be visualized as in fig. 4c.

♠

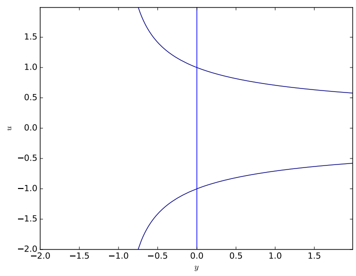
Blowing up singular surfaces or curves can be great fun and a lot of interesting pictures arise. We now take a look at an example where blowing up the surface in a single point is not sufficient for removing the singularities.

4.3 Blowing up a line of singularity.

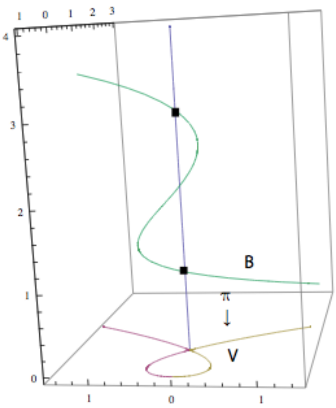
In this section we will consider the algebraic surface



(a) The affine chart resulting from setting $u = 1$ and eliminating y in the original equation.



(b) The affine chart resulting from setting $v = 1$ and eliminating x in the original equation



(c) A visual depiction of the blowup surface that clearly shows how the singularity in the origin is resolved. Taken from [Whi08].

Figure 4: The affine charts of the Node. The vertical line plotted in slightly brighter blue, is called the *exceptional divisor* and corresponds to the blowup of what was the origin in the original surface. Note that the curves intersect the exceptional divisor in exactly two points. This corresponds to the double point singularity in the original surface.

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