

MANDATORY ASSIGNMENT 1

MAT2410

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Exercise 1.

- a) We wish to show that a circle $C \subset \mathbb{C}$ can be expressed equivalently as the set of points $z \in \mathbb{C}$ that satisfy

$$C = \{z \in \mathbb{C} \mid z\bar{z} - \bar{z}_0 z - z_0 \bar{z} + |z_0|^2 - r^2 = 0\},$$

where the typical expression reads

$$C = \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

Taking the expression $|z - z_0| = r$, squaring both sides and using the identity $|w|^2 = w\bar{w}$ for any complex number w we achieve

$$\begin{aligned} |z - z_0|^2 - r^2 &= (z - z_0)(\bar{z} - \bar{z}_0) - r^2 \\ (1) \quad &= z\bar{z} - \bar{z}_0 z - z_0 \bar{z} + |z_0|^2 - r^2 \\ &= 0. \end{aligned}$$

This is what we wanted to show.

- b) We now wish to show that the set of points $z \in \mathbb{C}$ satisfying

$$(2) \quad \left| \frac{z - z_1}{z - z_2} \right| = k,$$

is a circle where $z_1, z_2 \in \mathbb{C}$ with $z_1 \neq z_2$ and $k > 0$. We also want to find the center and radius of the circle.

Multiplying through by $|z - z_2|$ and squaring both sides we achieve

$$\begin{aligned} |z - z_1|^2 - k^2 |z - z_2|^2 &= (z - z_1)(\bar{z} - \bar{z}_1) - k^2 ((z - z_2)(\bar{z} - \bar{z}_2)) \\ &= z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 - k^2 (z\bar{z} - z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2) \\ &= (1 - k^2)z\bar{z} + (k^2\bar{z}_2 - \bar{z}_1)z + (k^2 z_2 - z_1)\bar{z} + (|z_1|^2 - k^2 |z_2|^2) \end{aligned}$$

Under the assumption that $k \neq 1$, we divide by $(1 - k^2)$. Now we define $w = (k^2 z_2 - z_1) / (1 - k^2)$ and $\psi = k^2 |z_2|^2 - |z_1|^2$. We can then rewrite the above equations as

$$\begin{aligned} |z - z_1|^2 - k^2 |z - z_2|^2 &= z\bar{z} - \bar{w}z - w\bar{z} + |w|^2 - (\psi + |w|^2) \\ &= z\bar{z} - \bar{w}z - w\bar{z} + |w|^2 - r^2 = 0 \end{aligned}$$

where $r^2 = \psi + |w|^2$. We have now rewritten eq. (2) on the form eq. (1). Hence our original equation describes a circle centered at w with radius $r = \sqrt{\psi + |w|^2}$.

- c) If we assume $k = 1$, then the set described in eq. (2) is just the line equidistant from z_1 and z_2 . In other words, it's the set of points that are just as far away from z_1 as z_2 .

Exercise 2.

- a) We wish to solve the quadratic equation $z^2 + 4z + 16 = 0$ and write the solution in polar form. A simple application of the *abc*-formula for quadratic equations yields

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 16}}{2} = -2 \pm 2\sqrt{3}i.$$

Hence the two roots of the equations are given by

$$z_1 = -2 + 2\sqrt{3}i \quad \text{and} \quad z_2 = -2 - 2\sqrt{3}i.$$

In order to find the polar form for these two numbers, we use the relations

$$r = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}, \quad \cos \theta = \frac{\operatorname{Re}(z)}{r}, \quad \sin \theta = \frac{\operatorname{Im}(z)}{r}.$$

These yield

$$z_1 = 4e^{2\pi i/3},$$

and

$$z_2 = 4e^{4\pi i/3}.$$

We can now solve the equation

$$(3) \quad z^6 = \frac{i}{1-i} = -\frac{1}{2} + \frac{i}{2}.$$

We are looking for the complex number z such that when taken to the sixth power we achieve the complex number $-1/2 + i/2$. This has to be a number with one sixth of the argument and the 6'th root of the modulus of $-1/2 + i/2$. So, using the relations above, we find that

$$|z^6| = \frac{\sqrt{2}}{2}, \quad \text{and} \quad \arg(z^6) = \frac{3\pi}{4}.$$

Hence, our z must be the number

$$z = \sqrt[6]{\frac{\sqrt{2}}{2}} e^{\frac{\pi i}{8}},$$

which when plugged back into eq. (3) yields $-1/2 + i/2$.

Exercise 3.

- a) We wish to determine where in the complex plane the function

$$f(z) = e^{z^2}$$

is holomorphic. By definition of holomorphism, we wish to examine the limit

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \frac{e^{(z+h)^2} - e^{z^2}}{h}.$$