MAT2410 - MANDATORY ASSIGNMENT

IVAR HAUGALØKKEN STANGEBY

Exercise 1. Let $f(z)$ be the complex function

$$f(z) = \frac{1}{1+z^2}.$$

Let r be a real number, $r > 0$ and let L_r be the line from the point $-r$ to r in $\mathbb{C}$. Let γ_r be the upper half circle with radius r and center in 0, i.e.

$$\gamma_r = \{z \in \mathbb{C} : |z| = r \text{ and } \operatorname{Im}(z) \geq 0\},$$

with positive orientation.

a) Compute $\int_{L_r} f(z)\, dz$.
b) Compute $\int_{\gamma_r} f(z)\, dz$ when $r < 1$.
c) Compute $\int_{\gamma_r} f(z)\, dz$ when $r > 1$.

Solution.

a) For this we recall that $(\arctan x)' = 1/(1+x^2)$. We then see that

$$\int_{L_r} f(z)\, dz = \int_{-r}^{r} f(x)\, dx = \arctan(r) - \arctan(-r)$$
$$= 2\arctan r.$$

b) We note that $f(z)$ has poles at $z = \pm i$. For $r < 1$ we are the pole at $z = i$ is not in the interior of our contour. Since $f(z)$ is holomorphic in the interior, Cauchy's theorem tells us that the integral $\int_{\gamma_r + L_r} f(z)\, dz = 0$ where $\gamma_r + L_r$ denotes the closed curve consisting of the line between $-r$ and r joined to the half circle from r to $-r$ with positive orientation. We then have

$$\int_{\gamma_r + L_r} f(z)\, dz = \int_{\gamma_r} f(z)\, dz + 2\arctan(r) = 0.$$

Consequently,

$$\int_{\gamma_r} = -2\arctan(r)$$

when $r < 1$.

On the other hand, when $r > 1$ we have a pole in our interior, hence we need to compute the residue at $z = i$. Since this is a simple pole, the residue formula tells us that

$$\operatorname{res}_{z=i} f = \lim_{z \to i} (z - i) f(z) = \frac{1}{2i}.$$

We now know that our integral over $\gamma_r + L_r$ must equal $\operatorname{res}_{z=i} f\, 2\pi i$, hence

$$\int_{\gamma_r + L_r} f(z)\, dz = \pi,$$

and it then follows that

$$\int_{\gamma_r} f(z)\, dz = \pi - 2\arctan(r)$$

when $r > 1$.

Exercise 2. Let
$$f(z) = e^{1/z}.$$
Show directly that for every $w \neq 0 \in \mathbb{C}$ and every real $\varepsilon > 0$ there exists an infinite number of complex numbers z with $|z| < \varepsilon$ such that $f(z) = w$.

Solution. Fix $w \in \mathbb{C}$ with $w \neq 0$ and let let $\varepsilon > 0$ be given. Then we have
$$w = re^{i\theta} = e^{\ln r} e^{i\theta}.$$
Setting $f(z) = w$ and equating exponents we see that
$$z = \frac{1}{\ln r + i\theta} = \frac{1}{\ln r + i(2\pi n)}.$$
Taking absolute value we achieve
$$|z| = \frac{1}{|\ln r + i(2\pi n)|} = \frac{1}{\sqrt{\ln^2 r + (2\pi n)^2}}.$$
This last expression we can force smaller than any ε by choosing n large enough to make the denominator arbitrarily large. Hence, there exists an $N \in \mathbb{N}$ such that $|z| < \varepsilon$ for all $n \geq N$.

Exercise 3. Compute the residue of
$$f(z) = \frac{1 - e^{2z}}{z^4}$$
at 0.

Solution. We have a singularity at $z = 0$ so we examine $f(z)$ expressed using the power series expansion of e^{2z}:
$$f(z) = \frac{1 - \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}}{z^4} = -2z^{-3} - 2z^{-2} - \frac{4}{3}z^{-1} - \dots$$
The residue of f at 0 is given by the coefficient of the z^{-1} term in the Laurent series. Hence
$$\text{res}_{z=0} f = \frac{-4}{3}.$$

Exercise 4. Let
$$f(z) = \frac{z}{z^3 - (2-i)z^2 + (1-2i)z + i}.$$

a) Verify that 1 is a root of $z^3 - (2-i)z^2 + (1-2i)z + i$ and find the other roots.
b) Use the residue formula to evaluate $\int_\gamma f(z)\, dz$ where γ is the circle with center 0 and radius 2, equipped with the positive orientation.

Solution. a) In order to verify that 1 is a root of the denominator, we simply plug in $z = 1$ and get
$$1 - (2 - i) + (1 - 2i) + i = 0.$$
Hence 1 is a root of the denominator, and consequently $(z - 1)$ is a factor in the denominator. In order to find the two other roots, we simply perform polynomial division by the factor we just found. This gives us
$$z^2 - z + iz - i = (z + i)(z - 1).$$
We therefore have $z = 1$ as a root with multiplicity 2, and $z = -i$ as a simple root.

b) We now wish to compute the integral over γ. Since the interior of γ contains our poles the residue formula tells us that this integral is

$$I = \int_\gamma f(z)\,dz = 2\pi i(\mathrm{res}_{z=1} f + \mathrm{res}_{z=-i} f),$$

so we simply compute these:

$$\mathrm{res}_{z=-i} f = \lim_{z\to -i} \frac{z}{(z-1)^2} = -\frac{1}{2}$$
$$\mathrm{res}_{z=i} f = \lim_{z\to 1} \frac{(z+i)-z}{(z+i)^2} = \frac{1}{2}$$

We are then left with the integral being

$$I = 2\pi i\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

and we are done.