EULER'S POLYHEDRON FORMULA

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Abstract. In this short paper we look at Euler's polyhedron formula. This formula was one of my first experiences with more abstract mathematics, and I have loved it ever since. The proof is very simple, and extremely elegant.

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1. Introduction and motivation

As a vivid reader of popular science one often comes across the problem regarding the *Seven Bridges of Köningsberg*. This city, located in what was then known as Prussia had a river running through the middle. This river naturally divided the city into the mainland — on either side — as well as two large islands. Connecting these islands with the mainland were seven bridges. This is where the famous question arises:

> *Can the seven bridges of the city of Köningsberg, over the river Preger, all be traversed in a single trip without doubling back and have the trip end in the same place it began?*

Eminent mathematician Leonhard Euler proved that the problem stated above has no solution. In working with the solution to this problem however, he dabbled in a type of mathematics later to be known as *graph theory*. One of the more remarkable results (in my humble opinion) to arise from this new field of mathematics is the discovery of a relationship between the number of vertices, edges and faces in a polyhedron. It is exactly this relationship that Euler's polyhedron formula captures.

Theorem 1 (Euler's polyhedron formula)**.** *Any convex polyhedron's surface satisfies the equation*

$$V - E + F = 2$$

where V, E and F denote the number of vertices, edges and faces, respectively.

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2. Preliminaries

In order to get a firm grasp of this formula, we first need to recall some elementary geometry. We start with the definition of the *polyhedron*.

Definition 1 (Polyhedron)**.** A *polyhedron* is a solid in three dimensions with flat polygonal faces, straight edges and sharp corners or vertices.[1]

Polyhedra are named based on the number of faces using the greek prefix system. Some examples include the *hexahedron*, otherwise known as the cube with its six faces; the *octahedron* or *double pyramid* with its eight faces; and the *icosahedron* with its 20 faces.

An important tool when studying platonic solids is that of the *Schlegel diagram*. It is essentially a way of projecting a d-dimensional object into $d-1$ dimensions. In our case, we want to study the three dimensional platonic solids in the two dimensional plane. Each platonic solid, when projected into two dimensions, form what we call a *planar graph*.

Definition 2 (Planar graph)**.** A *planar graph*, is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

3. Proof of the formula

The proof of Euler's polyhedron formula is very simple and applies to any convex polyhedron. We will use a cube as our example. We first take note of the number of vertices, faces and edges. A cube has six faces, $F = 6$, twelve edges, $E = 12$, and eight vertices, $V = 8$. This yields

$$V - E + F = 8 - 12 + 6 = 2. \tag{1}$$

Our goal now is to progressively remove edges, faces and vertices until we are left with a very simple polygon. We will use transformations that preserve the relation shown in equation 1. We start by projecting the polyhedron into the plane using a Schlegel diagram. This is done by removing one face and squishing the cube into the plane, as shown in the leftmost image in figure 1. Note that removing one face does not alter the number of vertices and edges, hence we now have $V - E + F = 1$.

We now triangulate any face with more than three edges until all faces are triangular. This is done by adding one edge, but doing so also adds a new face. Hence the quantity $V - E + F$ is unchanged.

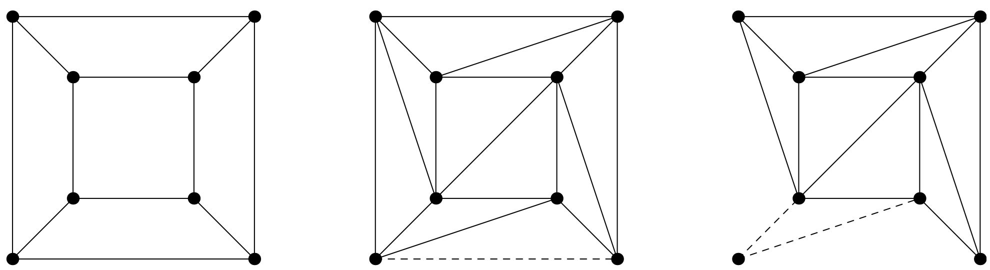

Figure 1. First iterations of proof; using a cube.

[1] This definition is not very precice, but it will suffice for our purposes.

We now have two possible transformations to iterate:

(1) Remove a triangle with one edge adjacent to the exterior as shown in the middle image in figure 1.
(2) Remove a triangle with two edges adjacent to the exterior as shown in the rightmost image in figure 1.

Note that neither of these operations change the quantity of interest. One removes one edge and one face and the other removes one vertex, two edges, and one face. We apply these transformations while always keeping the requirement that the outer edges form a simple cycle. We will eventually end up with a triangle with three vertices, three edges and one face.