# Algebraic Surfaces

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MAT2000 - Spring 2016

**Abstract**

In this paper we give a brief overview of some of the things one can be looking at when studying algebraic surfaces. We use a fairly naive approach without the very abstract mathematical machinery that this field has to offer. In section 1 we give some of the needed mathematical preliminaries. Section 2 motivates the definition of an algebraic surface, the main object of study. In section 3 we look at one of the main properties of algebraic surfaces, namely the singular points, and proceed by showing how to resolve them i section 4. Finally, we give an example of how to construct a singular surface with a certain symmetry in section 5. The images are rendered using the software Surfer unless specified otherwise. Some auxiliary Python and Mathematica scripts has also been used.

## Contents



# 1 Preliminaries

In algebraic geometry the central objects of study is the set of solutions to polynomial equations. Finding the zeros of, as an example, the quadratic equation

$$ax^2 + bx + c = 0$$

is something most people that have been exposed to elementary mathematics have done. Later on, one learns how to find the zeroes of more general functions, of either one or several variables over some general field of coefficients. Of special interest to us, are the polynomials over one or several variables, given as elements of a *polynomial ring* over some field which we denote $K[x_1, x_2, \ldots, x_n]$.

As an example, the most familiar polynomial ring is $\mathbb{R}[x]$. Elements of this ring include

$$f_1(x) = x^2, \qquad f_2(x) = \pi, \qquad f_3(x) = x^{17} + x^3 - e^4, \qquad f_4(x) = x^2 + 1.$$

Note that some of these have zeroes in the real numbers $\mathbb{R}$ while some does not. Most notably is the function $f_4$ whose zeroes lie in the realm of complex numbers, .

Given a set of polynomials, $f_1, f_2, \ldots, f_m$ in a polynomial ring $K[x_1, x_2, \ldots, x_n]$ we can look at their common zeroes. It is in our interest to restrict our attention to polynomial rings where the field of coefficients is algebraically closed.

Given an algebraically closed field $K$, we define *affine $n$-space* over $K$ to be the set of all $n$-tuples of elements of $K$. We denote affine $n$-space by $\mathbb{A}_K^n$ or $\mathbb{A}^n$ if the field of coefficients is clear from context. For our intents and purposes, we will be working over the algebraically closed field of complex numbers .

We now define the mathematical object that is of utmost importance in the study of zeroes of polynomials, namely the variety.

**Definition 1** (Affine algebraic variety [Har77, p. 3])**.** Given a set of polynomials $f_1, f_2, \ldots, f_m$ from a ring of polynomials, the set of points $(x_1, x_2, \ldots, x_n)$ in the affine space $\mathbb{A}_K^n$ that satisfy

$$f_i(x_1, x_2, \ldots, x_n) = 0 \quad \text{for } i = 1, 2, \ldots, m$$

is called an *affine algebraic variety* and is denoted $\mathbb{V}(f_1, f_2, \ldots, f_m)$. $\diamondsuit$

Given the polynomial $f(x) = x^2 - 1$, the affine algebraic variety $\mathbb{V}(f)$ is then the set $\{1, -1\}$ since $f(1) = f(-1) = 0$. Given an additional polynomial $g(x) = x - 1$ we have $\mathbb{V}(f, g) = \{1\}$ since $g(1) = 0$, but $g(-1) \neq 0$. Finally, if we also consider the polynomial $h(x) = x^2$, then $\mathbb{V}(f, g, h) = \emptyset$. We see as a simple fact, and state without proof the following.

**Corollary 1.** Given a set of polynomials $f_1, f_2, \ldots, f_m$, the variety $\mathbb{V}(f_1, f_2, \ldots, f_m)$ can be expressed as a finite intersection

$$\mathbb{V}(f_1, \ldots, f_m) = \bigcap_{i=1}^{m} \mathbb{V}(f_i).$$

It is also of interest, especially when aiming to create and visualize interesting figures, that the following hold:

**Corollary 2.** Given a set of polynomials $f_1, f_2, \ldots, f_m$, the affine algebraic variety of the product $f_1 f_2 \cdots f_m$ is expressible as a finite union

$$\mathbb{V}(f_1 f_2 \cdots f_m) = \bigcup_{i=1}^{m} \mathbb{V}(f_i).$$

With this in mind, we define one of the more interesting properties of varieties, and one we will work with a lot in this article, namely the singularities. A singular point is, strictly speaking, a point on a surface where the behaviour of the surface is slightly strange, say the surface passes through itself. We are often interested in studying how a surface behaves locally near a point $p$. We do this using *linearization at $p$*. This is what we do when we consider the tangent line to a point for a single-valued function, the tangent plane for a general surface in three dimensions, and a tangent *hyperplane* for general surfaces in $n$ dimensions. The linearization of a surface given by $f(x_1, x_2, \ldots, x_n) = 0$, at a point $p_0$ is given by

$$L(p) = f(p_0) + J_f(p_0)(p - p_0)$$

where $J_f(p_0)$ is the Jacobian matrix of $f(x_1, x_2, \ldots, x_n)$ evaluated at $p_0$. If this matrix has rank less than $n$, then the linearization is not uniquely defined, i.e., there are more than one plane satisfying the linearization. This leads us to the next definition.

**Definition 2** (Singularity [Whi08, p. 3])**.** A singularity of the variety $V = \mathbb{V}(f_1, f_2, \ldots, f_m) \subset \mathbb{A}_K^n$ is a point $p \in V$ such that the Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (p)$$

has rank strictly less than $\min(m, n)$. $\diamondsuit$

A very useful consequence of this definition deal with the varieties of single polynomials and will aid us tremendously in the search for singularities.

**Corollary 3.** A point $p$ is a singularity of the variety $V = \mathbb{V}(f) \subset \mathbb{A}^n_K$ if and only if the partial derivatives of $f$ vanish identically at $p$.

*Proof.* We recall that the rank of a matrix is the dimension of its column space, that is for a matrix $A$, $\text{rank}(A) = \dim(\text{Col}(A))$. Assume first that $p$ is a singularity of $V$. Then $\text{rank}(A) < min(n, m) = 1$. Consequently, we must have $\text{rank}(A) = 0$, and therefore $A$ equal to the zero matrix. All the partial derivatives are therefore zero. For the opposite implication, assume that the partials vanish identically at $p$. Then the Jacobian matrix reads

$$A = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix},$$

and consequently $A$ has rank strictly less than one. Hence $p$ is a point of singularity of $V$. □

Finally, we want to define the notion of a projective space, as this will aid us in the resolution of singularities later on. Informally, projective space generalizes the notion of parallel lines intersecting at infinity. The general definition of projective $n$-space is the set of lines in $\mathbb{A}^{n+1}$ passing through the origin. For our purposes we will take $\mathbb{A}^{n+1} =^{n+1}$.

**Definition 3** (The projective space of dimension $n$)**.** Let $\sim$ be a relation on ${}^{n+1} \setminus \{0\}$ defined by

$$(x_1, \ldots, x_{n+1}) \sim (x'_1, \ldots, x'_{n+1}) \iff (x'_1, \ldots, x'_{n+1}) = \lambda(x_1, \ldots, x_{n+1})$$

where $\lambda$ is some non-zero complex number. This relation can be shown to be an equivalence relation. We denote the equivalence class of $(x_1, x_2, \ldots, x_{n+1})$ by $[x_1 : x_2 : \ldots : x_{n+1}]$ and define the *projective space of dimension* $n$ as the set

$$\mathbb{P}^n_{=\{[x_1:\ldots:x_{n+1}] | (x_1,\ldots,x_{n+1}) \in^{n+1} \setminus \{0\}\}.}$$

The elements in $\mathbb{P}_n$ are called *homogeneous coordinates.* ♢

We now have the tools needed to start looking at algebraic surfaces.

# 2 Introduction to algebraic surfaces

In this section, we try to motivate the definition of an algebraic surface, and look at some examples. Hopefully, this will give an intuitive introduction to exactly what we will we looking at. We start by looking at real algebraic curves, as a good entry point.

## 2.1 Algebraic Curves

From elementary mathematics, one learns about real valued functions, $f(x)$, and how to plot these functions by setting $y = f(x)$ and plotting points in the $(x, y)$-plane. This yields a curve of sorts, moving around in the plane. Having an equation $y = f(x)$ we can form the *equation of a curve at zero.* We define a new function in two variables and equate it to zero. This yields

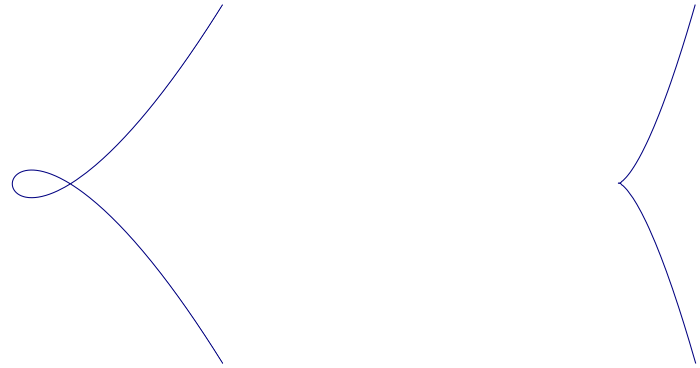
$$g(x, y) = y - f(x) = 0.$$

The set of points along the curve described by this equation is precisely the points that lie in the variety $\mathbb{V}(g(x, y))$. If the function $f(x)$ is a polynomial in the variable $x$, i.e., $f(x) \in K[x]$, we call the function curve of $f(x)$ algebraic. Now, all the curves formed in this way corresponds to some function. There are however curves that does not correspond to functions and these are given as general polynomials.

Plotting the result when setting $y^2 - x^3$, the plotted curve does not pass the vertical line test and is therefore something different from a function graph. Similarity, the equation $y^2 = x^3 + x^2$ defines another curve that is not a function graph. These are shown in figure 1, and are prime examples of curves that exhibit a singularity. We will get back to the curve shown in figure 1a later.

## 2.2 Moving up a dimension

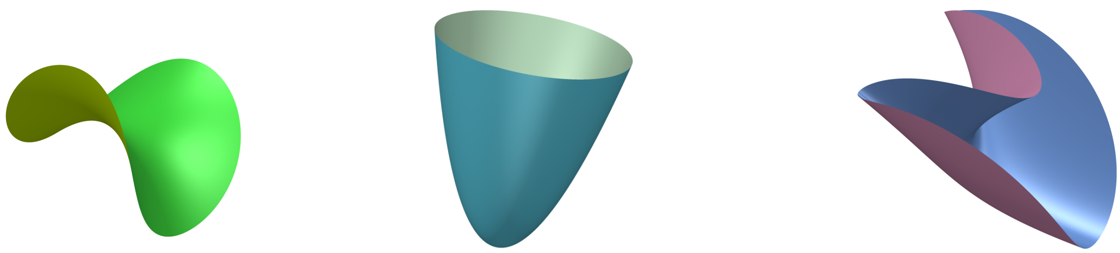
With what we have done so far, we can generalize and move up a dimension. Instead of considering curves in the $(x, y)$-plane, we can instead look at surfaces in the three-dimensional $(x, y, z)$-space. Following the same procedure as above, we define a new function $h(x, y, z) = z - g(x, y)$. This function equated at zero yields a surface. Completely analogous to the definition made for curves, we call the surface *algebraic* if $h(x, y, z)$ is a polynomial in the variables $x, y, z$. That is $h(x, y, z) \in K[x, y, z]$. We give some examples.



(a) The curve given by the equation $y^2 - x^3 - x^2 = 0$. This curve is commonly known as “The Node”. It exhibits a singularity at the point $p = (0, 0)$ which can be verified by computing the partial derivatives at $p$.

(b) The curve given by $y^2 - x^3 = 0$, or more commonly known as “The Cusp”, is also an example of a singular algebraic curve. The point $p = (0, 0)$ is a point of singularity.

Figure 1: First examples of algebraic curves. Visualized using PYTHON. Code can be seen in appendix B.



(a) $h_1(x, y, z) = 0$ (b) $h_2(x, y, z) = 0$ (c) $h_3(x, y, z) = 0$

Figure 2: The surfaces that fell under the category algebraic from example 1.

**Example 1.** The following equations satisfy the definition of an algebraic surface:

1. $h_1(x, y, z) = z - xy = 0$,
2. $h_2(x, y, z) = z - x^2 - y^2 = 0$,
3. $h_3(x, y, z) = z - (2x^2 - y)(y - x^2) = 0$,

while the following does *not*:

4. $g_1(x, y, z) = \sin(xy) + z = 0$, and
5. $g_2(x, y, z) = \tan^2(x) + \pi zy$.

These examples are visualized in figure 2. ♠

We can of course continue this process of introducing new functions and move up a dimension for each time indefinitely. We complete this section by formally defining an algebraic surface. In any case, the surfaces are defined by the variety of some defining polynomial.

**Definition 4** (Algebraic surface)**.** Given a set of polynomials $f_1, f_2, \ldots, f_m$ in three variables, i.e., in $K[x, y, z]$ we define an *algebraic surface* to be the algebraic variety $\mathbb{V}(f_1, f_2, \ldots, f_m)$. We define the *degree* of an algebraic surface to be the highest combined power in the defining polynomials terms, and denote this $d$. That is,

$$d = \max_{\alpha} \{\alpha_1 + \alpha_2 + \ldots \alpha_n\}$$

where $\alpha_i$ is the power of $x_i$ in a term. ♢

We now take a brief look at some of the properties of these surfaces in relation to the singularities.

# 3 Singularities

When studying algebraic surfaces we typically differentiate between two types of surfaces, namely the singular ones and the non-singular ones. These two categories are each interesting in their own respect. A lot of mathematical theory concerning surfaces only hold for surfaces that exhibit no singularities at all. It is therefore often of interest to somehow remove these singularities from the surface in order to apply known results.

The theory of singular surfaces on the other hand contains a lot of interesting theory that deals with: the classification of singularities, what different types of singularities are there; resolution of singularities; and deformation theory.

## 3.1 The number of singularities

For an algebraic surface of degree $d$ we let $\mu(d)$ denote the maximal number of singularities such a surface can have. Determining $\mu(d)$ for even relatively small $d$ has proven to be surprisingly difficult. [Lab14] gives a very intuitive introduction to the world of singularity-hunting and the search for the largest possible $\mu(d)$. We shall repeat some of the results here. We start with a non-example of a singular surface.

**Proposition 1** (Singularities of the plane)**.** Let $V = \mathbb{V}(f(x,y,z)) = \mathbb{V}(ax+by+cz+d)$ be a plane, then $V$ contains no singular points.

*Proof.* Assume for contradiction that $p \in V$ is a singular point. Then the Jacobian matrix of $ax + by + cz + d$ evaluated at $p$ must have rank strictly less than 1, or equivalently, by corollary 3, the partial derivatives vanish at $p$. The partial derivatives of our defining polynomial is given by

$$\frac{\partial f}{\partial x} = a, \qquad \frac{\partial f}{\partial y} = b, \qquad \frac{\partial f}{\partial z} = c,$$

however, none of these are dependent on the point we chose. Hence, for these partials to vanish at $p$, we need $a = b = c = 0$. However, our variety $V$ then reduces to $\mathbb{V}(d)$ where $d$ is some constant. If $d \neq 0$, then $\mathbb{V}(d) = \emptyset$, and therefore does not contain $p$ at all, which contradicts our assumption. If $d = 0$ on the other hand, then we are left with the equation $0 = 0$ which does not describe a plane. Consequently, $p$ cannot be a singular point of $V$, and the plane therefore exhibits no singularities. $\square$

The argument above tells us that for a general algebraic surface of degree $d = 1$, we can have no singularities, hence $\mu(1) = 0$.

We now look at a surface with four singularities.

**Example 2** (The Cayley cubic)**.** The Cayley cubic, a surface of degree $d = 3$, is defined by the polynomial

$$f(x,y,z) = xyz + (1 - x - y - z)(yz + xz + xy). \tag{1}$$

We are interested in finding the singularities of this surface, if there are any. We first compute the partial derivatives and find that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -(y+z)(2x+y+z-1), \\
\frac{\partial f}{\partial y} &= -(x+z)(x+2y+z-1), \\
\frac{\partial f}{\partial z} &= -(x+y)(x+y+2z-1).
\end{aligned}$$

By inspection, we see that the three partials vanish in the point $(1,0,0)$. Due to the symmetric nature[1] of the defining equation, we see that $(0,1,0)$ and $(0,0,1)$ also are zeroes of these polynomials. Finally, the partials also vanish in the point $(0,0,0)$. We also see that $f(1,0,0) = f(0,1,0) = f(0,0,1) = f(0,0,0) = 0$ so these are indeed singularities of the Cayley cubic. The symmetries in the defining polynomial is seen reflected in the visualized surface shown in figure 3.

♠

[1]The study of symmetric functions are also of great interest, which we will not go into detail here. However, as a brief mention, given a function $f(x_1, x_2, \ldots, x_n)$ and a permutation $\sigma \in S_n$, where $S_n$ is the symmetry group on $n$ letters, one can consider the set $\mathcal{S} = \left\{\sigma \in S_n \mid f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right) = f(x_1, x_2, \ldots, x_n)\right\}$ of permutations that fixes the function $f$. For the Cayley cubic discussed above, any $\sigma \in S_3$ fixes the function, and hence the zeros were easily found by symmetry.

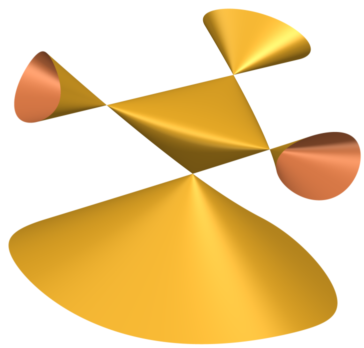


Figure 3: The Cayley cubic, a degree three surface given by the equation $f(x, y, z) = xyz+(1-x-y-z)(yz+xz+xy)$. It is a good example of a surface whose set of singularities show a lot of symmetry. It also exhibit the maximal number of singularities for a degree 3 surface, namely 4.

From [Lab14, p. 7] we have the following upper bound on the maximum number of singularities for $d \geq 3$,

$$\mu(d) \leq \frac{1}{2}(d(d-1)^2 - 3)$$

and plugging in $d = 3$ yields $\mu(3) \leq 4$. However, we have already presented a surface of degree $d = 3$ with four singularities, so in fact $\mu(d) = 4$. The theory of finding the maximal number of singularities of a given degree and constructing surfaces that exhibit all is a very technical matter. The thesis [Lab05] gives a good introduction to the construction and visualization of such surfaces.

We have now had a brief look at singularities of surfaces, and how to find them, we now consider how to remove, or *resolve*, such singularities.

# 4 Resolving singularities

We call a variety that exhibits no singularities *smooth*. It is often possible to study a smooth copy of a variety with singularities if we move from affine space to projective space. This process can be defined in many different ways, with the seemingly most prevalent one being through maps from various spaces to each other. In this paper we will try to keep it as intuitive as possible, by using the more geometrically based procedure.

When we blow up, we blow up along a *center* of the variety. By making sure that this center is contained in the singular locus of the variety, we do not change the surrounding surface. We first consider the case where the blowup center is a single point, and then an example where the center is a line.

## 4.1 Blowing up a point

We start by examining resolutions of single points. Informally, when we blow up a point $p$ in $\mathbb{A}^n$, we replace the point $p$ by an entire copy of a projective space $\mathbb{P}^{n-1}$ and try to leave the rest of the space $\mathbb{A}^n$ "unchanged". We will give a few examples, that hopefully will illustrate the geometrical aspect of this, but first a formal definition.

**Definition 5** (Blowup of $\mathbb{A}^n$ at a point $p$.)**.** We let the *blowup-surface* $\mathrm{Bl}_p \mathbb{A}^n$ be the set of all pairs consisting of a point $p \in \mathbb{A}^n$ and a line from $p$ through the origin. The surface is embedded in the space $\mathbb{A}^n \times \mathbb{P}^{n-1}$. We write

$$\mathrm{Bl}_p \mathbb{A}^n = \{(p, L) \mid p \in \mathbb{A}^n \text{ and } p \in L\}.$$

$\diamondsuit$

Now, this is a fairly technical definition, so where do we go from here? When blowing up a surface at a point $p$ we are essentially giving the surface "more room" so that the points creating the singularities can pass each other without hassle. In order to aid us with the above definition, we show the following claim that tells us under what circumstances the point $p$ lie on the line $L$:

**Proposition 2.** Given a point $x = (x_1, x_2, \dots, x_n)$ in $\mathbb{A}^n$ and a point $y = [y_1 : y_2 : \dots : y_n]$ in the space $\mathbb{P}^{n-1}$, the point $x$ is a multiple of $y$ if and only if the matrix

$$A = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix}$$

has rank less than or equal to 1. Similarly, the rank is less than or equal to 1 if and only if each $2 \times 2$-minor $x_i y_j - x_j y_i = 0$ for $1 \leq i < j \leq n$.

*Proof.* In order to verify this, we recall that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$, so we chose to work with the transposed matrix instead. Assume that $x$ is a multiple of $y$, so $y = \lambda x$ for some scalar $\lambda$. Then

$$A^T = \begin{bmatrix} x_1 & \lambda x_1 \\ x_2 & \lambda x_2 \\ & \vdots \\ x_n & \lambda x_n \end{bmatrix}$$

is the associated matrix. If $x = 0$, then $A^T$ is the zero matrix and has therefore rank 0. If $x \neq 0$, then we have exactly one pivot column in this matrix, hence $\dim(\operatorname{Col}(A^T)) = 1$. In any case, $\operatorname{rank}(A) = \operatorname{rank}(A^T) \leq 1$. For the opposite implication. If $A$ has rank 0, then $A$ is the zero matrix, and hence trivially, $x$ is a multiple of $y$, both being zero. If $A$ has rank 1, then there is one linearly independent column in $A^T$, hence the other column can be expressed as a multiple of the first. In other words, $x$ is a multiple of $y$.

If there were a non-zero minor $r \times r$ for $r > 1$, then this suggests that there are $r$ linearly independent column vectors, but this can't occur since $\operatorname{rank}(A^T) \leq 1$. $\square$

With the above result, we can rewrite the definition of the blowup-surface $\operatorname{Bl}_p \mathbb{A}^n$ as

$$\operatorname{Bl}_p \mathbb{A}^n = \{(x_1, x_2, \dots, x_n; y_1 : y_2 : \dots : y_n) \mid x_i y_j = x_j y_i \text{ for } 1 \leq i < j \leq n\}$$

which is computationally easier to work with.

We now give some examples of the computations involved when finding blowup-surfaces. We start with the resolution of a singularity in an algebraic curve.

**Example 3** (The node)**.** The Node, shown in figure 1a, is a singular algebraic curve with a singularity in the point $p = (0, 0)$. The defining polynomial of the node is given by

$$f(x, y) = y^2 - x^2 - x^3, \tag{2}$$

with the curve $\mathcal{C} = \mathbb{V}(f(x, y, z))$. This curve lies in the affine plane $\mathbb{A}^2$, while the resulting blowup-surface of $\mathcal{C}$ at the point $(0, 0)$ lies in the space $\mathbb{A}^2 \times \mathbb{P}^1$. For a point $(x, y) \in \mathcal{C}$ we need to figure out what line passes through both $(x, y)$ and the origin $(0, 0)$. However, proposition 2 tells us exactly under what circumstances this occurs. Let $[u : v]$ be a point in $\mathbb{P}^1$. Then, the matrix

$$A = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$$

has rank less than or equal to one only when $xv - yu = 0$, our only $2 \times 2$-minor in this case. This gives us the equation $xv = yu$. We chose the point $(u, v)$ as a representative for the equivalence class $[u : v]$ however, we might as well chose the point $(u, 1)$ or $(1, v)$, since we can set either $u = 1$ or $v = 1$ and then multiply the representative by $1/u$ or $1/v$ respectively. This makes us able to eliminate either $u$ or $v$ from $xv = yu$. This gives us two different cases:

**Case 1:** Setting $u = 1$ we get the equation $y = xv$. This can be substituted back into equation (2) in order to eliminate $y$. This gives us a new equation, namely

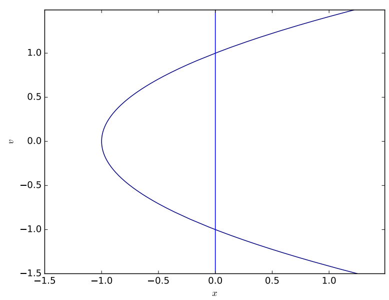
$$g(x, v) = x^2(v^2 - 1 - x) = 0.$$

The variety $\mathbb{V}(g(x, v))$ is what we call an *affine chart* of the blowup-surface $\operatorname{Bl}_p \mathbb{A}^2$. It can be visualized in the $(x, v)$-plane.

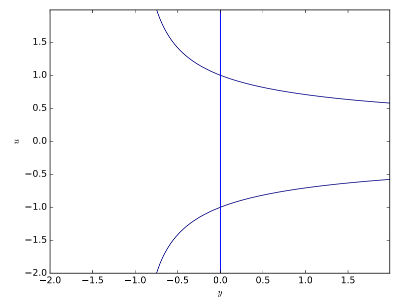
**Case 2:** Setting $v = 1$ yields the equation $x = yu$ which makes us able to eliminate $x$ from equation (2). This gives us a new equation

$$h(y,u) = y^2(1 - u^2 - u^2 y) = 0.$$

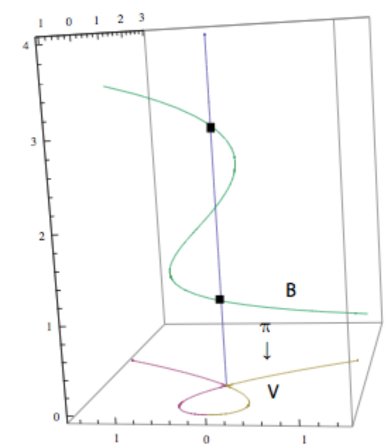
This is the other affine chart of $\mathrm{Bl}_p\,\mathbb{A}^2$ and can be visualized in the $(y,u)$-plane. The affine charts given by $h(x,y,z)$ and $g(x,y,z)$ are shown visualized in figure 4. The relation between these two charts, and the original surface can be visualized as in figure 4c.



(a) The affine chart resulting from setting $u = 1$ and eliminating $y$ in the original equation.



(b) The affine chart resulting from setting $v = 1$ and eliminating $x$ in the original equation




(c) A visual depiction of the blowup surface that clearly shows how the singularity in the origin is resolved. Taken from [Whi08].

Figure 4: The affine charts of the Node. The vertical line plotted in slightly brighter blue, is called the *exceptional divisor* and corresponds to the blowup of what was the origin in the original surface. Note that the curves intersect the exceptional divisor in exactly two points. This corresponds to the double point singularity in the original surface. The top row is visualized in Python.

♠

Blowing up singular surfaces or curves can be great fun and a lot of interesting pictures arise. We now take a look at an example where blowing up the surface in a single point is not sufficient for removing the singularities.

## 4.2 Blowing up a line

In this section we will consider the algebraic surface given by the polynomial

$$f(x,y,z) = x^2 y - z^2. \tag{3}$$

The associated variety is $\mathbb{V}(f(x,y,z))$. This surface is commonly known as the "Whitney Umbrella" and is shown visualized in figure 5. As we will see shortly, when trying to blow up this surface in a single point, say the origin, the resulting surface does not seem to be any "less singular".

Figure 5: The "Whitney Umbrella" defined by the polynomial $f(x, y, z) = x^2y - z^2$. This surface exhibits singularities all along the $y$-axis, as can easily be pointed out on the picture.

**Example 4** (Blowup of the Whitney Umbrella in a single point $p$)**.** If we were to proceed by trying to blow up the origin, we quickly encounter a problem. Following the standard procedure that we used in the previous example, we get the defining equations $xs = yr$, $xt = zr$ and $yt = zs$ with $r, s$ and $t$ our homogeneous coordinates. Contrary to the previous example, we here get three affine charts, one for each homogeneous coordinate.

**Case 1:** Setting $r = 1$ gives us $y = xs$ and $z = xt$. When plugged into equation (3) gives the equation

$$g(x, s, t) = x^2(xs - t^2) = 0.$$

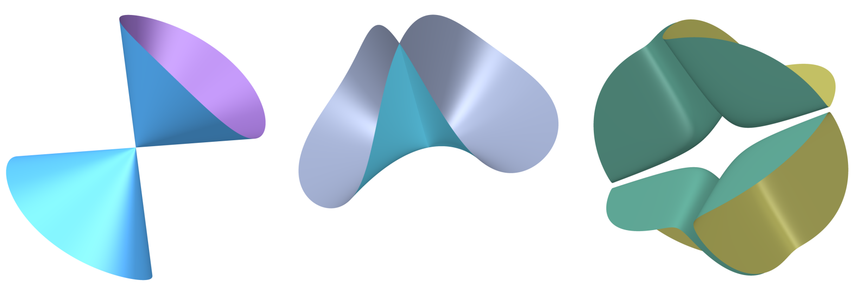
**Case 2:** We start by setting $s = 1$, which gives us $x = yr$ and $z = yt$. Eliminating $x$ and $y$ from equation (3) yields the equation

$$h(y, r, t) = y^2(yr^2 - t^2) = 0.$$

**Case 3:** Finally, setting $t = 1$ yields $x = zr$ and $y = zs$, giving an equation

$$i(z, r, s) = z^2(r^2sz - 1) = 0.$$

These can be visualized, and the surfaces are shown in figure 6. As we see in figure 6b the blowup fails to remove the singularities.



(a) $g(x, s, t) = 0$ (b) $h(y, r, t) = 0$ (c) $i(z, r, s) = 0$

Figure 6: The affine charts of the blowup of the Whitney Umbrella at the point $p = (0, 0, 0)$. As we can see the singularities improve in two of the charts, and in one case the chart is smooth. However, the middle surface is identical to our original surface, so the blow-up has failed to resolve all the singularities of the original variety.

♠

As seen, in this case it is not sufficient to blow up only in one single point. We may have to blow up the surface at a singular line, singular curve or in general, any other set. We make a definition of the blowup of a space in a line $L$, analogous to the blowup in a point, We specialize this definition for $\mathbb{A}^3$.

**Definition 6** (Blowup of $\mathbb{A}^3$ at a line $L$)**.** Given a point $p$ and a line $L$, we are interested in the unique plane $H$ that contains both $p$ and $L$. Since it contains $L$, the plane can be parametrized by a point $\lambda \in \mathbb{P}^1$.

We let the blowup-surface $\mathrm{Bl}_L \mathbb{A}^3$ be the set of all pairs consisting of a point $p \in \mathbb{A}^3$ and a point $\lambda \in \mathbb{P}^1$ that parametrizes the unique plane $H$ that contains both $p$ and $L$. We write

$$\mathrm{Bl}_L \mathbb{A}^3 = \left\{(p,\lambda) \in \mathbb{A}^3 \times \mathbb{P}^1 \mid p, L \in H\right\}.$$

$\diamondsuit$

Again, this is a fairly technical definition, so where do we go from here? Assuming $L$ is parametrized by $x = at, y = bt, z = ct$, with $a, b$ and $c$ non-zero, we can form the symmetric equations

$$t = \frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

The line $L$ is then completely determined by two of the three possible relations $xb - ya = 0$, $xc - za = 0$ and $yc - zb = 0$. In order to find the respective plane, we are interested in knowing when the plane $rx + sy + tz = 0$ can be expressed in terms of the line given by $xb - ya = 0$ and $yc - zb = 0$. This can be solved as a matrix problem. Given the matrix

$$A = \begin{bmatrix} b & 0 & r \\ -a & c & s \\ 0 & -b & t \end{bmatrix}$$

one of the columns of $A$ is linearly dependent of the two others if $\det(A) = 0$. Solving $\det(A) = 0$ gives the equation

$$\det(A) = b(ct + sb) + rab = 0.$$

We can now solve for either one of $r, s$ and $t$. Solving for $r$ we get $r = -(ct + sb)/a$ which we can substitute back into the equation for our plane. The plane is then determined by

$$\frac{-ct - sb}{a}x + sy + tz = 0.$$

Multiplying through by $a$ we achieve

$$t(za - cx) + s(ya - bx) = 0,$$

which gives us the restrictions that we need. The $\lambda$ that parametrizes the plane $H$ is then the point $[t : s] \in \mathbb{P}^1$.

We illustrate this definition by again considering the Whitney Umbrella.

**Example 5** (Blowup of the Whitney Umbrella at a line $L$)**.** The singularities of the Whitney Umbrella lie along the $y$-axis, which is described by the equations $x = z = 0$. Now, our goal here is to find under what circumstances a plane $H$ contains the point $p = (x, y, z)$ as well as the line $L$. In other words, we want both the $y$-axis and $p$ to lie in the plane $rx + sy + tz = 0$.

We first find the planes that contain the $y$-axis. Choosing the vector $(0, 1, 0)$ as a representative for the $y$-axis we compute the cross product between this vector and the point $(r, s, t)$ as a representative for the plane. This yields a normal vector given by

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r & s & t \\ 0 & 1 & 0 \end{vmatrix} = (-t, 0, r).$$

A plane containing the $y$-axis is then given by the equation $-tx + 0y + rz = rz - tx = 0$. The point $\lambda$ is in this case the point $[r : t] \in \mathbb{P}^1$. We can now consider the two affine charts by setting $r = 1$ and $t = 1$ respectively.

**Case 1:** Setting $r = 1$ yields $z = tx$ which when plugged into $x^2y - z^2 = 0$ gives us $x^2y - (t^2x^2) = x^2(y - t^2) = 0$. This affine chart can be visualized in the $(x, y, t)$-plane.

**Case 2:** Setting $t = 1$ yields $x = rz$ which when plugged into $x^2y - z^2 = 0$ gives the equation $r^2z^2y - z^2 = r^2(z^2y - 1) = 0$. This can be visualized in the $(r, y, z)$-plane.

The two affine charts are shown visualized in figure 7.

$\spadesuit$

(a) The surface given by the equation $x^2(y - t^2) = 0$. The curve of intersection between this surface and the exceptional divisor looks like it can be "pinched" into a double line, i.e., the original line of singularity.

(b) The surface given by the equation $r^2(z^2y - 1) = 0$. The intersection between this surface and the exceptional divisor can also be "fused" together to form a double line, by laying the surfaces flat on top of each other.

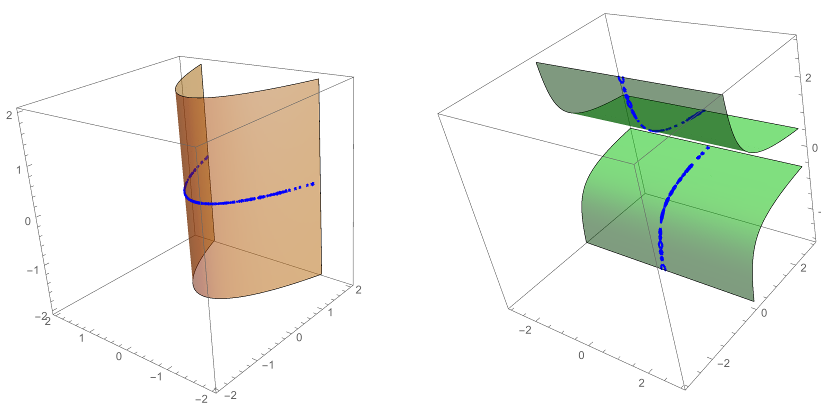


Figure 7: The two affine charts of the variety $\mathbb{V}(x^2y - z^2)$. The top row is rendered by Surfer, and the bottom row rendered by Mathematica. The exceptional divisor in the top row has been divided out and is therefore not shown because Surfer does a poor job at rendering planes. As seen, the charts are smooth and exhibits no singularities. The intersection with the exceptional divisor is rendered in blue in the bottom row. The bottom row is visualized using Mathematica.

# 5 Constructing singular surfaces

In this section we take a brief look at how one can construct singular surfaces with a certain symmetry. The way we will be doing this is based on the set theoretic results given in corollary 1 and corollary 2. This is purely exploratory and is merely a matter of trying things out and seeing what the results are.

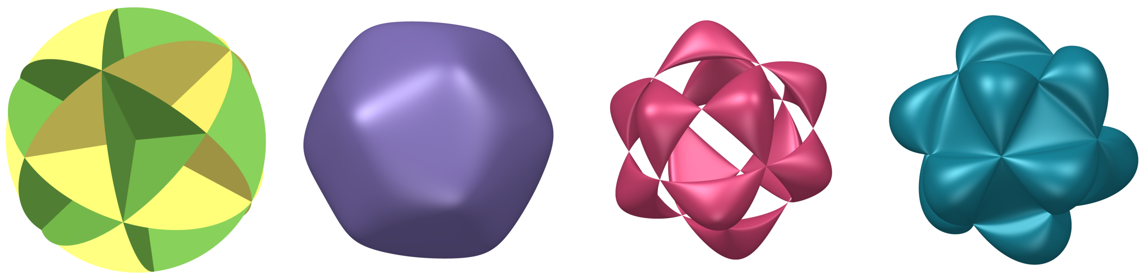
As a baseline for our surface, we wish to use the zero set of the function

$$f(x, y, z) = (x - y)(x + y)(x - z)(x + z)(y - z)(y + z).$$

We let $\mathcal{B} = \mathbb{V}(f(x, y, z))$ This surface is the intersection of six planes, as shown in figure 8a. This surface has a very nice set of singularities, namely on the intersection between the planes. It also has a nice symmetry that we want our final surface to inherit.

We can now take $\mathcal{B}$ and "flesh it out" by adding another polynomial to the defining function. We want to add a surface that has a group of symmetry that contains the symmetry group of $\mathcal{B}$. In our case, the unit sphere seems a reasonable choice. The unit sphere is given by $g(x, y, z) = x^2 + y^2 + z^2 - 1$. We let $\mathcal{S} = \mathbb{V}(g(x, y, z))$. Define $\mathcal{B} + \mathcal{S} = \mathbb{V}(f(x, y, z) + g(x, y, z))$. This surface has now gone from singular to smooth, but, while a bit hard to see, it does retain all the symmetries of $\mathcal{B}$ as shown in figure 8b.

In order to regain the set of singularities, we can instead add powers of $\mathcal{S}$ to $\mathcal{B}$. By powers of $\mathcal{S}$ we mean $\mathcal{S}^n = \mathbb{V}(g(x, y, z)^n)$. If $n$ is an even number, we get a surface with holes as shown in figure 8c. If we instead were to take $n$ an odd number, we get a solid surface with no holes, as in figure 8d. The higher the value we chose for $n$, the more protruding the nodes of the surface become.



(a) $\mathcal{B}$ (b) $\mathcal{B}+\mathcal{S}$ (c) $\mathcal{B}+\mathcal{S}^n$ for $n$ even. (d) $\mathcal{B}+\mathcal{S}^n$ for $n$ odd.

Figure 8: Constructing a singular surface with a given symmetry and set of singularities. The surface $\mathcal{B}$ is the product of six planes. The surface $\mathcal{S}$ is the unit sphere. By adding various powers of $\mathcal{S}$ to $\mathcal{B}$ we achieve a surface that retains the symmetries and singularities of $\mathcal{B}$.

# 6 Conclusion

This paper was meant as a way of introducing the reader to some of the aspects of algebraic geometry, specifically dealing with algebraic surfaces. The intention was to, by restricting our attention to specific examples, avoid having to take into account the fairly theoretical mathematics that lie behind the scenes. We have briefly touched upon different kinds of theory related to these algebraic surfaces, which might have piqued an interest. Algebraic surfaces, while being very intuitive objects in some sense, trying to figure out what they really are can quickly prove a challenge because the mathematics is so involved.

Algebraic surfaces can also work as a solid entry point into visualization techniques and computational mathematics, and while not talked about here, we have seen some end results of the mathematical endeavours related to this area of expertise, through the usage of Surfer, Mathematica and Python.

As the author of this text, it has been great fun trying to wrap my head around these things, and I hope I will get the opportunity to explore this field of mathematics in greater detail.

# A Installing visualization software

The algebraic surface visualization industry is definitely not a million dollar business, hence there are only a select few "out of the box" visualization softwares out there. Most notably are Surf, Surf-Ex, Spicy and Surfer. While fairly modest in its features, Surfer is the one we have chosen for this paper, due to its ease of installation and ease of use. In that regard, it is perfect for this kind of introductory/exploratory text. The software can be downloaded from

https://imaginary.org/program/surfer#all-downloads

and is compiled for use with Windows, OS-X and Linux operating systems. Debian based Linux distributions will want to download the .deb package, where as Red-Hat systems will want to use the .rpm package.

If one want more options, then Surf-Ex is the software of choice, however this has proven to be very difficult to install as it uses Surf as a back end. In any case, the software can be found on

http://surfex.algebraicsurface.net

# B Code snippets

## Plotting algebraic curves

```
import matplotlib.pyplot
from numpy import arange
from numpy import meshgrid

delta = 0.025
xrange = arange(-5, 5.0, delta)
yrange = arange(-2.5, 2.5, delta)
X, Y = meshgrid(xrange,yrange)

# F is one side of the equation, G is the other
F = X**2*(1 - Y**2 - X*Y**3)
G = 0

matplotlib.pyplot.axis('on')
matplotlib.pyplot.grid('on')
matplotlib.pyplot.contour(X, Y, (F - G), [0])

matplotlib.pyplot.savefig('output.pdf')
```

Listing 1: curves.py -http://stackoverflow.com/a/2484594

Plotting the intersection between to surfaces

```
(* affine chart and exceptional divisor *)
h = z^2 y - 1;
g = x^2;

ContourPlot3D[
    {h == 0, g == 0},
    (* bounding box dimensions *)
    {x, -3, 3},
    {y, -3, 3},
    {z, -3, 3},
    MeshFunctions -> {
        Function[ {x, y, z, f}, h - g ]
    },
    MeshStyle -> {
        { Thick, Blue }
    },
    Mesh -> {{0}},
    ContourStyle -> Directive[
            Green, Opacity[0.5], Specularity[White, 30]
    ],
    PlotPoints -> 60,
    SphericalRegion -> True
]
```

Listing 2: intersection.nb - http://community.wolfram.com/groups/-/m/t/178218?p_p_auth=lT7er3Rd

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