Visualizing Algebraic Surfaces

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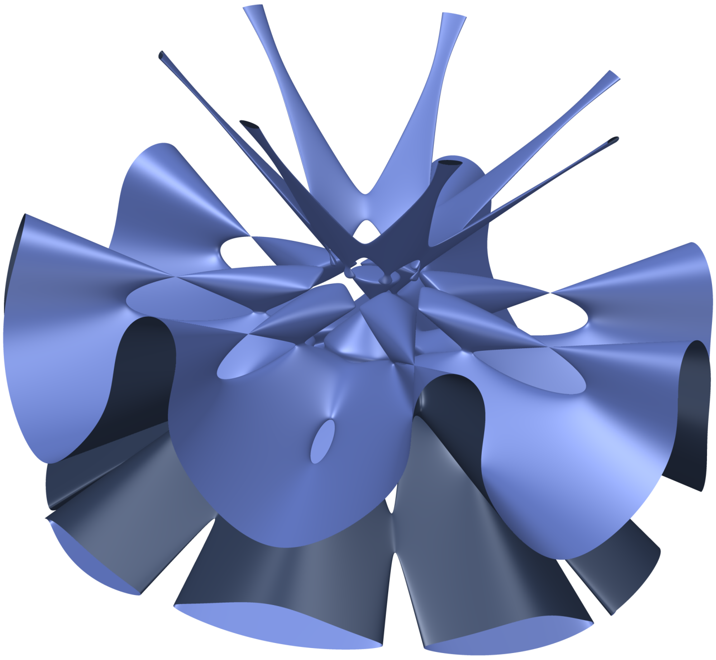

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1 An Informal Introduction

In this section we briefly look at the natural construction of the mathematical objects we are interested in studying in the rest of this paper.

1.1 Real Algebraic Curves

From elementary mathematics one learns about real valued functions, $f(x)$, and how to graph these functions by setting $y = f(x)$ and plotting points in the (x, y)-plane. Now, the *graph of a function* is something of a peculiarity, because it comes with some restrictions. Not all curves in the (x, y)-plane correspond to functions. The method for verifying whether a certain "graph" corresponds to a function or not typically taught in school is the *vertical line test*.

Having an equation $y = f(x)$ we can form what we call the *equation of a curve at zero*. We define a new function in two variables

$$g(x, y) = y - f(x) = 0. \tag{1}$$

If the function $f(x)$ is a polynomial in the variable x with certain coefficients we call the function graph of $f(x)$ *algebraic*.

Generally speaking, we call a curve defined by eq. (1) an *algebraic curve* if the function $g(x, y)$ is a polynomial in two variables, x and y. Mathematically, this can be expressed as

$$g(x, y) = \sum_{i,j} a_{i_j} x^i y^i. \tag{2}$$

However, if this distinction between a graph and a curve is to be justified there has to be some curves that are not graphs. One of the first examples one encounter of a curve not corresponding to a function is what you get when you set $y^2 = x^3$. This curve does not pass the vertical-line-test and is therefore something different from a graph. Similarly, the equation $y^2 = x^3 - x^2$ defines a curve that again is not a graph. These are shown in section 1.1. These curves exhibit *singular points* at $(0, 0)$.

With what we have so far, we can move up a dimension. Instead of considering curves in the (x, y)-plane we can look at surfaces in the (x, y, z)-space. We, as we did with $g(x, y)$ above, define a function $h(x, y, z) = z - g(x, y)$.

(a) $g(x, y) = y^2 - x^3 = 0$ (b) $g(x, y) = y^2 - x^3 + x^2 = 0$

Figure 1: Examples of curves that are *not* graphs given by their functions $g(x, y)$
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The surfaces are given by equations on the form $h(x, y, z) = 0$. These surfaces are called algebraic if $h(x, y, z)$ is a polynomial in the variables x, y, z. Again, mathematically, this is expressed as

$$h(x, y, z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k. \tag{3}$$

1.2 Going Complex

The surfaces considered so far are over the real numbers, i.e., the coefficients a_{ijk} are real numbers. We can instead work over the complex numbers where the surfaces are given by an equivalent equation, but where the coefficients now are complex numbers[1]:

$$\{(x, y, z) \in \mathbb{C} \mid h(x, y, z) = 0\}. \tag{4}$$

Again, doing the same trick (one-trick ponies) we can now define $w = h(x, y, z)$ and look at the equations $w - h(x, y, z) = 0$ in the (x, y, z, w)-space. Now, we are stuck with an equation in four variables. With only three degrees of freedom to work with when visualizing these objects we encounter an obstacle. How do we know what these surfaces look like? This brings us to the world of *projective geometry.*

2 Projective Geometry

The way we work these surfaces in four variables is by introducing a new variable making the polynomial terms all have the same degree. This is called *homogenizing* the polynomial $h(x, y, z)$. We first want to introduce some definitions.

Quite informally, projective space formalizes the notion of parallel lines intersecting at infinity. In order to get a better understanding of the general projective space $\mathbb{P}^n$ we will first construct the *projective plane* $\mathbb{P}^2(\mathbb{C})$.

[1]We will come back to why we make this transition when we talk about sets being algebraically closed or not.

2.1 The Projective Plane

According to [Wik16] there are three equivalent definitions:

1. The set of all lines in $\mathbb{C}^3$ passing through the origin $(0,0,0)$. Every such line meets the *sphere* of radius one centered in the origin exactly twice, say in $P = (x, y, z)$ and its antipodal point $(-x, -y, -z)$.

2. The points on the sphere S^2, where every point P and its antipodal points are not distinguished. For example, the point $(1,0,0)$ is identified with the point $(-1,0,0)$.

3. The set of equivalence classes of $\mathbb{C}^3 \setminus \{0\}$ where two points P and P' are considered equivalent if and only if there is a non-zero $\lambda \in \mathbb{C}$ such that $P = \lambda P'$.

The last definition is the one seemingly most used, and is therefore the one we will employ here. Note that the elements in the projective plane are equivalence classes of points in $\mathbb{C}^3$. We denote an element in $\mathbb{P}^2(\mathbb{C})$ as $[x : y : z]$. These elements are commonly referred to as *homogenous coordinates.* We state this formally in a definition.

Definition 1 (The projective plane)**.** Let $\sim$ be the equivalence relation on $\mathbb{C}^3$ defined by

$$(x, y, z) \sim (x', y', z') \iff (x', y', z') = \lambda(x, y, z),$$

where λ is some non-zero scalar in $\mathbb{C}$. We denote the equivalence class of (x, y, z) as $[x : y : z]$ and we define the *projective plane* as the set

$$\mathbb{P}^2(\mathbb{C}) = \left\{ [x : y : z] \mid (x, y, z) \in \mathbb{C}^3 \setminus \{0\} \right\}.$$

We can now look at an interesting subset of $\mathbb{P}^2(\mathbb{C})$, namely the set of points $[x : y : z]$ where $z \neq 0$. Remembering the identification from the above definition, these are equivalent to $\left[\frac{x}{z} : \frac{y}{z} : 1\right]$ where $\lambda = 1/z$. If we consider definition item 2 from above, this corresponds to the set of points on the northern hemisphere of S^2 not including the circle of intersection in the (x, y)-plane. Similarly, we can look at the set of points $[x : y : 0]$. This corresponds, in S^2, to the circle of intersection in the (x, y)-plane.

We can now take the notion of the projective plane and generalize this to n dimensions.

2.2 Projective Space

The general definition of projective n-space is completely analogous to the definition of the projective plane, the set of lines in $\mathbb{C}^{n+1}$ passing through the origin, however we include a formal definition for completeness.

Definition 2 (The projective space of dimension n)**.** Let $\sim$ be an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$(x_1, \ldots, x_{n+1}) \sim \left(x'_1, \ldots, x'_{n+1}\right) \iff \left(x'_1, \ldots, x'_{n+1}\right) = \lambda\left(x_1, \ldots, x_{n+1}\right)$$

where λ is some non-zero complex number. We then denote the equivalence class of $(x_1, \ldots, x_{n+1})$ by $[x_1 : \ldots : x_{n+1}]$ and define the *projective space of dimension* n as the set

$$\mathbb{P}^n\mathbb{C} = \left\{ [x_1 : \ldots : x_{n+1}] \mid (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\} \right\}.$$

3 Algebraic Surfaces

We are now equipped with what we need to again look at our algebraic surfaces defined by $h(x,y,z) = 0$. The solutions to this equation are points in $\mathbb{C}^4$. We homogenize this polynomial by introducing the variable w and evaluating h at the points $x \to x/w$, $y \to y/w$ and $z \to z/w$ and then multiply by w^d where d is the degree of the polynomial $h(x,y,z)$.

Definition 3 (Homogenized polynomial)**.** Let $h(x,y,z)$ be a polynomial in x,y,z of degree d. The homogenized polynomial $h_{\text{hom}}(w,x,y,z)$ is given by

$$h_{\text{hom}}(w,x,y,z) = w^d h\left(\tfrac{x}{w}, \tfrac{y}{w}, \tfrac{z}{w}\right).$$

It follows that each non-zero term in this polynomial has the same degree.

The property that each non-zero term has the same degree in a homogeneous polynomial gives rise to a very important property which relates directly to the projective geometry introduced above.[2]

Corollary 1. *If h_{hom} is an homogeneous polynomial formed by homogenizing the polynomial $h(x,y,z)$ of degree d, then*

$$h_{\text{hom}}(\lambda w, \lambda x, \lambda y, \lambda z) = \lambda^d h_{\text{hom}}(w,x,y,z).$$

Proof. We simply compute the left hand side:

$$\begin{aligned} h_{\text{hom}}(\lambda w, \lambda x, \lambda y, \lambda z) &= (\lambda w)^d h\left(\tfrac{\lambda x}{\lambda w}, \tfrac{\lambda x}{\lambda w}, \tfrac{\lambda x}{\lambda w}\right) \\ &= \lambda^d h_{\text{hom}}(w,x,y,z). \end{aligned}$$

□

Since we are working with homogeneous coordinates, the condition $h_{\text{hom}}(w,x,y,z) = 0$ is well defined on the equivalence class discussed above, strictly due to this first corollary.

Now, if we can homogenize a polynomial, we can certainly dehomogenize it. Given a homogeneous polynomial $h_{\text{hom}}(w,x,y,z)$ we can dehomogenize the polynomial by *fixing* one of the variables. By setting $w = f(x,y,z)$ in the polynomial $h_{\text{hom}}(w,x,y,z)$ we get a new polynomial $h(x,y,z)$ which is not necessarily homogeneous.

3.1 Initial examples of algebraic surfaces

Although mentioned before, we repeat the definition of an algebraic surface.

Definition 4 (Algebraic surface)**.** A surface given by the equation $h(x,y,z) = 0$ is called *algebraic* if $h(x,y,z)$ is a polynomial in the variables x,y,z. Mathematically,

$$h(x,y,z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k.$$

[2] This following property can be seen directly by visualizing the homogeneous polynomial formed by a polynomial $g(x,y)$ and noticing that the figure does not change when zooming in or out.

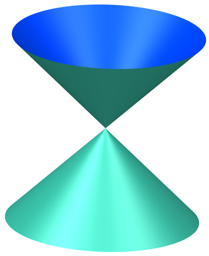

Figure 2: The equation $h(x, y, z) = x^2 + y^2 - z^2 = 0$ gives rise to a double-cone shaped algebraic surface.

We are also interested in classifying the surfaces according to their *degree*.

Definition 5 (The degree of a surface)**.** Given a surface defined by the equation $h(x, y, z) = 0$ we define the *degree*, denoted d, of the surface to be the largest sum of powers in the terms of the polynomial. Mathematically,

$$d = \max\{i + j + k\}$$

where i, j, k are the powers of x, y and z in the terms.

Now, what do these surfaces actually look like? We give some examples that hopefully is going to illustrate the above definition.

Example 1. Let h be a polynomial in x, y, z given by

$$h(x, y, z) = x^2 + y^2 - z^2.$$

We then look at the set of solutions to the equation $h(x, y, z) = 0$. Plotting these yields a double-cone shaped figure. We see there is an interesting quirk at the point $(0, 0, 0)$ where the surface seems to pass through itself, formally we say that the surface exhibits a *singularity* at the point $(0, 0, 0)$, but we will come back to that later. This surface has degree $d = 2$, and since all the terms have the same degree the defining polynomial is homogeneous, and hence the surface is invariant under scaling.

While there is a lot of fun to be had with algebraic surfaces of degree 2 there is a limit as to how exciting they are. Things get much more fun when we look at surfaces of higher degree. Consider for example the following surface of degree 3.

Example 2 (The ding-dong cubic)**.** Let $h(x, y, z) = x^2 + y^2 - (1 - z)z^2$. Then the equation $h(x, y, z) = 0$ defines a surface of degree $d = 3$ (attributed to the $(1 - z)z^2$ term). This surface also exhibits a singularity at the point $(0, 0, 0)$ where it passes through itself.

Another very interesting third-degree surface, or *cubic*, is the Cayley cubic.

Example 3. This is given by the homogeneous equation

$$h_{\text{hom}}(w, x, y, z) = \frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

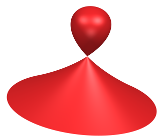

Figure 3: The ding-dong cubic given by the equation $h(x, y, z) = x^2 + y^2 - (1 - z)z^2 = 0$.

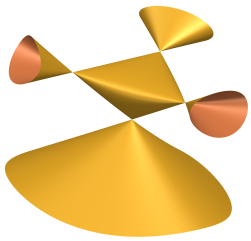

Figure 4: The Cayley-cubic given by the homogeneous equation $h(x, y, z) = 1/w + 1/x + 1/y + 1/z = 0$. This specific "version" is obtained by setting $w = 1 - x - y - z$.

In order to visualize this, we dehomogenize the polynomial by multiplying through by the common denominator and setting $w = f(x, y, z)$ for some polynomial f. It happens to be so that setting $w = 1 - x - y - z$ yields a very visually pleasing surface. This surface is named the *Cayley cubic* after the mathematician Arthur Cayley. Setting w to various functions f yield different visualizations of these surfaces.

The equation $xyz + (1 - x - y - z)(xy + xz + yz) = 0$ yields the following surface: Again, we see singularities of the surface, however this surface exhibits more than just one.

One of the more extreme surfaces in terms of its singularities is the Barth decic.

Example 4 (Barth decic)**.** The Barth Decic is given by the polynomial

$$\begin{aligned} &h(x, y, z) = \\ &8(x^2 - \varphi^4 y^2)(y^2 - \varphi^4 z^2)(z^2 - \varphi^4 x^2)(x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2) \\ &\qquad + (3 + 5\varphi)(x^2 + y^2 + z^2 - w^2)^2 \left[x^2 + y^2 + z^2 - (2 - \varphi)w^2\right]^2 w^2 \\ &\qquad\qquad = 0, \end{aligned} \tag{5}$$

where φ denotes the golden ratio. Again, we dehomogenize this polynomial, by setting $w = 1$. This yields the following picture.

We have now seen examples of surfaces exhibiting so called *singularities*, but what are these? We discuss this in the following section.

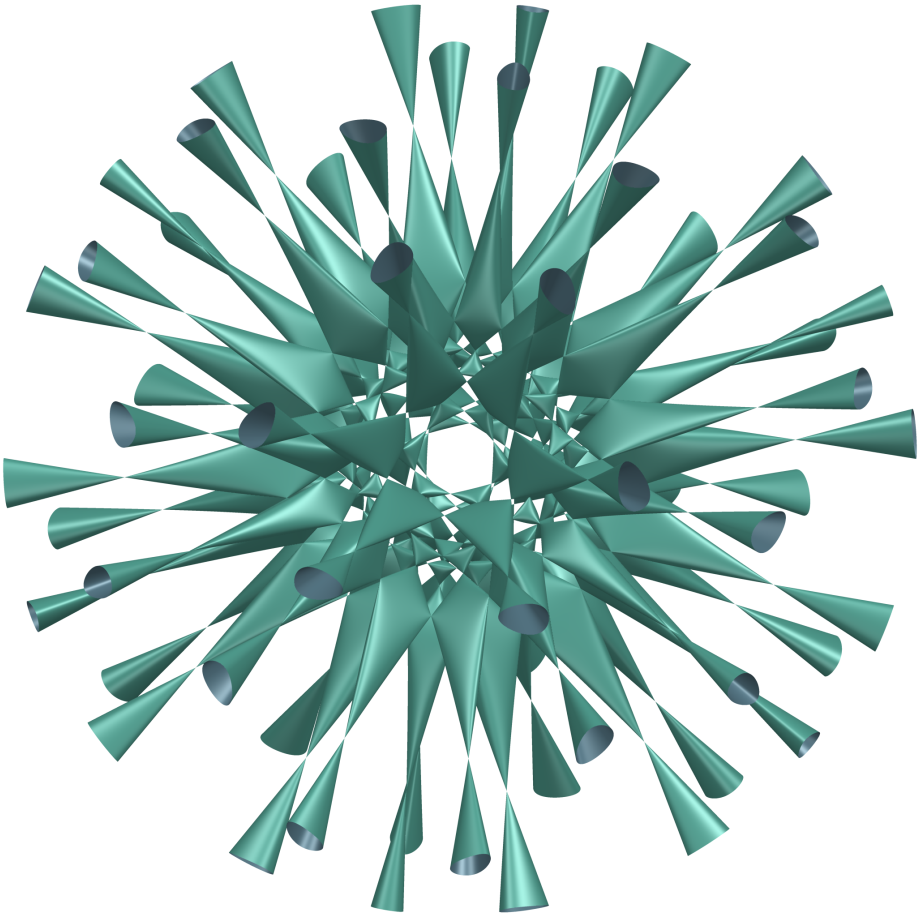

Figure 5: The Barth decic in all its glory. For $w = 1$ it exhibits 300 singularities.

3.2 The singularities of an algebraic surface.

Definition 6. A *singularity* of a surface is a point (x, y, z) where the defining polynomial $h(x, y, z)$ and all the partial derivatives of h vanish identically. Mathematically, (x, y, z) is a singularity if $h(x, y, z) = 0$ and

$$\frac{\partial}{\partial x}h(x, y, z) = \frac{\partial}{\partial y}h(x, y, z) = \frac{\partial}{\partial z}h(x, y, z) = 0.$$

One interesting question to consider is whether there is a limit to how many singularities a surface of degree d can have. Given a degree d, we let $\mu(d)$ denote the maximum number of singularities possible. It happens to be that $\mu(d)$ is unknown for large values of d.

Definition 7. Let $V = \mathbb{V}(f_1, ..., f_m) \subset \mathbb{A}^n$. A singularity, or singular point, of V is a point p such that the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}(p)$$

has rank strictly less than $\min(m, n)$.

References

[Wik16] Wikipedia. *Projective space — Wikipedia, The Free Encyclopedia.* [Online; accessed 19-February-2016]. 2016. URL: https://en.wikipedia.org/wiki/Projective_space.