Mandatory Assignment 4
MATINF4130

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Introduction

In this assignment we take a look at the PageRank algorithm, that helped establish Google as a powerful search engine. The algorithm assigns to each web page its "popularity" in a way that mimics how a human would define a popular web page. Before discussing the algorithm itself, we start with a mathematical intermezzo.

Exercise 1 We first establish some notation. Let $\mathcal{S}$ denote the unit simplex

$$\mathcal{S} := \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}. \tag{1}$$

For later, we note that this is a closed and bounded set, which in $\mathbb{R}^n$ is equivalent to compact. We let $\boldsymbol{A}$ be a real $n \times n$ matrix with non-negative elements $a_{ij} \geq 0$, whose columns sum to one, and refer to this as a *stochastic matrix*. The image of $\mathcal{S}$ under $\boldsymbol{A}$ is denoted

$$\boldsymbol{A}(\mathcal{S}) := \{ \boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{S} \}. \tag{2}$$

a) If $\boldsymbol{y} \in \boldsymbol{A}(\mathcal{S})$ then $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ for some $\boldsymbol{x}$. Note that since both x_i and a_{ij} are non-negative, we must have y_i non-negative for $i = 1, \ldots, n$. The sum

$$\sum_{i=1}^{n} y_i = \sum_{j=1}^{n} x_j \Big(\sum_{i=1}^{n} a_{ij} \Big) = \sum_{j=1}^{n} x_j = 1 \tag{3}$$

tells us that $\boldsymbol{y} \in \mathcal{S}$ and consequently $\boldsymbol{A}(\mathcal{S}) \subseteq \mathcal{S}$.

b) Considering $\boldsymbol{A} \colon \mathcal{S} \to \mathcal{S}$ as a linear operator, it suffices to show that it is bounded to show continuity. We have that

$$\|\boldsymbol{A}\|_1 = \max_{\|x\|=1} \|\boldsymbol{A}\boldsymbol{x}\|_1 = 1 \tag{4}$$

so $\boldsymbol{A}$ is bounded, and therefore also continuous in the $\|\cdot\|_1$ norm.

c) Assume that $(\lambda, \boldsymbol{v})$ is an eigenpair for $\boldsymbol{A}$. Since $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} \in \mathcal{S}$, we must have $|\lambda| \leq 1$. Since $\mathcal{S}$ is closed and bounded it is compact, and since $\mathcal{S}$ is continuous, it follows by Brouwer's fixed-point theorem that there exists a $\boldsymbol{w}$ such that

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{w}. \tag{5}$$

Consequently, $(1, \boldsymbol{w})$ is a right eigenpair for $\boldsymbol{A}$.

From now on, we assume that the matrix entries a_{ij} are all strictly positive. Denote by $\mathcal{S}^\star$ the *interior* of $\mathcal{S}$:

$$\mathcal{S}^\star := \{\boldsymbol{x} \in \mathcal{S} \mid x_i > 0 \text{ for } i = 1, \ldots, n\}. \tag{6}$$

d) Let $\boldsymbol{x} \in \mathcal{S}$ and set $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Since at least one of the elements x_i are non-negative, and all a_{ij} are strictly positive, we have $y_i > 0$ for all $i = 1, \ldots, n$. This means that $\boldsymbol{y} \in \mathcal{S}^\star$, so $\boldsymbol{A}$ maps $\mathcal{S}$ to its interior.

e) Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$ be two distinct vectors. Since the components of $\boldsymbol{x}$ and the components of $\boldsymbol{y}$ sum to one, we have that the components of $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{y}$ sum to zero. This means that since $\boldsymbol{x}$ and $\boldsymbol{y}$ are different, $\boldsymbol{z}$ is non-zero, hence $z_j < 0$ for at least one j. We will need this fact to achieve a strict inequality. We have that

$$\begin{aligned} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{y}\|_1 = \|\boldsymbol{A}\boldsymbol{z}\|_1 &= \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij} z_j| \\ &< \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} |z_j| \\ &= \sum_{j=1}^{n} |z_j| = \|\boldsymbol{z}\|_1 = \|\boldsymbol{x} - \boldsymbol{y}\|_1 \end{aligned} \tag{7}$$

Consequently, $\boldsymbol{A}\colon \mathcal{S} \to \mathcal{S}$ is a *contraction* in the $\|\cdot\|_1$ norm. Assume that $\boldsymbol{w}_1 \neq \boldsymbol{w}_2$ are two distinct eigenvectors with eigenvalue one. Then

$$\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_1 = \|\boldsymbol{A}\boldsymbol{w}_1 - \boldsymbol{A}\boldsymbol{w}_2\|_1 < \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_1, \tag{8}$$

which is a contradiction. We can therefore conclude that the geometric multiplicity $g(\lambda)$ of the eigenvalue $\lambda = 1$ is one.