Assignment 1
MATINF4170
Spline methods

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Abstract

In this assignment we discuss two methods for curve generation, Neville-Aitken for interpolating curves, and de Casteljau for Bézier curves. We also prove that B-splines has compact support, given explicitly by the degree of the B-spline. Finally we use the recursive definition of B-splines of degree p to evaluate a specific B-spline.

Exercise 1.3

In this exercise we take a look at the Neville-Aitken algorithm for evaluating points on an interpolating curve. In the general setup we are given $p+1$ control points, $\{c_j\}_{j=0}^p$, as well as $p+1$ strictly increasing parameter values $\{t_j\}_{j=0}^p$. The integer p ends up being the degree of the interpolating polynomial. The algorithm was implemented in PYTHON and the source code is given in Listing 1. We wish to interpolate N points on the semicircle $\mathcal{C} := \{(\cos(t), \sin(t)) \mid t \in [0, \pi]\}$ and study the effect of varying the number of interpolation points N, as well as the effect of varying the parameter values.

We chose two distinct parametrizations, one being the uniform parametrization given by

$$t_{j+1} - t_j = 1$$

for $j = 0, \ldots, N-1$ and with $t_0 = 0$. The other one being tailored to misbehave for increasing N was generated using a cumulative sum of random numbers, that is:

$$t_{j+1} - t_j = r_j$$

where r_j is a random integer in $\mathbb{Z}_5$ and $t_0 = 0$. The uniform parametrization behaved quite nicely under increasing N, as can be seen in Figure 1. The random parametrization on the other hand was very unstable under increasing N and hence yields a very bad interpolation to the semi circle for large N, as seen in Figure 2.

Exercise 1.4

In this exercise we look at the de Casteljau algorithm for computing points on a Bézier curve. It is very similar to the Neville-Aitken algorithm, however
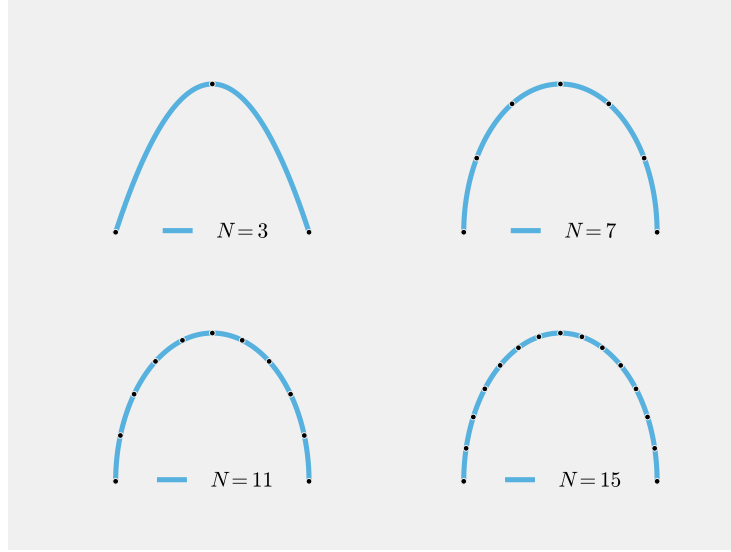
Figure 1: Uniform parametrization of the parameter values, i.e.,

$$\{t_j\}_{j=0}^{N-1} = \{0, 1, \ldots, N-1\}.$$

As N increases, we see that the interpolating polynomial converges fairly quickly to the unit semi circle. The uniform parametrization yields a good interpolation.
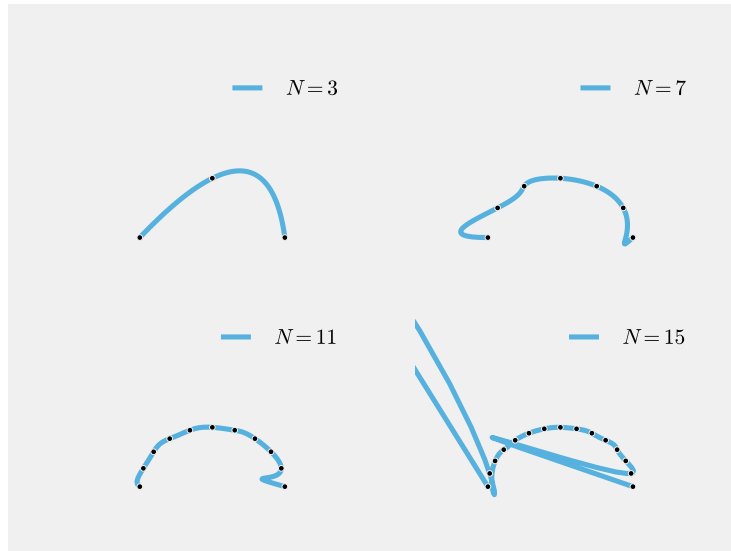

Figure 2: This parametrization was generated using a random generator. The parameter values for $N = 15$ ended up being

$$\{t_j\}_0^{14} = \{2, 4, 5, 6, 8, 10, 12, 14, 16, 17, 18, 20, 21, 23, 25\},$$

and for lower N a subsequence of this was used.

```python
def NevAit(T, c_points, t_points, eps=1.0E-14):
    """
    The Neville Aitkens Given p + 1 control points c_points and p+1
    strictly increasing parameter values t_points, computes the polynomial
    curve of degree p that interpolates the control points.
    """
    t = t_points
    c = np.array(c_points, dtype=np.float64)
    p = len(c) - 1
    for k in range(1, p + 1):
        for j in range(0, p - k + 1):
            denum = (t[j+k] - t[j])
            if abs(denum) <= eps:
                c[j] = 0
                continue
            l_one = (t[j+k] - T) / denum
            l_two = (T - t[j]) / denum
            c[j] = l_one*c[j] + l_two * c[j+1]
    return c[0]
```

Listing 1: The Neville-Aitken algorithm implemented in PYTHON. The algorithm uses in-place replacement of the newly computed values in order to save memory, as well as time. No other numerical optimizations has been made.

there is one big difference. While the Neville-Aitken algorithm parametrizes on adjacent parameter intervals, yielding convex combinations only in the last step of the algorithm, the de Casteljau algorithm parametrizes each line segment over the same interval. This has the added benefit of convex combinations in every step of the algorithm and yields higher numerical stability. Due to this, the resulting curve interpolates only the first and the last control point, hence the interpolatory nature of the Neville-Aitken is lost. The algorithm was implemented in PYTHON and the code can be seen in Listing 2.

Using the same data as in the previous exercise the resulting curve can be seen in Figure 3. As we can see the interpolation property has been lost for the interior control points.

Exercise 1.7

We wish to show that the jth B-spline of degree p, $B_{j,p}$ only relies on the knots $t_j, \ldots, t_{j+p+1}$. In addition we want to show that $B_{j,p}$ vanishes outside these knots. We first recall that $B_{j,0}$ is defined as

$$B_{j,0}(t) = \begin{cases} 1, & t_j \leq t < t_{j+1}; \\ 0, & \end{cases} \tag{1}$$

and that $B_{j,p}$ is defined recursively as

$$B_{j,p}(t) = \frac{t - t_j}{t_{j+k} - t_j} B_{j,p-1}(t) + \frac{t_{j+1+k} - t}{t_{j+k+1} - t_{j+1}} B_{j+1,k-1}(t). \tag{2}$$

```python
def deCasteljau(T, c_points, interval_start=0, interval_stop=1):
    """
    The de Casteljau algorithm.
    Given p + 1 control points c_points, computes the point on the bezier curve
    at parameter value T.
    """
    T = (interval_stop - T) / float(interval_stop - interval_start)
    c = np.array(c_points, dtype=np.float64)
    p = len(c_points) - 1

    for k in range(1, p+1):
        for j in range(0, p - k + 1):
            c[j] = (1-T)*c[j] + T * c[j+1]

    return c[0]
```

Listing 2: The de Casteljau algorithm for computing points on a Bézier curve. Again, the implementation uses in place replacement of newly computed values. No other numerical optimizations has been made.

We apply strong induction on p. The base case with $p = 0$ follows immediately from the definition given in Equation (1). Assume inductively that the hypothesis holds for $p = 0, 1, \ldots, k - 1$. We wish to show that it must then also hold for $p = k$. However, since the two lower degree B-splines depend only on the knots $t_j, \ldots, t_{j+k-1}$ and $t_{j+1}, \ldots, t_{j+k}$, and we only introduce the new knot t_{j+k+1} in the recurrence relation, the claim holds. This closes the induction.

To show that $B_{j,p}$ vanishes outside of the knot interval discussed above, we first discuss the case where $t < t_j$ and in this case $B_{j,0} = 0$. In addition, since the knots are assumed to be increasing $t_j \leq t_{j+1}$, we also have that $B_{k,0} = 0$ for all $k > j$. Consequently, all terms in the recurrence relation given in Equation (2) are zero, hence $B_{j,p} = 0$. Assume now that $t > t_{j+p+1}$, by the same argument above, $B_{k,0}$ are zero for all $k < j + p + 1$. Hence $B_{j,p} = 0$.

This result gives us a computational advantage, as for any given parameter value t, we only need to evaluate the B-splines that are "active" on the respective knot intervals. This is illustrated in Figure 4.

Exercise 2.1

We wish to evaluate the B-spline $B[0, 3, 4, 6](x)$. We first recall that for notational purposes, we often write
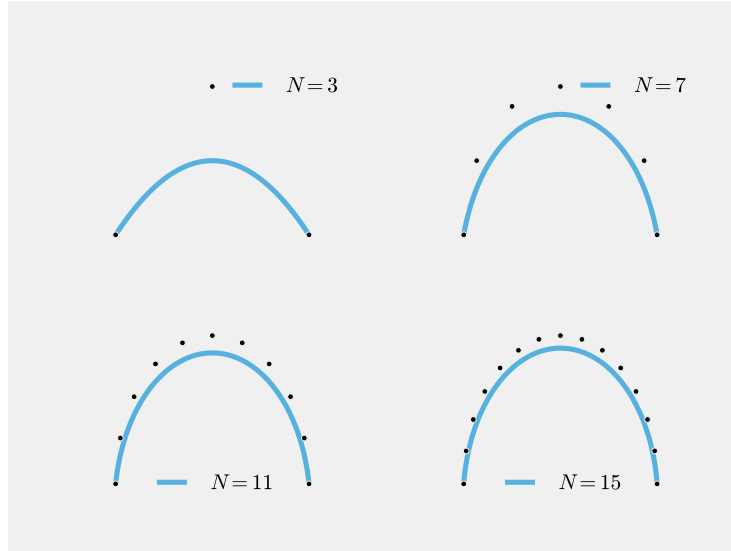
$$B_{j,p} = B[t_j, \ldots, t_{j+p+1}]$$

4

Figure 3: Computing convex combinations of the line segments yields what we call a Bézier curve. The curve has nice geometric properties due to it always lying in the convex hull of its control points. It therefore resembles the control polygon nicely. It does however not interpolate the interior control points.
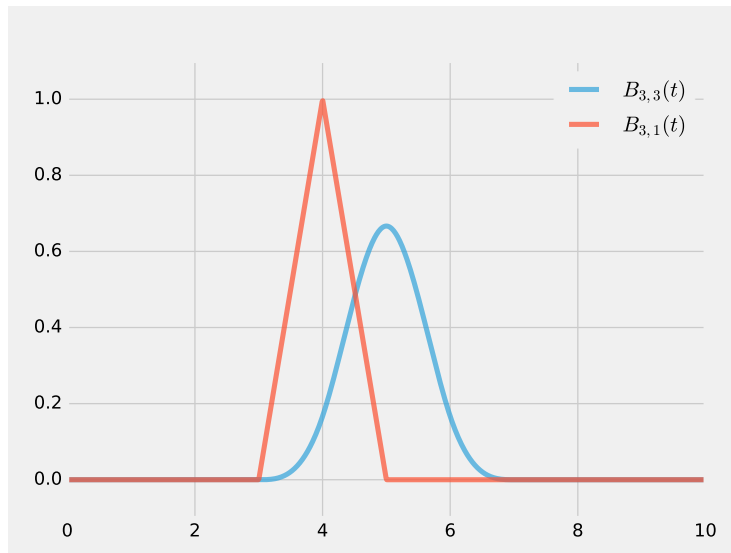

Figure 4: The third B-splines of degree three and one respectively. The knots are uniform. Here we clearly see the support of the basis splines, and how each B-spline has a set of knots where it is non-zero.

if the knots have to be emphasized. Using Equation (2) we compute directly:

$$
\begin{aligned}
B[0,3,4,6](x) &= \frac{x}{4}B[0,3,4](x) + \frac{6-x}{3}B[3,4,6](x) \\
&= \frac{x}{4}\left(\frac{x}{3}B[0,3](x) + \frac{x(4-x)}{4}B[3,4](x)\right) + \frac{(6-x)}{3}\left((x-3)B[3,4](x) + \frac{(6-x)}{2}B[4,6](x)\right) \\
&= \frac{1}{12}x^2 B[0,3](x) + \frac{1}{12}(-7x^2 + 48x - 72)B[3,4](x) + \frac{1}{6}(6-x)^2 B[4,6](x).
\end{aligned}
$$