Notes MAT4200 — Commutative Algebra

Ivar Stangeby

September 18, 2016

Contents

Exercises

Chapter 1

Exercise (1)**.** Assume that x is nilpotent, and that $1 + x$ is *not* a unit in A. Hence, $1 + x$ is contained in a maximal ideal $\mathfrak{m}$. Since any maximal ideal is prime, and x is nilpotent, we have $x^n = 0 \in \mathfrak{m} \implies x \in \mathfrak{m}$. Any ideal is an additive subgroup, so $1 \in \mathfrak{m}$ which contradicts the fact that $\mathfrak{m}$ is maximal.

Now assume u a unit and x nilpotent. Assume for the sake of contradiction that $u + x$ is *not* a unit in A. Then $u + x$ is contained in a maximal ideal $\mathfrak{m}$. Since x is nilpotent we have $x \in \mathfrak{m}$, hence $u \in \mathfrak{m}$ so $\mathfrak{m} = (1)$, again contradicting the fact that $\mathfrak{m}$ is maximal.

Exercise (4)**.** We want to show that in $A[x]$ we have $\mathfrak{N} = \mathfrak{R}$. We have trivially that $\mathfrak{N} \subseteq \mathfrak{R}$, so we only need to show the opposite inclusion.

Let $f \in \mathfrak{R}$ with $f = \sum_{i=0}^{n} a_i x^i$, so by proposition 1.9 we have $1 - fg$ a unit for all $g \in A[x]$. Let $g = x$ be an element in $A[x]$. Then the function

$$1 - a_0 x - a_1 x^2 - \ldots - a_n x^{n+1}$$

is a unit in $A[x]$. By exercise 1.2.(i) we have that $a_0, \ldots, a_n$ are nilpotent in A. By exercise 1.2.(ii) we have that f is nilpotent, so $f \in \mathfrak{N}$. Hence $\mathfrak{N} = \mathfrak{R}$.

Exercise (6)**.** Let A be a ring such that any ideal not contained in $\mathfrak{N}$ contains a non-zero idempotent element. We want to show that the nilradical and the Jacobson radical coincide in this case. We have the inclusion $\mathfrak{N} \subseteq \mathfrak{R}$ trivially. For the opposite inclusion we argue contrapositively. Let $c \notin \mathfrak{N}$. Then $(c) \not\subseteq \mathfrak{N}$. By assumption, (c) contains an idempotent element $a = cx$ for some $x \in A$. We wish to use proposition 1.9 again. Consider the element $1 - a$, and note that $a(1 - a) = a - a = 0$, so $1 - a$ is *not* a unit in A since it is a zero divisor. By proposition 1.9 we have $a \notin \mathfrak{R}$, so $(c) \not\subseteq \mathfrak{R}$. Consequently, $\mathfrak{R} \subseteq \mathfrak{N}$.

Exercise (7)**.** Let A be a ring in which every element satisfies $x^n = x$ for some $n \geq 2$ dependent on x. We want to show that the nilradical $\mathfrak{N}$ and the Jacobson radical $\mathfrak{R}$ coincide. The inclusion $\mathfrak{N} \subseteq \mathfrak{R}$ is trivial as any maximal ideal is prime. We show the opposite inclusion by a contrapositive argument.

Assume that $x \notin \mathfrak{N}$. Our plan is to show that $1 - xg$ is *not* a unit for any $g \in A$. Consider the element $1 - x \cdot x^{n-2}$. This is a zero divisor as shown by multiplying by x from the left. Hence $1 - xg$ is *not* a unit with $g = x^{n-2}$. By proposition 1.9 we then have $x \notin \mathfrak{R}$. This shows contrapositively that $\mathfrak{R} = \mathfrak{N}$.

Note that I did not prove that every prime ideal $\mathfrak{p}$ is also a maximal ideal as the exercise requested. Trying again below.

We seek to show that any prime ideal $\mathfrak{p}$ of A is maximal. Let $\varphi : A \to A/\mathfrak{p}$ be the canonical homomorphism. Let $x \in A$ be any element. We know that $x^n = x$ for some n. Let $\bar{x} = \varphi(x) \in A/\mathfrak{p}$. Note that $\bar{x} = \bar{x}^n$. Since $A/\mathfrak{p}$ is an integral domain due to $\mathfrak{p}$ being prime, we know that the cancellation law for multiplication holds. So $\bar{x}^n = 1x$ implies that $\bar{x}^{n-1} = 1$. So, $\bar{x}$ has inverse $\bar{x}^{n-2}$. Hence, $A/\mathfrak{p}$ is a field, which implies that $\mathfrak{p}$ is a maximal ideal.

Exercise (8)**.** Let A be a non-zero ring. We wish to show that the set of prime ideals of A has a minimal element with respect to set inclusion. This can be solved by an application of Zorn's Lemma. Let Σ denote the set of all prime ideals of A. Let $\mathfrak{q} \leq \mathfrak{p}$ if $\mathfrak{p} \subseteq \mathfrak{q}$ (note the reverse inclusion). This set is partially ordered with respect to this relation. We need to show that any chain Γ in Σ has an upper bound in Σ. Let $\mathfrak{P} = \bigcap_\alpha \mathfrak{p}_\alpha$. We claim that the intersection of all prime ideals $\mathfrak{P}$ is an element of Σ and an upper bound for Γ.

Let $xy \in \mathfrak{P}$. Then xy is an element of every prime ideal $\mathfrak{p}_\alpha$ in the chain Γ. Assume now that $x \notin \mathfrak{P}$. In this case we need to show that $y \in \mathfrak{P}$. Let $\mathfrak{p}_i$ be a prime ideal not containing x. It therefore contains y instead. Since Γ is totally ordered, we can consider all the elements $\mathfrak{p}_i \leq \mathfrak{p}'$, i.e., $\mathfrak{p}' \subseteq \mathfrak{p}_i$. Since $x \notin \mathfrak{p}_i$ we have $x \notin \mathfrak{p}'$. It then follows that all such $\mathfrak{p}'$ must contain y.

Consider now the elements $\mathfrak{p}' \geq \mathfrak{p}_i$, that is $\mathfrak{p}_i \subseteq \mathfrak{p}'$. Since $y \in \mathfrak{p}_i$ we have $y \in \mathfrak{p}'$. Consequently, we have y in all prime ideals, hence also in $\mathfrak{P}$. This shows that $\mathfrak{P}$ is an element of Σ.

To show that $\mathfrak{P}$ is an upper bound for Γ, let $I \in \Gamma$ be a prime ideal. Then by definition of $\mathfrak{P}$ we have $\mathfrak{P} \subseteq I$, hence $I \leq \mathfrak{P}$. Zorn's Lemma then guarantees the existence of a maximal element with respect the order $\leq$, and consequently we have shown the existence of a minimal element with respect to set inclusion.

Exercise (9)**.** Assume that $\mathfrak{a} = r(\mathfrak{a})$. By proposition 1.14 the radical of any ideal $\mathfrak{a}$ is the intersection of the prime ideals containing $\mathfrak{a}$. It therefore directly follows from our assumption that $\mathfrak{a}$ is an intersection of prime ideals. Assume now that $\mathfrak{a}$ is *not* an intersection of prime ideals. Then it cannot be equal to the radical, as the radical *is* an intersection of prime ideals.

Exercise (10)**.** Let A be a ring, and $\mathfrak{N}$ its nilradical. We wish to show that the following are equivalent:

(i) A has exactly one prime ideal;

(ii) every element of A is either a unit or nilpotent;

(iii) $A/\mathfrak{N}$ is a field.

We first show (i) $\implies$ (ii). Assume that A has exactly one prime ideal. Let x be an element in $\mathfrak{N}$. In that case it is nilpotent. Assume therefore that x is not an element in $\mathfrak{N}$. If we further assume that x is *not* a unit, then it is contained in a maximal ideal $\mathfrak{m}$. Since A has exactly one prime ideal, and any maximal ideal is prime, we must have $\mathfrak{m} = \mathfrak{N}$. This contradicts the fact that $x \notin \mathfrak{N}$. Consequently, x must be a unit. So any element x in A is either nilpotent, or a unit.

We now consider the implication (ii) $\implies$ (iii). Assume that every element of A is either a unit or nilpotent. We seek to show that $A/\mathfrak{N}$ is a field. In principle, we only need to show that $\mathfrak{N}$ is in fact a maximal ideal. Let $\mathfrak{a}$ be an ideal containing $\mathfrak{N}$. We need to show that $\mathfrak{a} = \mathfrak{N}$, or $\mathfrak{a} = A$.

If $\mathfrak{a} = \mathfrak{N}$, then we are done. Assume therefore that $\mathfrak{a} \neq \mathfrak{N}$. Then there is an element $x \in \mathfrak{N}$ that is not in $\mathfrak{a}$. Then x is *not* a nilpotent element, so by assumption it must be a unit. Since $\mathfrak{a}$ is an ideal containing a unit, we must have $\mathfrak{a} = A$. Hence $\mathfrak{N}$ is maximal and $A/\mathfrak{N}$ is a field.

We now show the final implication (iii) $\implies$ (i). Assume that $A/\mathfrak{N}$ is a field. Then $\mathfrak{N}$ is a maximal ideal in A. Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{N} \subseteq \mathfrak{p}$. If $\mathfrak{N} = \mathfrak{p}$ we are done. If not, since $\mathfrak{N}$ is maximal, we must have $\mathfrak{p} = A$, hence A contains *exactly* one prime ideal.

Exercise (11)**.** Let A be a boolean ring (i.e., $x^2 = x$ for all $x \in A$). We want to show that the following properties hold:

(i) $2x = 0$ for all $x \in A$;

(ii) every prime ideal $\mathfrak{p}$ in A is maximal, and $A/\mathfrak{p}$ is a field with two elements; and

(iii) every finitely generated ideal in A is principal.

For (i), let x be an element in A and let a be the additive inverse of x. That is $a + x = x + a = 0$. Multiplying both sides by $x - a$ yield

$$x^2 - a^2 = 0 \iff x - a = 0 \iff x = a.$$

It then follows that $2x = x + x = x + a = 0$.

For (ii), note that this is just a special case of exercise 7 with $n = 2$ for every $x \in A$, hence any prime ideal $\mathfrak{p}$ is also maximal. It remains to show that $A/\mathfrak{p}$ has two elements. Let x be an element of A and assume that $x \in \mathfrak{p}$. Then $\varphi(x) = 0$ in $A/\mathfrak{p}$. If $x \notin \mathfrak{p}$ then φx has an inverse, so it makes sense to look at $\varphi(x)\varphi(x^{-1}) = 1$. Multiplying by $\varphi(x)$ on both sides yields

$$\varphi(x^2)\varphi(x^{-1}) = \varphi(x)\varphi(x^{-1}) = 1 = \varphi(x).$$

So $\varphi(x) = 1$ in $A/\mathfrak{p}$. Hence, $A/\mathfrak{p} = \{0, 1\}$, namely the additive and the multiplicative identities.

Exercise (12)**.** We wish to show that a local ring A has no idempotent element not equal to 0 or 1. So, let $x \in A$ be idempotent with $x \neq 0, 1$. We consider two cases — assume first that x is a unit. But then, we have $x = x^{-1}x^2 = x^{-1}x = 1$ which contradicts our initial assumption.

Assume therefore that x is *not* a unit. By proposition 1.5, we must have x contained in some maximal ideal $\mathfrak{m}$. Since A is local, there is only one maximal ideal, hence $x \in \mathfrak{m} \implies x \in \mathfrak{R}$. Since x is in the Jacobson radical, we know that $1 - xy$ is a unit in A for all y in A. Consider the fact that x is idempotent, so

$$x^2 = x \implies x(1 - x) = 0.$$

Since $x \neq 0, 1$ we have that $(1-x)$ is a zero divisor in A, contradicting the fact that it is also a unit.[1]

Consequently, if x is idempotent, it must be either 0 or 1.

Exercise (15. The prime spectrum of a ring)**.** Let A be a ring, and X the set of all prime ideals of A. For each subset E of A, define $V(E)$ to be the set of prime ideals of A containing E. We want to prove that the following holds:

(i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$;

(ii) $V(0) = X$, $V(1) = \emptyset$;

(iii) if $\{E_i\}_{i \in I}$ is any family of subsets of A, then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i);$$

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

For (i), assume that $\mathfrak{a}$ is generated by E. That is

$$\mathfrak{a} = \left\{ \sum_{i \in I} a_i e_i \mid a_i \in A, e_i \in E \right\}. \tag{*}$$

Let $\mathfrak{p} \in V(E)$ be a prime ideal containing E. Since any ideal is an abelian additive group closed under multiplication from A we see that any product $a_i e_i$ is in $\mathfrak{p}$ for any $a_i \in A, e_i \in E \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an additive group, it follows that any sum $\sum_{i \in I} a_i e_i$ is also in $\mathfrak{p}$. So, all the elements in $\mathfrak{a}$ are in $\mathfrak{p}$ hence $\mathfrak{p} \supseteq \mathfrak{a}$ and consequently $\mathfrak{p} \in V(\mathfrak{a})$. Conversely, assume that $\mathfrak{p} \in V(\mathfrak{a})$. Since $\mathfrak{p}$ contains $\mathfrak{a}$ it must also contain any elements on the form $\sum_{i \in I} e_i$, so it contains E; set $a_i = 1$ for all i in eq. (*). We have the first equality, $V(E) = V(\mathfrak{a})$.

Now, let $\mathfrak{p} \in V(\mathfrak{a})$ so $\mathfrak{p}$ contains $\mathfrak{a}$. Consequently $\mathfrak{p} \supseteq \bigcup_{i \in I} \mathfrak{p}_i$ where each $\mathfrak{p}_i$ contain $\mathfrak{a}$. But by proposition 1.14, it follows that $\mathfrak{p}_i$ contain $\sqrt{\mathfrak{a}}$, so $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$. Conversely, let $\mathfrak{p} \in V(\sqrt{\mathfrak{a}})$. Then $\mathfrak{p}$ contains $\sqrt{\mathfrak{a}}$. Since the radical of an ideal contains the ideal, it follows that $\mathfrak{p}$ contains $\mathfrak{a}$. Hence $\mathfrak{p} \in V(\mathfrak{a})$. This yields the final equality: $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

For (ii), note that any prime ideal $\mathfrak{p}$ contains the zerp ideal, so $V(0) = X$. Similarly, by definition, a prime ideal is not equal to (1), so $V(1) = \emptyset$.

For (iii), let $\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right)$. This means that $\mathfrak{p}$ contains E_i for each $i \in I$, so $\mathfrak{p} \in V(E_i)$ for each $i \in I$. It then follows that $\mathfrak{p} \in \bigcup_{i \in I} V(E_i)$. Conversely, let $\mathfrak{p} \in \bigcup_{i \in I} V(E_i)$. This means that $\mathfrak{p}$ contains E_i for each $i \in I$. Consequently, $\mathfrak{p}$ contains the union of all E_i, hence $\mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right)$.

Chapter 2

Exercise (1)**.** We want to show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime. By definition of coprime, we have $\gcd(m, n) = 1$. We can find numbers $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Let $x \otimes y$ be an element of the tensor product. We therefore have

$$x \otimes y = 1\,(x \otimes y) = (am + bn)(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = 0.$$

[1] Equivalently, since both x and $(1-x)$ are non-units, they must both lie in a maximal ideal. Since A is local they lie in the same maximal ideal. Since any ideal is an additive subgroup we have that $x + (1-x) = 1$ lie in the maximal ideal, hence the maximal ideal is the whole thing, (1), which is a contradiction.

Exercise (2)**.** We let A be a ring, $\mathfrak{a}$ an ideal, and M an A-module. We wish to show that $(A/\mathfrak{a}) \otimes_A M \simeq M/\mathfrak{a}M$. We right tensor the exact sequence

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

with M. This gives another exact sequence (proposition 2.18)

$$\begin{tikzcd} \mathfrak{a} \otimes M \arrow[r, "f"] \arrow[dr] & A \otimes M \arrow[r, "g"] \arrow[d, "\psi"] & (A/\mathfrak{a}) \otimes M \arrow[r] & 0 \\ & M & & \end{tikzcd}$$

where ψ is the canonical isomorphism. By the first module isomorphism theorem we have

$$A \otimes M / \ker(g) \simeq \operatorname{im}(g).$$

Since the sequence is exact, g is injective, hence $\operatorname{im}(g) = (A/\mathfrak{a}) \otimes M$. We identify $\ker(g)$ by the elements in $(\psi \circ f)(\mathfrak{a} \otimes M)$. Hence,

$$A \otimes M / \ker(g) \simeq M/\mathfrak{a}M \simeq (A/a) \otimes M$$

as we wanted to show.

Exercise (3)**.** Let A be a local ring, and assume that M and N are finitely generated A-modules. Assume that $M \otimes_A N = 0$. We wish to show that then either $M = 0$, or $N = 0$. Define $k = A/\mathfrak{m}$ where $\mathfrak{m}$ is the only maximal ideal in A. Furthermore, let $M_k = k \otimes M$ and $N_k = k \otimes N$ be the k modules obtained from M and N by extension of scalars.

From the previous exercise, we know that $M_k \simeq M/\mathfrak{m}M$ and $N_k \simeq N/\mathfrak{m}N$. If we can show that either $M_k = 0$ or $N_k = 0$, then by Nakayama's lemma we have

$$M_k = 0 \implies M/\mathfrak{m}M = 0 \implies M = \mathfrak{m}M \implies M = 0$$

since $\mathfrak{m}$ is contained in the Jacobson radical of A and M is finitely generated (similarly for N_k). In order to show either $M_k = 0$ or $N_k = 0$ we consider the fact that $M \otimes_A N = 0$. This gives[2]

$$M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0.$$

Since $M_k \otimes_k N_k = 0$ it has dimension zero. The dimension of $M_k \otimes_k N_k$ is the product of the dimensions of M_k and N_k. However, this means that the dimension of either M_k or N_k is zero. Hence, either $M_k = 0$ or $N_k = 0$. Therefore, by the above mention of Nakayama's lemma, we have either $M = 0$ or $N = 0$.

Exercise (5)**.** We let $A[x]$ be the ring of polynomials over the indeterminate x. We wish to show that $A[x]$ is a flat A-algebra. We proceed accordingly:

1. Show that A is a flat A-module;
2. show that $M_i Ax^i \simeq A$;
3. show that $\bigoplus_i M_i = A[x]$;

[2]I think we use the fact that direct sum and direct product coincide if the index set is finite, in order to apply proposition 2.14(iii)

4. apply exercise 2.4.

To show that A is a flat A-module we check that any exact sequence of A-modules is exact when tensored with A. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Tensoring yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' \otimes A & \longrightarrow & M \otimes A & \longrightarrow & M'' \otimes A & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

This sequence is exact, hence A is a flat A-module. Define now $M_i = Ax^i$. Under the map $f_i : M_i \to A$ defined by $ax^i \mapsto a$ we see that $M_i \simeq A$, therefore each M_i is flat. It then follows that any element in $A[x]$ can be written as a sum of elements in the M_i's. That is,

$$\bigoplus_i M_i = A[x].$$

Applying exercise 2.4, we know that since each M_i is flat, that $A[x]$ is flat.