MANDATORY ASSIGNMENT 1

MAT2410

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Exercise 1.

a) We wish to show that a circle $C \subset \mathbb{C}$ can be expressed equivalently as the set of points $z \in \mathbb{C}$ that satisfy

$$C = \left\{ z \in \mathbb{C} \mid z\bar{z} - \bar{z_0}z - z_0\bar{z} + |z_0|^2 - r^2 = 0 \right\},$$

where the typical expression reads

$$C = \{ z \in \mathbb{C} \mid |z - z_0| = r \}.$$

Taking the expression $|z - z_0| = r$, squaring both sides and using the identity $|w|^2 = w\bar{w}$ for any complex number w we achieve

$$\begin{aligned} |z - z_0|^2 - r^2 &= (z - z_0)(\bar{z} - \bar{z_0}) - r^2 \\ &= z\bar{z} - \bar{z_0}z - z_0\bar{z} + |z_0|^2 - r^2 \\ &= 0. \end{aligned} \tag{1}$$

This is what we wanted to show.

b) We now wish to show that the set of points $z \in \mathbb{C}$ satisfying

$$\left| \frac{z - z_1}{z - z_2} \right| = k, \tag{2}$$

is a circle where $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$ and $k > 0$. We also want to find the center and radius of the circle.

Multiplying through by $|z - z_2|$ and squaring both sides we achieve

$$\begin{aligned} |z - z_1|^2 - k^2|z - z_2|^2 &= (z - z_1)(\bar{z} - \bar{z_1}) - k^2\big((z - z_2)(\bar{z} - \bar{z_2})\big) \\ &= z\bar{z} - z\bar{z_1} - z_1\bar{z} + z_1\bar{z_1} - k^2\big(z\bar{z} - z\bar{z_2} - z_2\bar{z} + z_2\bar{z_2}\big) \\ &= (1 - k^2)z\bar{z} + (k^2\bar{z_2} - \bar{z_1})z + (k^2 z_2 - z_1)\bar{z} + (|z_1|^2 - k^2|z_2|^2) \end{aligned}$$

Under the assumption that $k \neq 1$, we divide by $(1-k^2)$. Now we define $w = \big(k^2 z_2 - z_1\big)/(1 - k^2)$ and $\psi = k^2|z_2|^2 - |z_1|^2$. We can then rewrite the above equations as

$$\begin{aligned} |z - z_1|^2 - k^2|z - z_2|^2 &= z\bar{z} - \bar{w}z - w\bar{z} + |w|^2 - (\psi + |w|^2) \\ &= z\bar{z} - \bar{w}z - w\bar{z} + |w|^2 - r^2 = 0 \end{aligned}$$

where $r^2 = \psi + |w|^2$. We have now rewritten eq. (2) on the form eq. (1) Hence our original equation describes a circle centered at w with radius $r = \sqrt{\psi + |w|^2}$

c) If we assume $k = 1$, then the set described in eq. (2) is just the line equidistant from z_1 and z_2. In other words, its the set of points that are just as far away from z_1 as z_2.

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Exercise 2.

a) We wish to solve the quadratic equation $z^2 + 4z + 16 = 0$ and write the solution in polar form. A simple application of the *abc*-formula for quadratic equations yields

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 16}}{2} = -2 \pm 2\sqrt{3}i.$$

Hence the two roots of the equations are given by

$$z_1 = -2 + 2\sqrt{3}i \quad \text{and} \quad z_2 = -2 - 2\sqrt{3}i.$$

In order to find the polar form for these two numbers, we use the relations

$$r = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}, \qquad \cos\theta = \frac{\mathrm{Re}(z)}{r}, \qquad \sin\theta = \frac{\mathrm{Im}(z)}{r}.$$

These yield

$$z_1 = 4e^{2\pi i/3},$$

and

$$z_2 = 4e^{4\pi i/3}.$$

We can now solve the equation

$$z^6 = \frac{i}{1-i} = -\frac{1}{2} + \frac{i}{2}. \tag{3}$$

We are looking for the complex number z such that when taken to the sixth power we achieve the complex number $-1/2 + i/2$. This has to be a number with one sixth of the argument and the 6'th root of the modulus of $-1/2 + i/2$. So, using the relations above, we find that

$$|z^6| = \frac{\sqrt{2}}{2}, \quad \text{and} \quad \arg(z^6) = \frac{3\pi}{4}.$$

Hence, our z must be the number

$$z = \sqrt[6]{\frac{\sqrt{2}}{2}} e^{\frac{\pi i}{8}},$$

which when plugged back into eq. (3) yields $-1/2 + i/2$.

Exercise 3.

a) We wish to determine where in the complex plane the function

$$f(z) = e^{z^2}$$

We see that this is the composition of two functions g and h, with $g(z) = e^z$ and $h(z) = z^2$. Hence $f = g \circ h$. We see immediately that g is holomorphic, since it is *analytic* on $\mathbb{C}$ and Theorem 2.6 tells us that it must then be holomorphic. We also see that h is holomorphic, since any complex polynomial is holomorphic, by Proposition 2.2. It then follows, again by Proposition 2.2, that f is holomorphic, and infact *entire.*

b) We now wish to determine if the function

$$f(z) = e^{\bar{z}}$$

is holomorphic. Again, we see that this is the composition of two functions, but $\bar{z}$ is not holomorphic. We can therefore not apply the same proposition as for the previous function. However, by Proposition 2.3, if f is holomorphic, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

However, we have that

$$\frac{\partial f}{\partial \bar{z}} = e^{\bar{z}} \neq 0,$$

hence f is not holomorphic.

Exercise 4. In this exercise we let γ_R be the curve parametrized by $z(t) = Re^{it}$ for $t \in [0, 2\pi]$ and $R > 2$, $R \in \mathbb{R}$. We want to show that

$$\left| \int_{\gamma_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}. \tag{4}$$

We also wish to show that the integral on the left tends to 0 as $R \to \infty$.

We first let $f(z) = (2z^2 - 1)/(z^4 + 5z^2 + 4)$ and observe that the denominator can be factored. Our function f can therefore be written equivalently as

$$f(z) = \frac{2z^2 - 1}{(z^2 - 1)(z^2 - 4)}.$$

In order to show this inequality, we want to employ the estimate

$$\left| \int_{\gamma_R} f(z)\, dz \right| \leq \sup_{t \in [0, 2\pi]} |f(z(t))| \cdot \mathrm{length}(\gamma_R).$$

The length of our curve, being a half circle in the upper half-plane with radius R, is $\mathrm{length}(\gamma_R)\pi R$. Our function f evaluated at $z(t)$ is then

$$f(z(t)) = \frac{2R^2 e^{2it} - 1}{(R^4 e^{4it} - 1)(R^4 e^{4it} - 4)}$$

with the absolute value being

$$|f(z(t))| = \frac{\left|2R^2 e^{2it} - 1\right|}{|(R^2 e^{2it} - 1)|\, |(R^2 e^{2it} - 4)|} \leq \frac{|2R^2 e^{2it} - 1|}{(R^2 - 1)(R^2 - 4)}$$

where we have used the inverse triangle inequality on the denominator, $||z| - |w|| \leq |z - w|$. One final application of the triangle inequality gives us

$$|f(z(t))| = \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}.$$

Hence the absolute value is independent of t, and therefore eq. (4) holds.

If we now let $R \to \infty$, the right hand side in eq. (4) has a R^4 term in the denominator which dominates, and hence the whole expression tends to 0. Since our integral is bounded by this expression, the integral must also tend to 0 as $R \to \infty$.