Commutative Algebra
Notes in MAT4200

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Abstract

This document contains attempted solutions to the exercises given in the book *Introduction to Commutative Algebra* by M. F. Atiyah and I. G. McDonald. Text highlighted in red is meant to signify logical passages where I feel I have no idea what I am doing, and the reason for leaving this in is to later be able to reflect on the thought process.

Contents

Exercises

Chapter 1: Rings and ideals

Exercise (1)**.** Assume that x is nilpotent, and that $1 + x$ is *not* a unit in A. Hence, $1 + x$ is contained in a maximal ideal $\mathfrak{m}$. Since any maximal ideal is prime, and x is nilpotent, we have $x^n = 0 \in \mathfrak{m} \implies x \in \mathfrak{m}$. Any ideal is an additive subgroup, so $1 \in \mathfrak{m}$ which contradicts the fact that $\mathfrak{m}$ is maximal.

Now assume u a unit and x nilpotent. Assume for the sake of contradiction that $u + x$ is *not* a unit in A. Then $u + x$ is contained in a maximal ideal $\mathfrak{m}$. Since x is nilpotent we have $x \in \mathfrak{m}$, hence $u \in \mathfrak{m}$ so $\mathfrak{m} = (1)$, again contradicting the fact that $\mathfrak{m}$ is maximal.

Exercise (4)**.** We want to show that in $A[x]$ we have $\mathfrak{N} = \mathfrak{R}$. We have trivially that $\mathfrak{N} \subseteq \mathfrak{R}$, so we only need to show the opposite inclusion.

Let $f \in \mathfrak{R}$ with $f = \sum_{i=0}^{n} a_i x^i$, so by proposition 1.9 we have $1 - fg$ a unit for all $g \in A[x]$. Let $g = x$ be an element in $A[x]$. Then the function

$$1 - a_0 x - a_1 x^2 - \ldots - a_n x^{n+1}$$

is a unit in $A[x]$. By exercise 1.2.(i) we have that $a_0, \dots, a_n$ are nilpotent in A. By exercise 1.2.(ii) we have that f is nilpotent, so $f \in \mathfrak{N}$. Hence $\mathfrak{N} = \mathfrak{R}$.

Exercise (6)**.** Let A be a ring such that any ideal not contained in $\mathfrak{N}$ contains a non-zero idempotent element. We want to show that the nilradical and the Jacobson radical coincide in this case. We have the inclusion $\mathfrak{N} \subseteq \mathfrak{R}$ trivially. For the opposite inclusion we argue contrapositively. Let $c \notin \mathfrak{N}$. Then $(c) \not\subseteq \mathfrak{N}$. By assumption, (c) contains an idempotent element $a = cx$ for some $x \in A$. We wish to use proposition 1.9 again. Consider the element $1 - a$, and note that $a(1-a) = a - a = 0$, so $1 - a$ is *not* a unit in A since it is a zero divisor. By proposition 1.9 we have $a \notin \mathfrak{R}$, so $(c) \not\subseteq \mathfrak{R}$. Consequently, $\mathfrak{R} \subseteq \mathfrak{N}$.

Exercise (7)**.** Let A be a ring in which every element satisfies $x^n = x$ for some $n \geq 2$ dependent on x. We want to show that the nilradical $\mathfrak{N}$ and the Jacobson radical $\mathfrak{R}$ coincide. The inclusion $\mathfrak{N} \subseteq \mathfrak{R}$ is trivial as any maximal ideal is prime. We show the opposite inclusion by a contrapositive argument.

Assume that $x \notin \mathfrak{N}$. Our plan is to show that $1 - xg$ is *not* a unit for any $g \in A$. Consider the element $1 - x \cdot x^{n-2}$. This is a zero divisor as shown by multiplying by x from the left. Hence $1 - xg$ is *not* a unit with $g = x^{n-2}$. By proposition 1.9 we then have $x \notin \mathfrak{R}$. This shows contrapositively that $\mathfrak{R} = \mathfrak{N}$.

Note that I did not prove that every prime ideal $\mathfrak{p}$ is also a maximal ideal as the exercise requested. Trying again below.

We seek to show that any prime ideal $\mathfrak{p}$ of A is maximal. Let $\varphi : A \to A/\mathfrak{p}$ be the canonical homomorphism. Let $x \in A$ be any element. We know that $x^n = x$ for some n. Let $\bar{x} = \varphi(x) \in A/\mathfrak{p}$. Note that $\bar{x} = \bar{x}^n$. Since $A/\mathfrak{p}$ is an integral domain due to $\mathfrak{p}$ being prime, we know that the cancellation law for multiplication holds. So $\bar{x}^n = 1x$ implies that $\bar{x}^{n-1} = 1$. So, $\bar{x}$ has inverse $\bar{x}^{n-2}$. Hence, $A/\mathfrak{p}$ is a field, which implies that $\mathfrak{p}$ is a maximal ideal.

Exercise (8)**.** Let A be a non-zero ring. We wish to show that the set of prime ideals of A has a minimal element with respect to set inclusion. This can be solved by an application of Zorn's Lemma. Let Σ denote the set of all prime ideals of A. Let $\mathfrak{q} \leq \mathfrak{p}$ if $\mathfrak{p} \subseteq \mathfrak{q}$ (note the reverse inclusion). This set is partially ordered with respect to this relation. We need to show that any chain Γ in Σ has an upper bound in Σ. Let $\mathfrak{P} = \bigcap_\alpha \mathfrak{p}_\alpha$. We claim that the intersection of all prime ideals $\mathfrak{P}$ is an element of Σ and an upper bound for Γ.

Let $xy \in \mathfrak{P}$. Then xy is an element of every prime ideal $\mathfrak{p}_\alpha$ in the chain Γ. Assume now that $x \notin \mathfrak{P}$. In this case we need to show that $y \in \mathfrak{P}$. Let $\mathfrak{p}_i$ be a prime ideal not containing x. It therefore contains y instead. Since Γ is totally ordered, we can consider all the elements $\mathfrak{p}_i \leq \mathfrak{p}'$, i.e., $\mathfrak{p}' \subseteq \mathfrak{p}_i$. Since $x \notin \mathfrak{p}_i$ we have $x \notin \mathfrak{p}'$. It then follows that all such $\mathfrak{p}'$ must contain y.

Consider now the elements $\mathfrak{p}' \geq \mathfrak{p}_i$, that is $\mathfrak{p}_i \subseteq \mathfrak{p}'$. Since $y \in \mathfrak{p}_i$ we have $y \in \mathfrak{p}'$. Consequently, we have y in all prime ideals, hence also in $\mathfrak{P}$. This shows that $\mathfrak{P}$ is an element of Σ.

To show that $\mathfrak{P}$ is an upper bound for Γ, let $I \in \Gamma$ be a prime ideal. Then by definition of $\mathfrak{P}$ we have $\mathfrak{P} \subseteq I$, hence $I \leq \mathfrak{P}$. Zorn's Lemma then guarantees the existence of a maximal element with respect the order $\leq$, and consequently we have shown the existence of a minimal element with respect to set inclusion.

Exercise (9)**.** Assume that $\mathfrak{a} = r(\mathfrak{a})$. By proposition 1.14 the radical of any ideal $\mathfrak{a}$ is the intersection of the prime ideals containing $\mathfrak{a}$. It therefore directly follows from our assumption that $\mathfrak{a}$ is an intersection of prime ideals. Assume now that $\mathfrak{a}$ is *not* an intersection of prime ideals. Then it cannot be equal to the radical, as the radical *is* an intersection of prime ideals.

Exercise (10)**.** Let A be a ring, and $\mathfrak{N}$ its nilradical. We wish to show that the following are equivalent:

(i) A has exactly one prime ideal;

(ii) every element of A is either a unit or nilpotent;

(iii) $A/\mathfrak{N}$ is a field.

We first show (i) $\implies$ (ii). Assume that A has exactly one prime ideal. Let x be an element in $\mathfrak{N}$. In that case it is nilpotent. Assume therefore that x is not an element in $\mathfrak{N}$. If we further assume that x is *not* a unit, then it is contained in a maximal ideal $\mathfrak{m}$. Since A has exactly one prime ideal, and any maximal ideal is prime, we must have $\mathfrak{m} = \mathfrak{N}$. This contradicts the fact that $x \notin \mathfrak{N}$. Consequently, x must be a unit. So any element x in A is either nilpotent, or a unit.

We now consider the implication (ii) $\implies$ (iii). Assume that every element of A is either a unit or nilpotent. We seek to show that $A/\mathfrak{N}$ is a field. In principle, we only need to show that $\mathfrak{N}$ is in fact a maximal ideal. Let $\mathfrak{a}$ be an ideal containing $\mathfrak{N}$. We need to show that $\mathfrak{a} = \mathfrak{N}$, or $\mathfrak{a} = A$.

If $\mathfrak{a} = \mathfrak{N}$, then we are done. Assume therefore that $\mathfrak{a} \neq \mathfrak{N}$. Then there is an element $x \in \mathfrak{N}$ that is not in $\mathfrak{a}$. Then x is *not* a nilpotent element, so by assumption it must be a unit. Since $\mathfrak{a}$ is an ideal containing a unit, we must have $\mathfrak{a} = A$. Hence $\mathfrak{N}$ is maximal and $A/\mathfrak{N}$ is a field.

We now show the final implication (iii) $\implies$ (i). Assume that $A/\mathfrak{N}$ is a field. Then $\mathfrak{N}$ is a maximal ideal in A. Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{N} \subseteq \mathfrak{p}$. If $\mathfrak{N} = \mathfrak{p}$ we are done. If not, since $\mathfrak{N}$ is maximal, we must have $\mathfrak{p} = A$, hence A contains *exactly* one prime ideal.

Exercise (11)**.** Let A be a boolean ring (i.e., $x^2 = x$ for all $x \in A$). We want to show that the following properties hold:

(i) $2x = 0$ for all $x \in A$;

(ii) every prime ideal $\mathfrak{p}$ in A is maximal, and $A/\mathfrak{p}$ is a field with two elements; and

(iii) every finitely generated ideal in A is principal.

For (i), let x be an element in A and let a be the additive inverse of x. That is $a + x = x + a = 0$. Multiplying both sides by $x - a$ yield

$$x^2 - a^2 = 0 \iff x - a = 0 \iff x = a.$$

It then follows that $2x = x + x = x + a = 0$.

For (ii), note that this is just a special case of exercise 7 with $n = 2$ for every $x \in A$, hence any prime ideal $\mathfrak{p}$ is also maximal. It remains to show that $A/\mathfrak{p}$ has two elements. Let x be an element of A and assume that $x \in \mathfrak{p}$. Then $\varphi(x) = 0$ in $A/\mathfrak{p}$. If $x \notin \mathfrak{p}$ then φx has an inverse, so it makes sense to look at $\varphi(x)\varphi(x^{-1}) = 1$. Multiplying by $\varphi(x)$ on both sides yields

$$\varphi(x^2)\varphi(x^{-1}) = \varphi(x)\varphi(x^{-1}) = 1 = \varphi(x).$$

So $\varphi(x) = 1$ in $A/\mathfrak{p}$. Hence, $A/\mathfrak{p} = \{0, 1\}$, namely the additive and the multiplicative identities.

Exercise (12)**.** We wish to show that a local ring A has no idempotent element not equal to 0 or 1. So, let $x \in A$ be idempotent with $x \neq 0, 1$. We consider two cases — assume first that x is a unit. But then, we have $x = x^{-1}x^2 = x^{-1}x = 1$ which contradicts our initial assumption.

Assume therefore that x is *not* a unit. By proposition 1.5, we must have x contained in some maximal ideal $\mathfrak{m}$. Since A is local, there is only one maximal ideal, hence $x \in \mathfrak{m} \implies x \in \mathfrak{R}$. Since x is in the Jacobson radical, we know that $1 - xy$ is a unit in A for all y in A. Consider the fact that x is idempotent, so

$$x^2 = x \implies x(1-x) = 0.$$

Since $x \neq 0, 1$ we have that $(1 - x)$ is a zero divisor in A, contradicting the fact that it is also a unit.[1] Consequently, if x is idempotent, it must be either 0 or 1.

Exercise (15. The prime spectrum of a ring)**.** Let A be a ring, and X the set of all prime ideals of A. For each subset E of A, define $V(E)$ to be the set of prime ideals of A containing E. We want to prove that the following holds:

(i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$;

(ii) $V(0) = X$, $V(1) = \emptyset$;

(iii) if $\{E_i\}_{i \in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i);$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

For (i), assume that $\mathfrak{a}$ is generated by E. That is

$$\mathfrak{a} = \left\{ \sum_{i \in I} a_i e_i \mid a_i \in A, e_i \in E \right\}. \tag{*}$$

Let $\mathfrak{p} \in V(E)$ be a prime ideal containing E. Since any ideal is an abelian additive group closed under multiplication from A we see that any product $a_i e_i$ is in $\mathfrak{p}$ for any $a_i \in A, e_i \in E \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an additive group, it follows that any sum $\sum_{i \in I} a_i e_i$ is also in $\mathfrak{p}$. So, all the elements in $\mathfrak{a}$ are in $\mathfrak{p}$ hence $\mathfrak{p} \supseteq \mathfrak{a}$ and consequently $\mathfrak{p} \in V(\mathfrak{a})$. Conversely, assume that $\mathfrak{p} \in V(\mathfrak{a})$. Since $\mathfrak{p}$ contains $\mathfrak{a}$ it must also contain any elements on the form $\sum_{i \in I} e_i$, so it contains E; set $a_i = 1$ for all i in eq. (*). We have the first equality, $V(E) = V(\mathfrak{a})$.

Now, let $\mathfrak{p} \in V(\mathfrak{a})$ so $\mathfrak{p}$ contains $\mathfrak{a}$. Consequently $\mathfrak{p} \supseteq \bigcup_{i \in I} \mathfrak{p}_i$ where each $\mathfrak{p}_i$ contain $\mathfrak{a}$. But by proposition 1.14, it follows that $\mathfrak{p}_i$ contain $\sqrt{\mathfrak{a}}$, so $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$. Conversely, let $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$. Then $\mathfrak{p}$ contains $\sqrt{\mathfrak{a}}$. Since the radical of an ideal contains the ideal, it follows that $\mathfrak{p}$ contains $\mathfrak{a}$. Hence $\mathfrak{p} \in V(\mathfrak{a})$. This yields the final equality: $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

For (ii), note that any prime ideal $\mathfrak{p}$ contains the zero ideal, so $V(0) = X$. Similarly, by definition, a prime ideal is not equal to (1), so $V(1) = \emptyset$.

For (iii), let $\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right)$. This means that $\mathfrak{p}$ contains E_i for each $i \in I$, so $\mathfrak{p} \in V(E_i)$ for each $i \in I$. It then follows that $\mathfrak{p} \in \bigcup_{i \in I} V(E_i)$. Conversely, let $\mathfrak{p} \in \bigcup_{i \in I} V(E_i)$. This means that $\mathfrak{p}$ contains E_i for each $i \in I$. Consequently, $\mathfrak{p}$ contains the union of all E_i, hence $\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right)$.

[1] Equivalently, since both x and $(1 - x)$ are non-units, they must both lie in a maximal ideal. Since A is local they lie in the same maximal ideal. Since any ideal is an additive subgroup we have that $x + (1 - x) = 1$ lie in the maximal ideal, hence the maximal ideal is the whole thing, (1), which is a contradiction.

Chapter 2: Modules

Exercise (1)**.** We want to show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime. By definition of coprime, we have $\gcd(m, n) = 1$. We can find numbers $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Let $x \otimes y$ be an element of the tensor product. We therefore have

$$x \otimes y = 1\,(x \otimes y) = (am + bn)(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = 0.$$

Exercise (2)**.** We let A be a ring, $\mathfrak{a}$ an ideal, and M an A-module. We wish to show that $(A/\mathfrak{a}) \otimes_A M \simeq M/\mathfrak{a}M$. We right tensor the exact sequence

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

with M. This gives another exact sequence (proposition 2.18)

$$\begin{tikzcd}
\mathfrak{a} \otimes M \arrow[r, "f"] \arrow[dr] & A \otimes M \arrow[r, "g"] \arrow[d, "\psi"] & (A/\mathfrak{a}) \otimes M \arrow[r] & 0 \\
& M & &
\end{tikzcd}$$

where ψ is the canonical isomorphism. By the first module isomorphism theorem we have

$$A \otimes M/\ker(g) \simeq \operatorname{im}(g).$$

Since the sequence is exact, g is injective, hence $\operatorname{im}(g) = (A/\mathfrak{a}) \otimes M$. We identify $\ker(g)$ by the elements in $(\psi \circ f)(\mathfrak{a} \otimes M)$. Hence,

$$A \otimes M/\ker(g) \simeq M/\mathfrak{a}M \simeq (A/\mathfrak{a}) \otimes M$$

as we wanted to show.

Exercise (3)**.** Let A be a local ring, and assume that M and N are finitely generated A-modules. Assume that $M \otimes_A N = 0$. We wish to show that then either $M = 0$, or $N = 0$. Define $k = A/\mathfrak{m}$ where $\mathfrak{m}$ is the only maximal ideal in A. Furthermore, let $M_k = k \otimes M$ and $N_k = k \otimes N$ be the k modules obtained from M and N by extension of scalars.

From the previous exercise, we know that $M_k \simeq M/\mathfrak{m}M$ and $N_k \simeq N/\mathfrak{m}N$. If we can show that either $M_k = 0$ or $N_k = 0$, then by Nakayama's lemma we have

$$M_k = 0 \implies M/\mathfrak{m}M = 0 \implies M = \mathfrak{m}M \implies M = 0$$

since $\mathfrak{m}$ is contained in the Jacobson radical of A and M is finitely generated (similarly for N_k). In order to show either $M_k = 0$ or $N_k = 0$ we consider the fact that $M \otimes_A N = 0$. This gives[2]

$$M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0.$$

Since $M_k \otimes_k N_k = 0$ it has dimension zero. The dimension of $M_k \otimes_k N_k$ is the product of the dimensions of M_k and N_k. However, this means that the dimension of either M_k or N_k is zero. Hence, either $M_k = 0$ or $N_k = 0$. Therefore, by the above mention of Nakayama's lemma, we have either $M = 0$ or $N = 0$.

[2] I think we use the fact that direct sum and direct product coincide if the index set is finite, in order to apply proposition 2.14(iii)

Exercise (4)**.** Let $\{M_i\}_{i\in I}$ be family of A-modules, and let $M = \bigoplus_{i\in I} M_i$. We wish to show that M being flat is equivalent to each M_i being flat.

Assume that each M_i is flat. Then by definition, if

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow N' \otimes M_i \xrightarrow{f_i} N \otimes M_i \xrightarrow{g_i} N'' \otimes M_i \longrightarrow 0$$

is an exact sequence. Let $\varphi\colon \bigoplus(N' \otimes M_i) \to \bigoplus(N \otimes M_i)$ be defined by

$$\varphi(x) = (f_1(x_1), f_2(x_2), \dots)$$

and $\psi\colon \bigoplus(N \otimes M_i) \to \bigoplus(N'' \otimes M_i)$ be defined by

$$\psi(y) = (g_1(y_1), g_2(y_2), \dots)\,.$$

By the injectivity of each f_i and the surjectivity of each g_i we have that φ is injective and ψ surjective. Hence the sequence

$$0 \longrightarrow N' \otimes M \xrightarrow{\varphi} N \otimes M \xrightarrow{\psi} N'' \otimes M \longrightarrow 0$$

is exact. To show that M flat implies M_i flat, we argue by contradiction. Assuming M flat implies that the above short sequence is exact. That is φ and ψ are injective and surjective respectively. If one of the M_i's were not flat, this would contradict the fact that φ is injective, or that ψ is surjective. Consequently all the M_i's must be flat.

Exercise (5)**.** We let $A[x]$ be the ring of polynomials over the indeterminate x. We wish to show that $A[x]$ is a flat A-algebra. We proceed accordingly:

1. Show that A is a flat A-module;
2. show that $M_i = Ax^i \simeq A$;
3. show that $\bigoplus_i M_i = A[x]$;
4. apply exercise 2.4.

To show that A is a flat A-module we check that any exact sequence of A-modules is exact when tensored with A. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Tensoring yields

$$\begin{tikzcd}
0 \arrow[r] & M' \otimes A \arrow[r] \arrow[d, "\simeq"] & M \otimes A \arrow[r] \arrow[d, "\simeq"] & M'' \otimes A \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

This sequence is exact, hence A is a flat A-module. Define now $M_i = Ax^i$. Under the map $f_i : M_i \to A$ defined by $ax^i \mapsto a$ we see that $M_i \simeq A$, therefore each M_i is flat. It then follows that any element in $A[x]$ can be written as a sum of elements in the M_i's. That is,

$$\bigoplus_i M_i = A[x].$$

Applying exercise 2.4, we know that since each M_i is flat, that $A[x]$ is flat.

Exercise (7)**.** Let $\mathfrak{p}$ be a prime ideal in a ring A. We want to show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. We can do this by showing that $A[x]/\mathfrak{p}[x]$ is an integral domain. Consider the canonical homomorphism

$$\phi : A[x] \to (A/\mathfrak{p})\,[x].$$

This is surjective with kernel $\mathfrak{p}[x]$. Hence by the first isomorphism theorem for modules we have

$$A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})\,[x].$$

Since $A/\mathfrak{p}$ is an integral domain, the polynomial ring with coefficients in this integral domain must be an integral domain. Hence, $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. This implication does not hold if we replace *prime* with *maximal*. Take for instance the maximal ideal $2\mathbb{Z}$ in $\mathbb{Z}$. The quotient $\mathbb{Z}/2\mathbb{Z}$ is a field. However, $x \in (\mathbb{Z}/2\mathbb{Z})\,[x]$ does *not* have a multiplicative inverse.

Exercise (8)**.** We want to show the following:

i) If M and N are A-flat, then so is $M \otimes_A N$;

ii) If B is a flat A-algebra and N is a flat B-module, then N is A-flat as an A-module.

Consider first M,N as A-flat modules. Using the isomorphism from proposition 2.14(ii). Then tensoring any exact sequence by $M \otimes_A N$ keeps the sequence exact as M and N are flat.

For (ii) we want to use the injectivity characterization of flatness. Let φ be any injective map from M' to M, with M, M' both being A-modules. Since B is a flat A-module we know that

$$\varphi \otimes \mathrm{id}_B \colon M' \otimes_A B \to M \otimes_A B$$

is injective. We know by assumption that N is a flat B-module, so the map

$$(\varphi \otimes \mathrm{id}_B) \otimes \mathrm{id}_N \colon\; (M' \otimes_A B) \otimes_B N \to (M \otimes_A B) \otimes_B N$$

is injective and this makes sense when regarding $M \otimes_A B$ as a B-module (by exercise 2.15). By proposition 2.14(iv) this is isomorphic to

$$(\varphi \otimes \mathrm{id}_B) \otimes \mathrm{id}_N \colon M' \otimes_A N \to M \otimes_A N.$$

This map is the map we want as the following diagram commutes:

$$\begin{tikzcd}
(m' \otimes_A b) \otimes_B n \arrow[rr] \arrow[d] & & (\varphi(m') \otimes_A b) \otimes_B n \arrow[d] \\
m' \otimes_A (b \otimes_B n) \arrow[d] & \circlearrowleft & \varphi(m') \otimes_A (b \otimes_B n) \arrow[d] \\
m' \otimes_A bn \arrow[rr] & & \varphi(m') \otimes_A bn
\end{tikzcd}$$

This map being injective means that N is flat as an A-module.

Exercise (8)**.** We want to show that if M'' and M' are finitely generated and $0 \to M' \to M \to M'' \to 0$ is exact, then it follows that M is finitely generated. We show the special case that M' is a submodule of M and that $M'' = M/M'$. Let M' have generating set $\{x_i\}$ and M'' generating set $\{y_i\}$.

Exercise (10)**.** Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical $\mathfrak{R}$. Let M be an A-module and N a finitely generated A-module. We want to show that $u\colon M \to N$ is surjective if the induced map $v\colon M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. Seeing the words *finitely generated* and *Jacobson radical* we want to apply Nakayama's lemma.

Consider $\operatorname{coker}(u) = N/\operatorname{im} u$. If we can show this to be zero, then $\operatorname{im} u = u(M) = N$ so u is surjective. Tensoring the exact sequence $M \to N \to \operatorname{coker}(u) \to 0$ with $(A/\mathfrak{a})$ yields

$$\begin{tikzcd} M \otimes (A/\mathfrak{a}) \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "\simeq"] & N \otimes (A/\mathfrak{a}) \arrow[r] \arrow[d, "\simeq"] & \operatorname{coker}(u) \otimes (A/\mathfrak{a}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ M/\mathfrak{a}M \arrow[r, "v"] & N/\mathfrak{a}N \arrow[r] & \operatorname{coker}(u)/\mathfrak{a}\operatorname{coker}(u) \arrow[r] & 0 \end{tikzcd}$$

The fact that $u \otimes \mathrm{id} = g$ took a while to figure out, but it happens in the abstract nonsense from exercise 2.2. Consequently $\operatorname{coker}(u) = \mathfrak{a}\operatorname{coker}(u)$, and since $\operatorname{coker}(u)$ is finitely generated and $\mathfrak{a}$ is contained in $\mathfrak{R}$, it follows from Nakayama's lemma that $\operatorname{coker}(u) = 0$ which implies that $u(M) = N$, hence u is surjective.

An equivalent proof using a corollary of Nakayama is way easier. Consider the composition $f\colon M \to M/\mathfrak{a}M \to N/\mathfrak{a}N$. This is surjective by assumption. We now examine the image of f in two different ways. First of all we have $\operatorname{im}(f) = N/\mathfrak{a}N$ using the surjectivity. We also have that $\operatorname{im}(f) = \operatorname{im}(u)/\mathfrak{a}N$. Now, since we are modding out by $\mathfrak{a}N$, we can simply add the submodule $\mathfrak{a}N$ to obtain $\operatorname{im}(f) = (\operatorname{im}(u) + \mathfrak{a}N)/\mathfrak{a}N$ in order to get something on the form of Nakayama. So we have $N/\mathfrak{a}N = (\operatorname{im}(u) + \mathfrak{a}N)/\mathfrak{a}N$, so $N = \operatorname{im}(u) + \mathfrak{a}N)$. By corollary 2.7 we have that $\operatorname{im}(u) = N$ hence u is surjective.

Chapter 3: Rings and modules of fractions

Exercise (1)**.** Given a ring A, a multiplicatively closed subset S of A and a finitely generated A-module M, we want to show that $S^{-1}M = 0$ if and only if there exists an $s \in S$ such that $sM = 0$.

Let M have generators $x_1, \ldots, x_n$. Assume first that $S^{-1}M = 0$. This means that $m/s = 0/1$ in $S^{-1}M$ for all $m \in M, s \in S$. This in turn, means that by definition there exists t in S such that $tm = 0$ in M. In particular, let t_i be the element such that $tx_i = 0$, i.e., an element of S that kills one of the generating elements of M. Since S is multiplicatively closed, take $t' = t_1t_2 \ldots t_n$. Then t' kills all generators of M. Hence, $t'm = t' \sum a_i x_i = 0$ and consequently, $t'M = 0$.

For the converse, assume that there exist an $s \in S$ such that $sM = 0$. Consider the element $m/1$ in $S^{-1}M$. Then
$$\frac{m}{1} = \frac{sm}{s} = \frac{0}{s} = \frac{0}{1},$$
so any element $m/1$ in $S^{-1}M$ is zero, hence $S^{-1}M = 0$.

Exercise (2)**.** Let $\mathfrak{a}$ be an ideal of a ring A. Furthermore, let $S = 1 + \mathfrak{a}$. We want to show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical $\mathfrak{R}$ of $S^{-1}A$.

We first show that S is infact a multiplicatively closed subset of A. We first require 1 to be in S, but since 0 is contained in any ideal, we have $1 + 0 = 1 \in S$. Taking two elements $(1 + a)$ and $(1 + b)$ in S we see that $(1 + a + b + ab)$ is in S as $\mathfrak{a}$ is an ideal. Consequently, S is a mulitplicative set.

In order to check whether $S^{-1}\mathfrak{a}$ is contained in the Jacobson, we pick an element x in $S^{-1}\mathfrak{a}$ and show that $1-xy$ is a unit in $S^{-1}A$ for all y in $S^{-1}A$. We have that

$$1-xy = \frac{1}{1} - \frac{aa'}{(1+b)(1+b')} = \frac{(1+b)(1+b')-aa'}{(1+b)(1+b')}$$

the numerator is on the form $(1+a'')$ where a'' is in $\mathfrak{a}$, and the denominator is on the form $(1+b'')$ with b'' in $\mathfrak{a}$. Consequently, both numerator and denominator are units in $S^{-1}A$, so $1-xy$ is a unit.

We now wish to use this fact along with Nakayama's lemma in order to prove corollary 2.5 without the use of determinants. So, let M be finitely generated with $\mathfrak{a}$ an ideal such that $\mathfrak{a}M = M$. We localize in $S = 1+\mathfrak{a}$ which yields

$$S^{-1}M = S^{-1}\mathfrak{a}M = \left(S^{-1}\mathfrak{a}\right)\left(S^{-1}M\right).$$

Now, from the previous result we know that $S^{-1}\mathfrak{a}$ is contained in $\mathfrak{R}$. Since we also have that $S^{-1}M$ is finitely generated, due to M being finitely generated, we can conclude from Nakayama's lemma that $S^{-1}M = 0$. By exercise 1 we know that this holds if and only if there exist an $s \in S$ such that $sM = 0$. This s is on the form $s = 1+a$ for some a in $\mathfrak{a}$, hence $s \equiv 1 \pmod{\mathfrak{a}}$.

Exercise (3)**.** Let A be a ring, and let S and T be two multiplicatively closed subsets of A. Let U be the image of T in $S^{-1}A$. We want to show that the rings $(ST)^{-1}A$ and $U^{-1}\left(S^{-1}A\right)$ are isomorphic. For brevity, we shall call these rings B and C respectively. We let f be the only sensible choice of map between these two sets and show that it is a bijection. That is $f\colon (ST)^{-1}A \to U^{-1}\left(S^{-1}A\right)$ defined by

$$f\left(a/(st)\right) = (a/s)/(t/1).$$

We first show that f is well defined. Take $a/(st) = a'/(s't')$ and consider the image of these two elements. We need to find a u in C such that $u\left(at'/s\right) = u\left(ta'/s'\right)$. But, by assumption there exists a u' in B such that $u'as't' = u'a'st$, so we can chose $u = u'/1$. Consequently, f is a well defined map.

We now need to show that f is a bijection. We see immediately that it is surjective, since any element on the form $(a/s)/(t/1)$ comes from an element on the form $a/(st)$ with $a \in A$ and $st \in ST$. To see that it is injective, take an element $a/(st)$ in the kernel of f. Then $(a/s)/(t/1) = (0/1)/(1/1)$, so there is a u such that

$$u\left((a/s)(1/1) - (t/1)(0/1)\right) = 0,$$

so $u(a/s) - (0/1) = 0$, but this means that $a/st = 0$ in B. So f is injective. It remains to show that f is a ring homomorphism.

Chapter 4: Primary Decomposition

Exercise (2)**.** We wish to show that if an ideal $\mathfrak{a}$ satisfies $\sqrt{\mathfrak{a}} = \mathfrak{a}$, then it has no embedded prime ideals. Since we are dealing with ideals having a minimal primary decomposition, we this be

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

We denote the radical $\sqrt{\mathfrak{q}_i}$ by $\mathfrak{p}_i$. Now, since $\mathfrak{a} = \sqrt{a}$, we know that $\mathfrak{a} = \bigcap \mathfrak{p}_j$ where p_j is a prime ideal containing $\mathfrak{a}$. Taking radicals we get that, on one hand,

$$\mathfrak{a} = \sqrt{\mathfrak{a}} = \sqrt{\bigcap \mathfrak{p}_j} = \bigcap \sqrt{\mathfrak{p}_j} = \bigcap \sqrt{\mathfrak{p}_j} = \bigcap \mathfrak{p}_j,$$

while on the other,

$$\sqrt{\mathfrak{a}} = \bigcap_{i=1}^{n} \sqrt{q_i} = \bigcap_{i=1}^{n} p_i.$$

I now need to properly figure out exactly *why* this means that none of the prime ideals belonging to $\mathfrak{a}$ are embedded.

Exercise (4)**.** We consider the polynomial ring $\mathbb{Z}[t]$. We want to show that the ideal $\mathfrak{m} = (2, t)$ is maximal, and that the ideal $\mathfrak{q} = (4, t)$ is $\mathfrak{m}$ primary, but *not* a power of $\mathfrak{m}$. Note first that the quotient $\mathbb{Z}[t]/\mathfrak{m}$ consists of all constant polynomials where the constant is either 0 or 1. Hence, this is isomorphic to the field $\mathbb{Z}_2$. Consequently, $\mathfrak{m}$ is maximal. In order to show that $\mathfrak{q}$ is $\mathfrak{m}$ primary, we first show that all the zero-divisors in $\mathbb{Z}[t]/\mathfrak{q}$ are nilpotent, and then show that the radical $\mathfrak{q}$ is equal to $\mathfrak{m}$. The quotient $\mathbb{Z}[t]/\mathfrak{q}$ is isomorphic to $\mathbb{Z}_4$, and consists therefore only of constant polynomials where the constant is either 0, 1, 2 or 3. The only zero-divisor in $\mathbb{Z}_4$ is 2, and this is also nilpotent, as $2^2 = 0$. Consequently, $\mathfrak{q}$ is primary. Consider now $\sqrt{\mathfrak{q}}$:

$$\sqrt{\mathfrak{q}} = \sqrt{(4, t)} = \sqrt{(4) + (t)} = \sqrt{\sqrt{(4)} + \sqrt{(t)}} = \sqrt{(2) + (t)} = (2) + (t)$$

since $(2) + (t)$ is maximal (and therefore also prime). This shows that $\mathfrak{q}$ is $\mathfrak{m}$-primary. It remains to show that $\mathfrak{q}$ is *not* a power of of $\mathfrak{m}$.

Exercise (5)**.**

Problem Sheet

Problem (1)**.** We let M be an A module, and let N be a submodule of M. We define the radical of N in M as

$$r_M(N) = \{x \in A \mid x^n \subseteq N \text{ for some } n > 0\}.$$

a) We first want to show that $r_M(N) = \sqrt{(N : M)}$. Recall that $(N : M)$ is the set of all $x \in A$ that multiplies M into N. Let $x \in r_M(N)$. Then $x^n m \in N$ for all $m \in M$. Hence $x^n \in (N : M)$, so $x \in \sqrt{(N : M)}$. Conversely, let $x \in \sqrt{(N : M)}$, then $x^n \in (N : M)$, so $x^n M \subseteq N$, hence $x \in r_M(N)$.

b) We now wish to show that $(Q : M)$ is a primary ideal in A under the assumption that Q is primary in M as a module. Let $ab \in (Q : M)$, and assume in addition that $a \notin (Q : M)$. We need to show that $b^n \in (Q : M)$ for some $n > 0$. By assumption, $a \notin (Q : M)$ tells us that there is some element $m' \in M$ such that $am' \notin Q$. We know in addition, that $bam' \in Q$ since $ab \in (Q : M)$.

   Consider now the map $\varphi_b : M \to M$ given by $\varphi_b(x) = xb$. We have for the element $am' \notin Q$ that $\varphi_b(am')$ is in Q. Hence, in the quotient M/Q, we map a non-zero element to a zero element, so the induced map $\bar{\varphi}_b$ has non-zero kernel, hence not injective. This means that $\bar{b}$ is a zero-divisor in M/Q, and consequently, since Q is primary in M, b is nilpotent. This means that there is an $n > 0$, such that $\varphi(b)^n = 0$, hence $b^n m \in Q$ for all m. We can therefore conclude that $b^n \in (Q : M)$.