Notes MAT4200
Commutative Algebra

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1 Various propositions and claims

Claim 1. If $f\colon A \to B$ is a ring homomorphism, and suppose $J \subseteq B$ is an ideal in B, then $f^{-1}(J)$ is an ideal in A.

Proposition 1 (Ideal correspondence)**.** There is a one-to-one , order preserving correspondence, between ideals J of A containing I, and the ideals $\bar{J}$ of A/I, given by $\bar{J} = f^{-1}\left(\bar{J}\right)$.

Proof. Let J be an ideal containing I. We want to show that $f(J)$ is an ideal of A/I, that is, $yf(J) \subseteq f(J)$ for all $y \in A/I$. Assume that $y = c + I$ for some $c \in A$. Pick any $j + I \in f(J)$. Then

$$y(j + I) = (c + I)(j + I) = cj + I$$

with $c \in A$, $j \in J$. So $cj \in J$ since J is an ideal. We therefore have $cj+I \in f(J)$ and consequently, $yf(J) \subseteq f(J)$.

Now let $\bar{J}$ be an ideal in A/I. By Claim 1, $f^{-1}(\bar{J})$ is an ideal in A. All ideals contain 0, so $0_{A/I} \in \bar{J}$. It then follows that

$$f^{-1}\left(0_{A/I}\right) = f = I \subseteq f^{-1}(\bar{J}).$$

So the ideal $f^{-1}(\bar{J})$ contains the ideal I.

We now show that the map f is a one-to-one correspondence. We do this by showing $f(f^{-1}(\bar{J})) = \bar{J}$ and $f^{-1}(f(J)) = J$. Note that

$$f(f^{-1}(\bar{J})) = f\left(\left\{a \in A \mid f(a) \in \bar{J}\right\}\right) = \bar{J}.$$

Now, let $a \in J$. Then $f(a) = a + I \in f(J)$, so $a \in f^{-1}(f(J))$ by definition. For the opposite inclusion, note that

$$\begin{aligned} f^{-1}(f(J)) &= \{a \in A \mid f(a) \in f(J)\} \\ &= \{a \in A \mid a + I \in f(J)\}. \end{aligned}$$

Chose $a + I \in f(J)$. Then $a + I = a' + I$ for some $a' \in J$. We do not yet know whether $a = a'$, just that they represent the same coset. By the definition of coset equality we have $a - a' = c \in I$. Therefore, $a = c + a'$. Since $c \in I \subseteq J$ and $a' \in J$, we have $a \in J$ since J is abelian group under addition. We have therefore shown the inclusion $f^{-1}f(J)) \subseteq J$. Consequently, f is a one-to-one correspondence between ideals in A containing I and ideals in A/I. $\square$

2 Exercises

Chapter 1

Exercise (1)**.** Assume that x is nilpotent, and that $1 + x$ is *not* a unit in A. Hence, $1 + x$ is contained in a maximal ideal $\mathfrak{m}$. Since any maximal ideal is prime, and x is nilpotent, we have $x^n = 0 \in \mathfrak{m} \implies x \in \mathfrak{m}$. Any ideal is an additive subgroup, so $1 \in \mathfrak{m}$ which contradicts the fact that $\mathfrak{m}$ is maximal.

Now assume u a unit and x nilpotent. Assume for the sake of contradiction that $u + x$ is *not* a unit in A. Then $u + x$ is contained in a maximal ideal $\mathfrak{m}$. Since x is nilpotent we have $x \in \mathfrak{m}$, hence $u \in \mathfrak{m}$ so $\mathfrak{m} = (1)$, again contradicting the fact that $\mathfrak{m}$ is maximal.

Exercise (4)**.** We want to show that in $A[x]$ we have $\mathfrak{N} = \mathfrak{R}$. We have trivially that $\mathfrak{N} \subseteq \mathfrak{R}$, so we only need to show the opposite inclusion.

Let $f \in \mathfrak{R}$ with $f = \sum_{i=0}^{n} a_i x^i$, so by proposition 1.9 we have $1 - fg$ a unit for all $g \in A[x]$. Let $g = x$ be an element in $A[x]$. Then the function

$$1 - a_0 x - a_1 x^2 - \ldots - a_n x^{n+1}$$

is a unit in $A[x]$. By exercise 1.2.(i) we have that $a_0, \ldots, a_n$ are nilpotent in A. By exercise 1.2.(ii) we have that f is nilpotent, so $f \in \mathfrak{N}$. Hence $\mathfrak{N} = \mathfrak{R}$.

Exercise (6)**.** Let A be a ring such that any ideal not contained in $\mathfrak{N}$ contains a non-zero idempotent element. We want to show that the nilradical and the Jacobson radical coincide in this case. We have the inclusion $\mathfrak{N} \subseteq \mathfrak{R}$ trivially. For the opposite inclusion we argue contrapositively. Let $c \notin \mathfrak{N}$. Then $(c) \not\subseteq \mathfrak{N}$. By assumption, (c) contains an idempotent element $a = cx$ for some $x \in A$. We wish to use proposition 1.9 again. Consider the element $1 - a$, and note that $a(1 - a) = a - a = 0$, so $1 - a$ is *not* a unit in A since it is a zero divisor. By proposition 1.9 we have $a \notin \mathfrak{N}$, so $(c) \not\subseteq \mathfrak{N}$. Consequently, $\mathfrak{R} \subseteq \mathfrak{N}$.

Exercise (7)**.** Let A be a ring in which every element satisfies $x^n = x$ for some $n \geq 2$ dependent on x. We want to show that the nilradical $\mathfrak{N}$ and the Jacobson radical $\mathfrak{R}$ coincide. The inclusion $\mathfrak{N} \subseteq \mathfrak{R}$ is trivial as any maximal ideal is prime. We show the opposite inclusion by a contrapositive argument.

Assume that $x \notin \mathfrak{N}$. Our plan is to show that $1 - xg$ is *not* a unit for any $g \in A$. Consider the element $1 - x \cdot x^{n-2}$. This is a zero divisor as shown by multiplying by x from the left. Hence $1 - xg$ is *not* a unit with $g = x^{n-2}$. By proposition 1.9 we then have $x \notin \mathfrak{R}$. This shows contrapositively that $\mathfrak{R} = \mathfrak{N}$.

Note that I did not prove that every prime ideal $\mathfrak{p}$ is also a maximal ideal as the exercise requested. Trying again below.

We seek to show that any prime ideal $\mathfrak{p}$ of A is maximal. Let $\varphi : A \to A/\mathfrak{p}$ be the canonical homomorphism. Let $x \in A$ be any element. We know that $x^n = x$ for some n. Let $\bar{x} = \varphi(x) \in A/\mathfrak{p}$. Note that $\bar{x} = \bar{x}^n$. Since $A/\mathfrak{p}$ is an integral domain due to $\mathfrak{p}$ being prime, we know that the cancellation law for multiplication holds. So $\bar{x}^n = 1x$ implies that $\bar{x}^{n-1} = 1$. So, $\bar{x}$ has inverse $\bar{x}^{n-2}$. Hence, $A/\mathfrak{p}$ is a field, which implies that $\mathfrak{p}$ is a maximal ideal.

Exercise (8)**.** Let A be a non-zero ring. We wish to show that the set of prime ideals of A has a minimal element with respect to set inclusion. This can be solved by an application of Zorn's Lemma. Let Σ denote the set of all prime ideals of A. Let $\mathfrak{q} \leq \mathfrak{p}$ if $\mathfrak{p} \subseteq \mathfrak{q}$ (note the reverse inclusion). This set is partially ordered with respect to this relation. We need to show that any chain Γ in Σ has an upper bound in Σ. Let $\mathfrak{P} = \bigcap_\alpha \mathfrak{p}_\alpha$. We claim that the intersection of all prime ideals $\mathfrak{P}$ is an element of Σ and an upper bound for Γ.

Let $xy \in \mathfrak{P}$. Then xy is an element of every prime ideal $\mathfrak{p}_\alpha$ in the chain Γ. Assume now that $x \notin \mathfrak{P}$. In this case we need to show that $y \in \mathfrak{P}$. Let $\mathfrak{p}_i$ be a prime ideal not containing x. It therefore contains y instead. Since Γ is totally ordered, we can consider all the elements $\mathfrak{p}_i \leq \mathfrak{p}'$, i.e., $\mathfrak{p}' \subseteq \mathfrak{p}_i$. Since $x \notin \mathfrak{p}_i$ we have $x \notin \mathfrak{p}'$. It then follows that all such $\mathfrak{p}'$ must contain y.

Consider now the elements $\mathfrak{p}' \geq \mathfrak{p}_i$, that is $\mathfrak{p}_i \subseteq \mathfrak{p}'$. Since $y \in \mathfrak{p}_i$ we have $y \in \mathfrak{p}'$. Consequently, we have y in all prime ideals, hence also in $\mathfrak{P}$. This shows that $\mathfrak{P}$ is an element of Σ.

To show that $\mathfrak{P}$ is an upper bound for Γ, let $I \in \Gamma$ be a prime ideal. Then by definition of $\mathfrak{P}$ we have $\mathfrak{P} \subseteq I$, hence $I \leq \mathfrak{P}$. Zorn's Lemma then guarantees the existence of a maximal element with respect the order $\leq$, and consequently we have shown the existence of a minimal element with respect to set inclusion.

Exercise (9)**.** Assume that $\mathfrak{a} = r(\mathfrak{a})$. By proposition 1.14 the radical of any ideal $\mathfrak{a}$ is the intersection of the prime ideals containing $\mathfrak{a}$. It therefore directly follows from our assumption that $\mathfrak{a}$ is an intersection of prime ideals. Assume now that $\mathfrak{a}$ is *not* an intersection of prime ideals. Then it cannot be equal to the radical, as the radical *is* an intersection of prime ideals.

Exercise (10)**.** Let A be a ring, and $\mathfrak{N}$ its nilradical. We wish to show that the following are equivalent:

(i) A has exactly one prime ideal;

(ii) every element of A is either a unit or nilpotent;

(iii) $A/\mathfrak{N}$ is a field.

We first show (i) $\implies$ (ii). Assume that A has exactly one prime ideal. Let x be an element in $\mathfrak{N}$. In that case it is nilpotent. Assume therefore that x is not an element in $\mathfrak{N}$. If we further assume that x is *not* a unit, then it is contained in a maximal ideal $\mathfrak{m}$. Since A has exactly one prime ideal, and any maximal ideal is prime, we must have $\mathfrak{m} = \mathfrak{N}$. This contradicts the fact that $x \notin \mathfrak{N}$. Consequently, x must be a unit. So any element x in A is either nilpotent, or a unit.

We now consider the implication (ii) $\implies$ (iii). Assume that every element of A is either a unit or nilpotent. We seek to show that $A/\mathfrak{N}$ is a field. In principle, we only need to show that $\mathfrak{N}$ is in fact a maximal ideal. Let $\mathfrak{a}$ be an ideal containing $\mathfrak{N}$. We need to show that $\mathfrak{a} = \mathfrak{N}$, or $\mathfrak{a} = A$.

If $\mathfrak{a} = \mathfrak{N}$, then we are done. Assume therefore that $\mathfrak{a} \neq \mathfrak{N}$. Then there is an element $x \in \mathfrak{N}$ that is not in $\mathfrak{a}$. Then x is *not* a nilpotent element, so by assumption it must be a unit. Since $\mathfrak{a}$ is an ideal containing a unit, we must have $\mathfrak{a} = A$. Hence $\mathfrak{N}$ is maximal and $A/\mathfrak{N}$ is a field.

We now show the final implication (iii) $\implies$ (i). Assume that $A/\mathfrak{N}$ is a field. Then $\mathfrak{N}$ is a maximal ideal in A. Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{N} \subseteq \mathfrak{p}$. If $\mathfrak{N} = \mathfrak{p}$ we are done. If not, since $\mathfrak{N}$ is maximal, we must have $\mathfrak{p} = A$, hence A contains *exactly* one prime ideal.

Exercise (11)**.** Let A be a boolean ring (i.e., $x^2 = x$ for all $x \in A$). We want to show that the following properties hold:

(i) $2x = 0$ for all $x \in A$;

(ii) every prime ideal $\mathfrak{p}$ in A is maximal, and $A/\mathfrak{p}$ is a field with two elements; and

(iii) every finitely generated ideal in A is principal.

For (i), let x be an element in A and let a be the additive inverse of x. That is $a + x = x + a = 0$. Multiplying both sides by $x - a$ yield

$$x^2 - a^2 = 0 \iff x - a = 0 \iff x = a.$$

It then follows that $2x = x + x = x + a = 0$.

For (ii), note that this is just a special case of exercise 7 with $n = 2$ for every $x \in A$, hence any prime ideal $\mathfrak{p}$ is also maximal. It remains to show that $A/\mathfrak{p}$ has two elements. Let x be an element of A and assume that $x \in \mathfrak{p}$. Then $\varphi(x) = 0$ in $A/\mathfrak{p}$. If $x \notin \mathfrak{p}$ then φx has an inverse, so it makes sense to look at $\varphi(x)\varphi(x^{-1}) = 1$. Multiplying by $\varphi(x)$ on both sides yields

$$\varphi(x^2)\varphi(x^{-1}) = \varphi(x)\varphi(x^{-1}) = 1 = \varphi(x).$$

So $\varphi(x) = 1$ in $A/\mathfrak{p}$. Hence, $A/\mathfrak{p} = \{0, 1\}$, namely the additive and the multiplicative identities.

Exercise (12)**.** We wish to show that a local ring A has no idempotent element not equal to 0 or 1. So, let $x \in A$ be idempotent with $x \neq 0, 1$. We consider two cases — assume first that x is a unit. But then, we have $x = x^{-1}x^2 = x^{-1}x = 1$ which contradicts our initial assumption.

Assume therefore that x is *not* a unit. By proposition 1.5, we must have x contained in some maximal ideal $\mathfrak{m}$. Since A is local, there is only one maximal ideal, hence $x \in \mathfrak{m} \implies x \in \mathfrak{R}$. Since x is in the Jacobson radical, we know that $1 - xy$ is a unit in A for all y in A. Consider the fact that x is idempotent, so

$$x^2 = x \implies x(1 - x) = 0.$$

Since $x \neq 0, 1$ we have that $(1-x)$ is a zero divisor in A, contradicting the fact that it is also a unit.[1]

Consequently, if x is idempotent, it must be either 0 or 1.

Exercise (15. The prime spectrum of a ring)**.** Let A be a ring, and X the set of all prime ideals of A. For each subset E of A, define $V(E)$ to be the set of prime ideals of A containing E. We want to prove that the following holds:

(i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$;

(ii) $V(0) = X$, $V(1) = \emptyset$;

(iii) if $\{E_i\}_{i \in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i);$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

For (i), assume that $\mathfrak{a}$ is generated by E. That is

$$\mathfrak{a} = \left\{ \sum_{i \in I} a_i e_i \mid a_i \in A, e_i \in E \right\}. \tag{*}$$

Let $\mathfrak{p} \in V(E)$ be a prime ideal containing E. Since any ideal is an abelian additive group closed under multiplication from A we see that any product $a_i e_i$ is in $\mathfrak{p}$ for any $a_i \in A, e_i \in E \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an additive group, it follows that any sum $\sum_{i \in I} a_i e_i$ is also in $\mathfrak{p}$. So, all the elements in $\mathfrak{a}$ are in $\mathfrak{p}$ hence $\mathfrak{p} \supseteq \mathfrak{a}$ and consequently $\mathfrak{p} \in V(\mathfrak{a})$. Conversely, assume that $\mathfrak{p} \in V(\mathfrak{a})$. Since $\mathfrak{p}$ contains $\mathfrak{a}$ it must also contain any elements on the form $\sum_{i \in I} e_i$, so it contains E; set $a_i = 1$ for all i in eq. (*). We have the first equality, $V(E) = V(\mathfrak{a})$.

Now, let $\mathfrak{p} \in V(\mathfrak{a})$ so $\mathfrak{p}$ contains $\mathfrak{a}$. Consequently $\mathfrak{p} \supseteq \bigcup_{i \in I} \mathfrak{p}_i$ where each $\mathfrak{p}_i$ contain $\mathfrak{a}$. But by proposition 1.14, it follows that $\mathfrak{p}_i$ contain $\sqrt{\mathfrak{a}}$, so $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$. Conversely, let $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$. Then $\mathfrak{p}$ contains $\sqrt{\mathfrak{a}}$. Since the radical of an ideal contains the ideal, it follows that $\mathfrak{p}$ contains $\mathfrak{a}$. Hence $\mathfrak{p} \in V(\mathfrak{a})$. This yields the final equality: $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

For (ii), note that any prime ideal $\mathfrak{p}$ contains the zerp ideal, so $V(0) = X$. Similarly, by definition, a prime ideal is not equal to (1), so $V(1) = \emptyset$.

[1] Equivalently, since both x and $(1-x)$ are non-units, they must both lie in a maximal ideal. Since A is local they lie in the same maximal ideal. Since any ideal is an additive subgroup we have that $x + (1-x) = 1$ lie in the maximal ideal, hence the maximal ideal is the whole thing, (1), which is a contradiction.