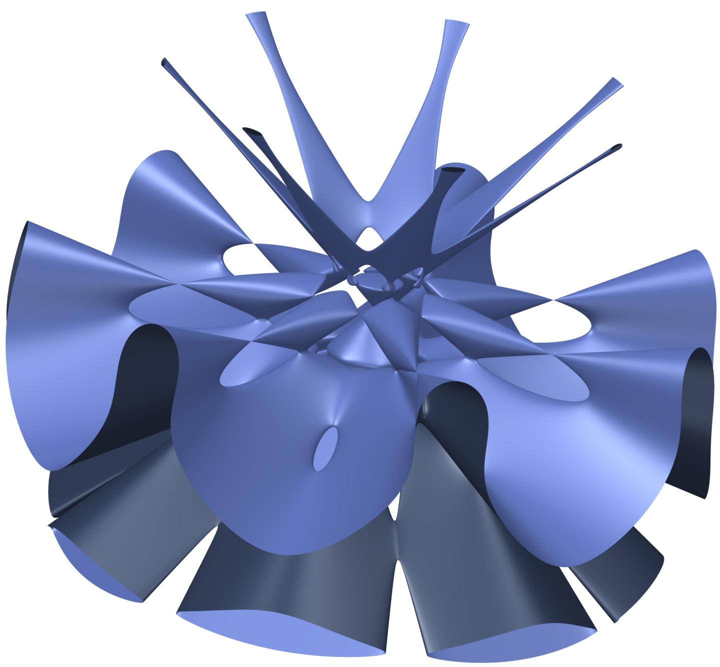
Visualizing Algebraic Surfaces

Ivar Haugaløkken Stangeby

April 12, 2016



Abstract

One of the most common ways people are exposed to mathematics are through the visually pleasing mathematical objects. The aim of this article is to use algebraic surfaces to highlight interesting areas of mathematics, and hopefully provide a way people easily can dive in and visualize these themselves. We take a look at the mathematical definition of algebraic surfaces. We then consider their singularities, how to remove their singularities, and the symmetries of such singularities.

Contents

1 Algebraic Surfaces

In order to understand mathematically what kind of object we are dealing with, namely the algebraic surfaces, we must first introduce some preliminaries. Starting with the notion of an *algebraic variety*.

Definition 1 (Affine Algebraic Variety)**.** Given a ring of polynomials in n variables $k[x_1, \ldots, x_n]$ where k is an algebraically closed field, let $S \subset k[x_1, \ldots, x_n]$ be a set of polynomials. We define the *affine algebraic variety associated to S* as the set of solutions

$$V(S) = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

This definition might seem a bit cryptic, but we can give a fairly intuitive example.

Example 1 (Roots of unity)**.** Let $k = \mathbb{C}$ and $S = \{f_n(z) = z^n + 1 \mid n \in \mathbb{N}\}$ then the algebraic variety associated to S will be the roots of unity on the circle in the complex plane.

An algebraic variety of *dimension two* is what we call an *algebraic surface*[1]. Since algebraic surfaces are polynomials, it is reasonable to talk about the *degree* of the algebraic surface. We let $d = \max(i + j + k)$ be the maximum sum of all powers of all terms $a_m x^{i_m} y^{j_m} z^{k_m}$.

Example 2 (The degree of a surface)**.**

One of the distinguishing features between a *manifold* and an algebraic variety is that the algebraic varieties may exhibit singularities. This is one of the more aesthetically pleasing properties of algebraic surfaces, hence something we are very much interested in. A singularity is strictly speaking a point

[1]Figuring out the dimension of a variety is a bit tricky, related to chains of ideals it seems.

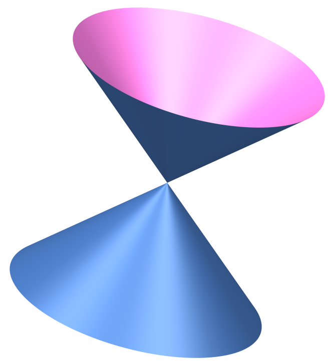


Figure 1: A double cone given by the equation $f(x, y, z) = ax^2 + bx^2 + cz^2$.

where the surface and the partial derivatives vanish identically. We define this formally:

Definition 2 (Point of Singularity **[OLabs2]**)**.** A point p in $\mathbb{C}^n$ is called a *singular point* or *singularity of the hypersurface* $f \in \mathbb{C}[x_1, \ldots, x_n]$ if $f(p) = 0$ and all the partial derivatives are zero. Mathematically:

$$\frac{\partial}{\partial x_i} f(p) = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

Again, we can give a few examples.

Example 3. Let our surface of degree 2 be defined by $f(x, y, z) = ax^2 + by^2 + cz^2$. We first take the look at the partial derivatives. These are given by

$$\frac{\partial f}{\partial x} = 2ax = 0, \qquad \frac{\partial f}{\partial y} = 2by = 0, \qquad \frac{\partial f}{\partial z} = 2cz = 0.$$

Solving these for x, y, z yield $x = y = z = 0$. We want f to vanish identically on the point $(0, 0, 0)$, so setting $f(x, y, z) = 0$ tells us that our surface f exhibit a singularity at the point $(0, 0, 0)$. The surface, with its singularity, is shown in fig. 1.

Now, it is interesting to ask whether there are different types of singularities, or are all singularities of the same nature? It turns out that there are several different kinds of singularities, namely; (a) isolated singularities, (b) double points. We will later learn how to characterize these singularities, but for now we will just look at them as singularities.

A Visualization Techniques

Index