MANDATORY ASSIGNMENT MAT3400

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Problem 1

Let Ω be a non-empty set, $\{x_n\}_n$ a sequence of distinct elements and $\{b_n\}_n$ a sequence of non-negative real numbers. For $E \subseteq \Omega$ define

$$\mu(E) = \sum_{\{n \in \mathbb{N} | x_n \in E\}} b_n.$$

Prove that μ is a measure on $\mathcal{P}(\Omega)$. Note that the function μ is defined as a sum of non-negative real numbers. Hence it follows directly that $\mu(E) \leq 0$ for all $E \in \mathcal{P}(\Omega)$. We also see that since $x_n \notin \emptyset$ for all n, so in the case of $\mu(\emptyset)$ we have no contributions. Hence $\mu(\emptyset) = 0$. For the countable disjoint union, we simply write it out:

$$\mu\left(\bigcup_n A_n\right) = \sum_{\left\{i | x_i \in \bigcup_n A_n\right\}} b_n$$

but since the sets are disjoint we can write this as distinct sums

$$= \sum_n \left(\sum_{\{i | x_i \in A_n\}} b_n\right) = \sum_n \mu(A_n).$$

Consequently, μ is a measure on $\mathcal{P}(\Omega)$ and the triplet $(\Omega, \mathcal{P}(\Omega), \mu)$ is a measure space.

Problem 2

Consider the measure space $([0,1], \mathcal{M}_{[0,1]}, \lambda)$ where λ is the Lebesgue measure restricted to $[0,1]$. Let f_n be defined on $[0,1]$ as follows

$$f_n(x) = \frac{n\sqrt{n}}{1 + n^2 x^2}.$$

We wish to compute $\lim_n \int_{[0,1]} f_n \, d\lambda$. For $x \in [0,1]$ we have that $f_n(x)$ is bounded by $1/(2\sqrt{x})$ and since $f_n(x) \to 0$ as $n \to \infty$ we can apply Dominated Convergence Theorem. This yields:

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\lambda = \int_{[0,1]} \lim_{n\to\infty} f_n \, d\lambda.$$

Using L'Hôpitals rule, we see that

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\lambda = 0.$$

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Problem 3

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to [0,1]$ an $\mathcal{A}$-measurable function. We wish to prove that

$$\lim_{n\to\infty} \int_\Omega f^{1/n}\, d\mu = \mu(f^{-1}((0,1])).$$

We also wish to show that under the assumption $\mu(\Omega) \leq \infty$, that

$$\lim_{n\to\infty} \int_\Omega f^{n}\, d\mu = \mu\left(f^{-1}(\{1\})\right).$$

Note that the function $f^{1/n}$ is monotone on Ω since f takes values in $[0,1]$. If we partition our set Ω into two disjoint sets Ω_1 and Ω_2, the set where f takes the value 0 and, the set where f takes values in $(0,1]$, respectively, then we can split our integral into two:

$$\lim_{n\to\infty} \int_\Omega f^{1/n}\, d\mu = \lim_{n\to\infty} \int_{\Omega_1} f^{1/n}\, d\mu + \lim_{n\to\infty} \int_{\Omega_2} f^{1/n}\, d\mu.$$

Applying the Monotone Convergence Theorem lets us interchange limits and integrals:

$$\lim_{n\to\infty} \int_\Omega f^{1/n}\, d\mu = \int_{\Omega_2} 1\, d\mu = \mu(\Omega_2) = \mu\left(f^{-1}(0,1]\right)$$

as we wanted to show. The integral over Ω_1 vanishes due to the pointwise limit in $x = 0$.

Let us now assume that $\mu(\Omega) < \infty$. We consider the function f^n. This is a monotone decreasing function on the interval $[0,1)$ and we again partition our integration domain into Ω_1 and Ω_2, the set where f attains the value 1 and the set where f takes values in $[0,1)$ respectively. Applying the Monotone Convergence Theorem for monotone decreasing sequences tells us that

$$\lim_{n\to\infty} \int_\Omega f^{n}\, d\mu = \int_{\Omega_1} \lim_{n\to\infty} f^n\, d\mu = \int_{\Omega_1} 1\, d\mu = \mu\left(f^{-1}(\{1\})\right).$$

Problem 4

Let $\sum_{k=0}^\infty a_k$ be a convergent series of non-negative real numbers and suppose that b_{n_k} are complex numbers such that $|b_{n_k}| \leq M \leq \infty$ for all $n, k \in \mathbb{N}$ and some positive real number m.

We wish to consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with μ the counting measure and write down a sequence $\{f_n\}_n$ of complex-valued $\mathcal{P}(\mathbb{N})$-measurable functions such that the following is satisfied:

$$\sum_{n=1}^\infty \int_{\mathbb{N}} |f_n|\, d\mu < \infty.$$

Integrating with respect to the counting measure is equivalent to summing over function values. Hence

$$\int_{\mathbb{N}} |f_n|\, d\mu = \sum_{i=1}^\infty |f_n(i)|.$$

If we define the function f_n by

$$f_n(k) = \frac{a_n b_{n_k}}{2^k}$$

then $\sum_{k=1}^\infty a_n b_{nk}/2^k \leq a_n M$. Our integral then becomes

$$\sum_{n=1}^\infty \int_{\mathbb{N}} |f_n(k)|\, d\mu = M \sum_{n=1}^\infty a_n < \infty.$$