Weekly Problems

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Week 9

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Problem 2.2

Let S be any non-empty set and let X be a normed space over $\mathbb{F}$. Let $F_b(S, X)$ be the linear subspace of $F(S, X)$ of all functions $f : S \to X$ such that $\{\|f(s)\|_b \mid s \in S\}$ is bounded. Show that $F_b(S, X)$ has a norm defined by

$$\|f\|_b = \sup \{\|f(s)\| \mid s \in S\}.$$

Solution. First recall that $\| \cdot \|$ is the norm on X, and consequently on S. Firstly, we see that $\sup \{\|f(s)\|\} \geq 0$ for all $s \in S$ with $\|f(s)\| = 0$ if and only if $f = 0$. Let $\alpha \in \mathbb{F}$. Then $\sup \{\|\alpha f(s)\|\} = \alpha \sup \{\|f(s)\|\}$. Lastly, let $f, g \in F_b(S, X)$. Then

$$\sup \{\|f + g\|\} \leq \sup \{\|f\|\} + \{\|g\|\}$$

It then follows that $\| \cdot \|_b$ is a norm on $F_b(S, X)$.

Problem 2.3

For each $n \in \mathbb{N}$ let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Find the norm of f_n in the following cases:

a) in the normed space $C_{\mathbb{R}}([0, 1])$;

b) in the normed space $L^1[0, 1]$.

Solution.

a) The *standard norm* on $C_{\mathbb{R}}([0, 1])$ is defined by

$$\|f\| = \sup \{|f(x)| \mid x \in [0, 1]\}.$$

   Therefore, $\|x^n\| = 1$.

b) The standard norm on $L^1[0, 1]$ is defined by

$$\|f\|_1 = \int_0^1 |f(x)| \, dx.$$

   Then $\|x^n\|_1 = \int_0^1 |x^n| \, dx = \frac{1}{n+1}$.

Problem 2.12

Let X be a normed linear space and, for any $x \in X$ and $r > 0$, let
$T = \{y \in X \mid \|y - x\| \leq r\}$ and $S = \{y \in X \mid \|x - y\| < r\}$.

a) Show that T is closed.

b) If $z \in T$ and $z_n = \left(r - n^{-1}\right)z$, for $n \in \mathbb{N}$, show that $\lim_{n\to\infty} z_n = z$ and
hence show that $\overline{S} = T$.[1]

Solution.

a) Let $\{y_n\}$ be a sequence in T that converge to $z \in X$. Then by definition $\|y_n - x\| \leq r$. Taking limits and applying Theorem 2.11 we have that

$$\lim_{n\to\infty} \|y_n - x\| = \|z - x\| \leq r$$

hence $z \in T$ and consequently T is closed.

b) We have that

$$\|z_n - z\| = \left\| \left((1 - n^{-1})z - z\right) \right\| = n^{-1}\|z\| = n^{-1}\|z - 0\| \leq n^{-1}$$

hence $\lim_{n\to\infty} z_n = z$.

Since $S \subseteq T$ and T is closed, $\overline{S} \subseteq T$. Conversely, if $z \in T$ and z_n defined as above then $\|z_n\| = \left(1 - n^{-1}\right)\|z_n\| \leq (1 - n^{-1})r < r$. Therefore z is a limit of a sequence of elements of S, so $z \in \overline{S}$. Hence $T \subseteq \overline{S}$ so $T = \overline{S}$.

[1] The proposed solution uses $z_n = \left(1 - n^{-1}\right)z$ so I am going to assume that there is an error in the problem text.

Problem 2.14

Let S be any non-empty set, let X be a Banach space over $\mathbb{F}$ and let $F_b(S, X)$
be the vector space given in Problem 2.2 with the norm
$\|f\|_b = \sup\{\|f(s)\| \mid s \in S\}$. Show that $F_b(S, X)$ is a Banach space.

Solution. Let $\{f_n\}$ be Cauchy in $F_b(S, X)$. Then there exists an N such that

$$\|f_n(s) - f_m(s)\| \leq \|f_n - f_m\|_b \leq \varepsilon.$$

It then follows that $\{f_n(s)\}$ is Cauchy in X, and since X is complete it is a convergent sequence. It is then meaningful to talk of the limit of this sequence, which we denote $f(s) = \lim_{n\to\infty} f_n(s)$. From the Cauchy property we have that

$$\lim_{m\to\infty} \|f_n(s) - f_m(s)\| = \|f_n(s) - f(s)\| \leq \varepsilon.$$

Applying the inverse triangle inequality yields

$$\|f(s)\| \leq \varepsilon + \|f_n(s)\| \leq \|f_n\|_b + \varepsilon.$$

Hence $\|f(s)\|$ is bounded and we can therefore conclude that $f \in F_b(S, X)$.[2] It then follows that $F_b(S, X)$ is a Banach space.

[2] The reason we did all this was to make sure that the limit function was in fact inside the space $F_b(S, X)$.

Problem 3.4

Give the proof of Lemma 3.14.

Solution. Lemma 3.14 states that if X is an inner product space with inner product $(\cdot \mid \cdot)$. Then for all $u, v, x, y \in X$:

a) $(u + v \mid x + y) - (u - v \mid x - y) = 2 (u \mid y) + 2 (v \mid x)$;

b) $4 (u \mid y) = (u + v \mid x + y) - (u - v \mid x - y) + i (u - iv \mid x - iy) - i (u - iv \mid x - iy)$. (for complex X).

For the first one, we simply perform the calculations, using the linearity in the first variable and the conjugate linearity in the second variable. For the second we simply observe that the right hand side closely resembles the identity in a) so we apply that and use the conjugate linearity in the second variable. We have that

$$(u + v \mid x + y) - (u - v \mid x - y) = 2 (u \mid y) + 2 (v \mid x)$$
$$i (u - iv \mid x - iy) - i (u - iv \mid x - iy) = -i^2 (2 (u \mid y) + 2 (u \mid x))$$

Adding these two together yields $4 (u \mid y)$ as we wanted to show.

Week 10

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Problem 6.2

Find the adjoint of the linear operator $T : \ell^2 \to \ell^2$ defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (0, 4x_1, x_2, 4x_3, x_4, \ldots).$$

Solution. Theorem 6.1 tells us that an unique adjoint T^* must exist. By the property of T^* we have that

$$(T\{x_n\} \mid \{y_n\}) = (\{x_n\} \mid \{z_n\})$$

with $\{z_n\} = \{Ty_n\}$. This can only be true in the case where $z_1 = 4y_2$, $z_2 = y_3$, $z_3 = 4y_4$, and so on. Hence, the adjoint T^* of T is defined uniquely through

$$T^*(\{y_n\}) = (4y_2, y_3, 4y_4, \ldots).$$

Problem 6.16

Let $T \in \mathcal{B}(\ell^2)$ be defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_1, -x_2, x_3, -x_4, \ldots).$$

a) Show that 1 and -1 are eigenvalues of T with eigenvectors $(1, 0, 0, \ldots)$ and $(0, 1, 0, 0, \ldots)$ respectively.

b) Find T^2 and hence show that $\sigma(T) = \{1, -1\}$.

Solution. For a) we simply plug in the values:

$$T(1, 0, 0, \ldots) = 1(1, 0, 0, \ldots)$$
$$T(0, 1, 0, \ldots) = -1(0, 1, 0, \ldots)$$

Hence 1 and -1 are eigenvalues with respective eigenvectors.

For b) we compute T^2:

$$T^2 x = TT x^2 = (x_1, x_2, x_3, x_3, x_4, \ldots)$$

so $T^2 = I$. Consequently, $\sigma(T^2) = \{1\}$.